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On some anisotropic elliptic problem with measure data

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Abstract

We prove optimal existence results for entropy solutions to some anisotropic boundary value problems like

$$\begin{cases} -\sum_{i=1}^{N} D^i A_i(x,w,\nabla w) = f - \operatorname{div} F(w) \text{ in } \Omega, & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0.1}$$

where $f \in L^1(\Omega)$, $F = (F_1, ..., F_N)$ satisfies $F \in (C^0(\mathbb{R}))^N$.and Ω is a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, and the function $A_i(x,s,\xi)$ verify the large monotonicity condition. The construction of the proof of our theorem is done by using Minty's Lemma in its modified version.

Keywords: Entropy solutions, nonlinear elliptic equations, anisotropic Sobolev spaces, entropy solutions
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1 Introduction

The study of anisotropic elliptic equations on bounded domain has been intensively studied by large number of scientists and researchers, this study is motived by the fact that this type of equations can intimate connections with some application in elasticity, in the process of image restoration and Stochastic Processes with constraints (see for instance [40, 12], and references therein).

In order to fix the ideas let us consider the strongly anisotropic elliptic problems as

$$\begin{cases} -\sum_{i=1}^{N} D^i A_i(x,w,\nabla w) = f - \operatorname{div} F(w) & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^1(\Omega)$, $F \in (C^0(\mathbb{R}))^N$, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$.

In large recent researches, existence result with some qualitative properties and regularity of nonlinear anisotropic elliptic equations where the data belonns to L^1- have been proved see the references [16] when Badr EL HAJI et al have been shown the existence result of entropy solution in weighted-Orlicz spaces, other works found by Youssef

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AKDIM et al. in their paper [1] devoted to study a degenerated problem (0.1) via Minty's Lemma in weigthed Orlicz-Sobolev space, in the similar direction faria et al (see [22]) have been treated the similar problem as (0.1) where the solution u of the elliptic problem studied depend on the gradient.

On the other hand, by using as main tool an L^1 version of Minty's lemma EL HAJI et al (see [2]) extending the main result under studies to the Musielak-orlicz spaces by giving an existence result for an entropy solutions of elliptic problem as

$$\begin{cases} L(w) = g(x) & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $g \in L^1(\Omega)$ and $L(w) = -\operatorname{div} l(x, w, \nabla w)$.

The mathematical researches dealing the existence of solutions to some problems parabolic and elliptic under a different assumptions is massive; we refer the reader to [7, 15, 20, 3, 4, 5, 8, 17, 10, 19, 9, 18, 14, 13, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] and the references therein.

Our goal in this paper is to solve the problem (0.1)(existence results) where the function $A_i(x, s, \xi)$ satisfy the large monotonicity condition and without adopting the almost everywhere convergence of the gradients, and in order to overcome this difficulties, we exploit the technique of Minty's lemma for proving the existence of an entropy solutions, However the approach that we used in the proof differs from that adopted by A. Benkirane et al used in [6]

The outline of this note is as follows. After giving the definition and some auxiliary results on anisotropic Sobolev space, we recall in Section 3 some essential assumptions which are necessary to have an existence solution, finally section 4 will be devoted to give our main results and their proofs.

2 Preliminaries

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$. Let $p_1, \ldots, p_N$ be N real constants numbers, with $\infty > p_i > 1$ for $i = 1, \ldots, N$. We set

$$\vec{p} = (p_1, \cdots, p_N), \underline{p} = p^-, p_0 = p^+ \quad D^0 w = w \quad \text{and} \quad D^i w = \frac{\partial w}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, N,$$

and we set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad p_0 = \max\{p_1, p_2, \ldots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ like :

$$W^{1,\vec{p}}(\Omega) = \left\{ w \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i w \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N \right\},$$

endowed with the norm

$$\|w\|_{\vec{p}} = \|w\|_{L^1(\Omega)} + \sum_{i=1}^{N} \left\| D^i w \right\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|w\|_{1,\vec{p}}\right)$ is a reflexive Banach (separable) space (cf [25]). We denote by $W_0^{1,\vec{p}}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to (2.1).

Proposition 2.1. (see. [21, 26]) Let $w \in W_0^{1,\vec{p}}(\Omega)$, we have
(i) : there exists $C_p > 0$, such that

$$\|w\|_{L^{p_i}(\Omega)} \leq C_p \sum_{i=1}^{N} \|D^i w\|_{L^{p_i}(\Omega)} \qquad \text{for any} \quad i = 1, \ldots, N.$$

(ii) : there exists $C_s > 0$, such that

$$\|w\|_{L^q(\Omega)} \leq \frac{C_s}{N} \sum_{i=1}^{N} \left\| \frac{\partial w}{\partial x_i} \right\|_{L^{p_i}(\Omega)},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geq N \end{cases}$$

Lemma 2.2. Let Ω be a bounded open set in $I\!R^N$ $(N \geq 2)$, we set

$$s = \max(q, \max_{1 \leq i \leq N} p_i),$$

therefore, the embedding listed below holds :

- if $\overline{p} < N$ so $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,
- if $\overline{p} = N$ therefore $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty[$,
- if $\overline{p} > N$ we have $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.

The proof of the above result (lemma 2.2) depends to the Proposition 2.1.

Definition 2.3. For $k > 0$, we give the following truncation $T_k(\cdot) : I\!R \longmapsto I\!R$, that will be used latter

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \{w : \Omega \mapsto I\!R \text{ measurable } / T_k(w) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Lemma 2.4. Let $w \in \mathcal{T}_0^{1,\vec{p}}(\Omega)$, there exists one function $v_i : \Omega \mapsto I\!R$ measurable with $i \in \{1, \ldots, N\}$, such that

$$\forall k > 0 \qquad D^i T_k(w) = v_i . \chi_{\{|w|<k\}} \qquad \text{a.e.} \quad x \in \Omega,$$

with χ_A be a characteristic function of a measurable set A. v_i define the weak partial derivatives of w denoted by $D^i w$. Therefore, if $w \in W_0^{1,1}(\Omega)$, then $(v_i = D^i w.)$

Lemma 2.5. (see [24], Theorem 13.47) Let $(w_n)_n$ be a sequence in $L^1(\Omega)$ and $w \in L^1(\Omega)$ such that

(i) $w_n \to w$ a.e. in Ω,

(ii) $w_n \geq 0$ and $w \geq 0$ a.e. in Ω,

(iii) $\displaystyle\int_\Omega w_n \, dx \to \int_\Omega w \, dx$,

then $w_n \to w$ in $L^1(\Omega)$.

Lemma 2.6. Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. If $w \in (W_0^{1,\vec{p}}(\Omega))^N$ then $\displaystyle\int_\Omega \operatorname{div}(w)dx = 0$.

Proof . Fix a vector $w = \left(w^1, \ldots, w^N\right) \in \left(W_0^{1,\vec{p}}(\Omega)\right)^N$. since $W_0^1 L_\varphi(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_0^{1,\vec{p}}(\Omega)$, then each term w^i can be approximated by a suitable sequence $w_k^i \in D(\Omega)$ such that w_k^i converges to w^i in $W_0^{1,\vec{p}}(\Omega)$. Moreover, due to the fact that $w_k^i \in C_0^\infty(\Omega)$, then the Green formula gives

$$\int_\Omega \frac{\partial w_k^i}{\partial x_i} = \int_{\partial\Omega} w_k^i \vec{n} ds = 0. \tag{2.2}$$

On the other hand, $\frac{\partial w_k^i}{\partial x_i} \to \frac{\partial w^i}{\partial x_i}$ in $L_\varphi(\Omega)$. Thus $\frac{\partial w_k^i}{\partial x_i} \to \frac{\partial w^i}{\partial x_i}$ in $L^1(\Omega)$, which gives in view of (2.2) that

$$\int_\Omega \operatorname{div}(w)dx = 0.$$

□

3 Essential assumptions

We consider a Leray-Lions operator $\mathbb{A} : W_0^{1,\vec{p}}(\Omega) \longmapsto W^{-1,\vec{p}'}(\Omega)$ modeled by

$$\mathbb{A}w = -\sum_{i=1}^{N} D^i A_i(x, w, \nabla w)$$

where $A_i : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \mapsto \boldsymbol{R}$ are Carathéodory functions, for $i = 1, \ldots, N$, which satisfy the hypothesis listed bellow as follows:

$$|A_i(x, s, \xi)| \leq \beta \left(R_i(x) + |s|^{p_i - 1} + |\xi|^{p_i - 1} \right) \quad \text{for} \quad i = 1, \ldots, N, \tag{3.1}$$

$$A_i(x, s, \xi)\xi_i \geq \alpha \, |\xi_i|^{p_i} \quad \text{for} \quad i = 1, \ldots, N, \tag{3.2}$$

$$(A_i(x, s, \xi) - A_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \text{ for } \xi_i \neq \xi_i', \tag{3.3}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, where $R_i(x) \in L^{p_i'}(\Omega)$ and $p_i - 1 > q_i > 0$ for $i = 1, \ldots, N$, where $R_i(x), \alpha, \beta > 0$.

$$f \in L^1(\Omega) \tag{3.4}$$

and $F = (F_1, ..., F_N)$ satisfies

$$F \in (C^0(\mathbb{R}))^N. \tag{3.5}$$

The following section devoted to stating our Main results and their proofs

4 Main results

The approach used by Boccardo [11] of entropy solution is given by the following notion.

Definition 4.1. A function w (mesurable) is named an entropy solution of (0.1) if $T_k(w) \in W_0^{1,\vec{p}}(\Omega)$ and satisfy

$$\sum_{i=1}^{N} \int_\Omega A_i(x, w, \nabla w) D^i T_k(w - \Phi) dx \leq \int_\Omega f T_k(w - \Phi) dx + \int_\Omega F(w) \nabla T_k(w - \Phi) dx$$

for any $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.2. Suppose that (3.1)-(3.4) are holds, then the problem (0.1) admit one entropy solution w.

4.1 The key Lemma

Lemma 4.3. Let w (mesurable function) such that $T_k(w) \in W_0^{1,\vec{p}}(\Omega)$ for every $k > 0$. Then

$$\sum_{i=1}^{N} \int_\Omega A_i(x, w, \nabla w) D^i T_k(w - \Phi) dx \; dx \leq \int_\Omega f \; T_k(w - \Phi) dx + \int_\Omega F(w) \nabla T_k(w - \Phi) dx \tag{4.1}$$

is equivalent to

$$\sum_{i=1}^{N} \int_\Omega A_i(x, w, \nabla w) D^i T_k(w - \Phi) \; dx = \int_\Omega f \; T_k(w - \Phi) dx + \int_\Omega F(w) \nabla T_k(w - \Phi) dx \tag{4.2}$$

for every $\Phi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$.

4.2 Proof of The key lemma

It's clear that The equation (4.2) implies (4.1). Now, by adding and subtracting

$$\sum_{i=1}^{N}\int_{\Omega} A_i(x,w,\nabla w)D^iT_k(w-\Phi)\ dx,$$

therefore by using assumption (3.2), we can prove that (4.1) implies (4.2). Let $h,k>0$, let $\lambda\in\,]-1,1[$ and $\Theta\in W_0^{1,\vec{p}}(\Omega)\cap L^{\infty}(\Omega)$. We take, $\Phi=T_h(w-\lambda T_k(w-\Theta))\in W_0^{1,\vec{p}}(\Omega)\cap L^{\infty}(\Omega)$ in (4.1), we get:

$$E_{hk}\le F_{hk}, \tag{4.3}$$

with

$$E_{hk} = \sum_{i=1}^{N}\int_{\Omega} A_i(x,w,D^iT_h(w-\lambda T_k(w-\Theta)))D^iT_k(w-T_h(w-\lambda T_k(w-\Theta))))\quad dx,$$

and

$$F_{hk}=\int_{\Omega} f\ T_k(w-T_h(w-\lambda T_k(w-\Theta))))\ dx+\int_{\Omega} F(w)\nabla T_k(w-T_h(w-\lambda T_k(w-\Theta))).$$

Put

$$S_{hk}=\{x\in\Omega, |w-T_h(w-\lambda T_k(w-\Theta))|\le k\},$$

and

$$T_{hk}=\{x\in\Omega, |w-\lambda T_k(w-\Theta)|\le h\}.$$

Then, we obtain

$$\begin{aligned}E_{hk}=&\sum_{i=1}^{N}\int_{S_{kh}\cap T_{hk}} A_i(x,w,D^iT_h(w-\lambda T_k(w-\Theta)))D^iT_k(w-T_h(w-\lambda T_k(w-\Theta)))\ dx\\
&+\sum_{i=1}^{N}\int_{S_{kh}\cap T_{hk}^C} A_i(x,w,D^iT_h(w-\lambda T_k(w-\Theta)))D^iT_k(w-T_h(w-\lambda T_k(w-\Theta))))\ dx\\
&+\sum_{i=1}^{N}\int_{S_{kh}^C} A_i(x,w,D^iT_h(w-\lambda T_k(w-\Theta)))D^iT_k(w-T_h(w-\lambda T_k(w-\Theta)))\ dx.\end{aligned}$$

Since $D^iT_k(w-T_h(w-\lambda T_k(w-\Theta)))\neq 0$ on S_{kh}, we get

$$\sum_{i=1}^{N}\int_{S_{kh}^C} A_i(x,w,D^iT_h(w-\lambda T_k(w-\Theta)))D^iT_k(w-T_h(w-\lambda T_k(w-\Theta)))\ dx=0. \tag{4.4}$$

Therefore, if $x\in T_{hk}^C$, we can get $D^iT_h(w-\lambda T_k(w-\Theta))=0$ and using (3.3), we conclude the following equality,

$$\begin{aligned}&\sum_{i=1}^{N}\int_{S_{kh}\cap T_{hk}^C} A_i(x,w,D^iT_h(w-\lambda T_k(w-\Theta)))D^iT_k(w-T_h(w-\lambda T_k(w-\Theta))))\ dx\\
&=\sum_{i=1}^{N}\int_{S_{kh}\cap T_{hk}^C} A_i(x,w,0)D^iT_k(w-T_h(w-\lambda T_k(w-\Theta))))\ dx=0.\end{aligned} \tag{4.5}$$

According to (4.4) and (4.5), we get

$$E_{hk}=\sum_{i=1}^{N}\int_{S_{kh}\cap T_{hk}} A_i(x,w,D^iT_h(w-\lambda T_k(w-\Theta)))D^iT_k(w-T_h(w-\lambda T_k(w-\Theta))))\ dx.$$

Let $h \to +\infty, |\lambda| \leq 1$, we obtain

$$S_{kh} \to \{x, |\lambda||T_k(w-\Theta)| \leq h\} = \Omega, \tag{4.6}$$

$$T_{hk} \to \Omega \quad \text{implies that} \quad S_{kh} \cap T_{hk} \to \Omega. \tag{4.7}$$

By applying the Lebesgue theorem, we obtain

$$\lim_{h\to+\infty} \sum_{i=1}^{N} \int_{S_{kh}\cap T_{hk}} A_i(x,u,D^i T_h(w-\lambda T_k(w-\Theta)))D^i T_k(w - T_h(w-\lambda T_k(w-\Theta)))) \, dx$$
$$= \lambda \sum_{i=1}^{N} \int_{\Omega} A_i(x,w,\nabla(w-\lambda T_k(w-\Theta)))D^i T_k(w-\Theta) \, dx. \tag{4.8}$$

thus implies that,

$$\lim_{h\to+\infty} E_{hk} = \lambda \sum_{i=1}^{N} \int_{\Omega} A_i(x,w,\nabla(w-\lambda T_k(w-\Theta)))D^i T_k(w-\Theta) \, dx. \tag{4.9}$$

Moreover, one has

$$F_{hk} = \int_{\Omega} f \; T_k(w - T_h(w-\lambda T_k(w-\Theta))) \, dx + \int_{\Omega} F(w)\nabla T_k(w - T_h(w-\lambda T_k(w-\Theta))) \, dx.$$

Then

$$\lim_{h\to+\infty} \int_{\Omega} f \; T_k(w - T_h(w-\lambda T_k(w-\Theta))) \, dx + \int_{\Omega} F(w)\nabla T_k(w - T_h(w-\lambda T_k(w-\Theta))) \, dx. \tag{4.10}$$
$$= \lambda(\int_{\Omega} fT_k(w-\Theta)dx + \int_{\Omega} F(w)\nabla(T_k(w-\Theta))) \, dx., \tag{4.11}$$

i.e.,

$$\lim_{h\to+\infty} F_{hk} = \lambda(\int_{\Omega} fT_k(w-\Theta)dx + \int_{\Omega} F(w)\nabla(T_k(w-\Theta))) \, dx. \tag{4.12}$$

Thanking to (4.9), (4.12) therefore by passing to the limit in (4.3), we can get,

$$\lambda\left(\sum_{i=1}^{N} \int_{\Omega} A_i(x,w,\nabla(w-\lambda T_k(w-\Theta)))D^i T_k(w-\Theta) \, dx\right) \leq \lambda\left(\int_{\Omega} fT_k(w-\Theta)dx + \int_{\Omega} F(w)\nabla(T_k(w-\Theta)) \, dx\right) \tag{4.13}$$

for every $\Theta \in W_0^{1,\vec{p}}(\Omega) \cap L^{\infty}(\Omega)$, and for every $k > 0$. Let us take $\lambda > 0$, dividing by λ, and $\lambda \longrightarrow 0$, we have

$$\sum_{i=1}^{N} \int_{\Omega} A_i(x,w,\nabla w)D^i T_k(w-\Theta) \, dx \leq \int_{\Omega} fT_k(w-\Theta)dx + \int_{\Omega} F(w)\nabla(T_k(w-\Theta))) \, dx. \tag{4.14}$$

for $\lambda < 0$, dividing by λ, and $\lambda \longrightarrow 0$, we have

$$\sum_{i=1}^{N} \int_{\Omega} A_i(x,w,\nabla w)D^i T_k(w-\Theta) \, dx \geq \int_{\Omega} fT_k(w-\Theta)dx + \int_{\Omega} F(w)\nabla(T_k(w-\Theta))) \, dx. \tag{4.15}$$

Thanking to (4.14) and (4.15), we deduce that :

$$\sum_{i=1}^{N} \int_{\Omega} A_i(x,w,\nabla w)D^i T_k(w-\Theta) \, dx = \int_{\Omega} fT_k(w-\Theta)dx + \int_{\Omega} F(w)\nabla(T_k(w-\Theta))) \, dx. \tag{4.16}$$

This achieve the demonstration of Lemma 4.3.

4.3 Proof of Main results

4.3.1 Approximate problem

For $n \in \mathbb{N}$, define $f_n := T_n(f)$. Let $w_n \in W_0^{1,\vec{p}}(\Omega)$ be solution of the approximate equation of the type

$$\begin{cases} A_n w_n = f_n - \operatorname{div} F_n(w_n) & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.17}$$

which exists according to ([23]). We take $T_k(w_n)$ as test in (4.17), we get

$$\sum_{i=1}^{N} \int_\Omega A_i\left(x, T_n\left(w_n\right), \nabla w_n\right) D^i T_k\left(u_n\right)\ dx = \int_\Omega f_n T_k(w_n)\ dx + \int_\Omega F_n(w_n) \nabla T_k(w_n)\ dx.$$

We claim that:

$$\int_\Omega F_n(w_n) \nabla T_k(w_n)\ dx = 0, \tag{4.18}$$

using $\nabla T_k(w_n) = \nabla w_{n\chi\{|w_n| \le k\}}$, define $\Theta(t) = F_n(t)\chi\{t| \le k\}$, and $\tilde{\Theta}(t) = \displaystyle\int_0^t \Theta(\tau)\ d\tau$, by using the divergence lemma we can get $\tilde{\Theta}(w_n) \in (W_0^1 L_\varphi(\Omega))^N$

$$\begin{aligned} \int_\Omega F_n(w_n) \nabla T_k(w_n)\ dx &= \int_\Omega F_n(w_n)\chi\{|w_n| \le k\} \nabla w_n\ dx \\ &= \int_\Omega \Theta(w_n) \nabla w_n\ dx = \int_\Omega \operatorname{div}(\tilde{\Theta}(w_n))\ dx = 0 \end{aligned} \tag{4.19}$$

(by lemma 2.6) which proves the claim. Now thanks to (3.3), we obtain

$$\sum_{i=1}^{N} \int_\Omega A_i\left(x, w_n, \nabla w_n\right) D^i T_k\left(w_n\right)\ dx \ge \alpha \int_\Omega \left|D^i T_k\left(w_n\right)\right|^{p_i}\ dx,$$

then

$$\int_\Omega \left|D^i T_k\left(w_n\right)\right|^{p_i}\ dx \le k\|f\|_{L^1(\Omega)}. \tag{4.20}$$

Then

$$\int_\Omega \left|D^i T_k\left(w_n\right)\right|^{p_i}\ dx \le C_1 k, \tag{4.21}$$

where C_1 is a constant independently of n.

4.3.2 Locally convergence of w_n in measure

Taking $\lambda|T_k(w_n)|$ in (4.17) and using (4.21), one has

$$\int_\Omega \lambda_1 \frac{|D^i T_k(w_n)|^{p_i}}{\lambda} dx \le \int_\Omega \lambda_1 |D^i T_k(w_n)|^{p_i} dx \le C_1 k. \tag{4.22}$$

by using (4.22), we can have

$$\begin{aligned} meas\{|w_n| > k\} &\le \frac{1}{\inf\limits_k \frac{k}{\lambda}} \int_{\{|w_n|>k\}} \frac{|w_n(x)|^{p_i}}{\lambda} dx \\ &\le \frac{1}{\inf\limits_k \frac{k}{\lambda}} \int_\Omega \frac{1}{\lambda} |T_k(w_n)|^{p_i} dx \\ &\le \frac{C_1 k}{\inf\limits_k \frac{k}{\lambda}} \quad \forall n, \quad \forall k \ge 0. \end{aligned} \tag{4.23}$$

For any $\beta > 0$, we have

$$meas\{|w_n - w_m| > \beta\} \le meas\{|w_n| > k\} + meas\{|w_m| > k\} + meas\{|T_k(w_n) - T_k(w_m)| > \beta\},$$

and so that

$$meas\{|w_n - w_m| > \beta\} \le \frac{2C_1 k}{\inf\limits_{x\in\Omega} \frac{k}{\lambda}} + meas\{|T_k(w_n) - T_k(w_m)| > \beta\}. \tag{4.24}$$

By Applying Poincaré inequality (proposition 2.1) and according to (4.21) we obtain the boundedness of $(T_k(w_n))$ in $W_0^{1,\vec{p}}(\Omega)$, therefore there exists $\omega_k \in W_0^{1,\vec{p}}(\Omega)$ such that $T_k(w_n) \rightharpoonup \omega_k$ weakly in $W_0^{1,\vec{p}}(\Omega)$, strongly in $L^{\underline{p}}(\Omega)$ and a.e. in Ω. So, we suppose that $(T_k(w_n))_n$ is a Cauchy sequence in measure in Ω. Let $\varepsilon > 0$, then by (4.24) and in view of $\frac{2C_1 k}{\inf\limits_{x\in\Omega} \frac{k}{\lambda}} \to 0$ as $k \to +\infty$ there exists some $k = k(\varepsilon) > 0$ such that

$$meas\{|w_n - w_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \ge h_0(k(\varepsilon), \lambda).$$

This proves that w_n is a Cauchy sequence in measure, thus, w_n converges almost everywhere to w (measurable function). Therefore, there exist a subsequence of $\{w_n\}_n$, still indexed by n, and a function $w \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} w_n \rightharpoonup w \quad \text{weakly in } W_0^{1,\vec{p}}(\Omega) \\ w_n \longrightarrow w \quad \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{4.25}$$

4.3.3 An intermediate Inequality

Here, we can show that, for $\Phi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, we get

$$\sum_{i=1}^N \int_\Omega A_i(x, w_n, \nabla\Phi) D^i T_k(w - \Phi)\, dx \le \int_\Omega f_n\, T_k(w_n - \Phi)\, dx + \int_\Omega F_n \nabla T_k(w_n - \Phi)\, dx. \tag{4.26}$$

Now, we take $T_k(w_n - \Phi)$ as test in (4.17), with Φ in $W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, we can obtain

$$\sum_{i=1}^N \int_\Omega A_i(x, w_n, \nabla\Phi) D^i T_k(w - \Phi)\, dx = \int_\Omega f_n T_k(w_n - \Phi)\, dx + \int_\Omega F_n \nabla T_k(w_n - \Phi)\, dx. \tag{4.27}$$

The term $\sum_{i=1}^N \int_\Omega A_i(x, w_n, \nabla\Phi) D^i T_k(w - \Phi)\, dx$ can be added and subtracted to the equation (4.27) we can obtain,

$$\sum_{i=1}^N \int_\Omega A_i(x, w_n, \nabla w_n) D^i T_k(w - \Phi)\, dx + \sum_{i=1}^N \int_\Omega A_i(x, w_n, \nabla\Phi) D^i T_k(w - \Phi)\, dx \tag{4.28}$$

$$-\sum_{i=1}^N \int_\Omega A_i(x, w_n, \nabla\Phi) D^i T_k(w - \Phi)\, dx = \int_\Omega f_n T_k(w_n - \phi) + \int_\Omega F_n \nabla T_k(w_n - \Phi)\, dx. - \Phi) dx.$$

By (3.2) and truncation function, we can get

$$\sum_{i=1}^N \int_\Omega (A_i(x, w_n, \nabla w_n) - A_i(x, w_n, \nabla\Phi)) D^i T_k(w - \Phi)\, dx \ge 0. \tag{4.29}$$

According to (4.28) and (4.29), we get (4.26).

4.3.4 Passing to the limit

We verify that for $\Phi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, one has

$$\sum_{i=1}^{N} \int_\Omega A_i(x, w_n, \nabla\Phi) D^i T_k(w_n - \Phi)\ dx \leq \int_\Omega f T_k(w - \Phi) dx + \int_\Omega F \nabla T_k(w - \Phi)\ dx.$$

Now, we show that

$$\sum_{i=1}^{N} \int_\Omega A_i(x, w_n, \nabla\Phi) D^i T_k(w_n - \Phi)\ dx \to \sum_{i=1}^{N} \int_\Omega A_i(x, w, \nabla\Phi) D^i T_k(w - \Phi)\ dx \text{ as } n \to +\infty.$$

as $T_M(w_n) \rightharpoonup T_M(w)$ weakly in $W_0^{1,\vec{p}}(\Omega)$, with $M = k + \|\Phi\|_\infty$, therefore

$$T_k(w_n - \Phi) \rightharpoonup T_k(w - \Phi) \text{ in } W_0^{1,\vec{p}}(\Omega), \tag{4.30}$$

then

$$\frac{\partial T_k}{\partial x_i}(w_n - \Phi) \rightharpoonup \frac{\partial T_k}{\partial x_i}(w - \Phi) \text{ weakly in } L^{\vec{p}}(\Omega) \qquad \forall i = 1, .., N. \tag{4.31}$$

Let us prove that

$$A_i(x, T_M(w_n), \nabla\Phi) \to A_i(x, T_M(w), \nabla\Phi) \text{ strongly in } (L^{\underline{p}}(\Omega))^N.$$

By(3.1), we get

$$|A_i(x, T_M(w_n), \nabla\Phi)| \leq \beta \left(R_i(x) + |T_M(w_n)|^{p_i - 1} + |\nabla\Phi|^{p_i - 1} \right),$$

with β be a positive constant. as $T_M(w_n) \rightharpoonup T_M(w)$ weakly in $W_0^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, therefore $T_M(w_n) \rightharpoonup T_M(w)$ strongly in $L^{\underline{p}}(\Omega)$ and a.e. in Ω, hence

$$|A_i(x, T_M(w_n), \nabla\Phi)| \to |A_i(x, T_M(w), \nabla\Phi)| \text{ a.e. in } \Omega.$$

and

$$\beta \left(R_i(x) + |T_M(w_n)|^{p_i - 1} + |\nabla\Phi|^{p_i - 1} \right) \to \beta \left(R_i(x) + |T_M(w)|^{p_i - 1} + |\nabla\Phi|^{p_i - 1} \right),$$

a.e. in Ω. Therefore, Vitali's theorem, implies

$$A_i(x, T_M(w_n), \nabla\Phi) \to A_i(x, T_M(w), \nabla\Phi) \text{ strongly in } (L^{\underline{p}}(\Omega))^N, \text{ as } n \to \infty. \tag{4.32}$$

According to (4.31) and (4.32), we can get

$$\int_\Omega A_i(x, w_n, \nabla\Phi) \nabla T_k(w_n - \Phi)\ dx \to \int_\Omega A_i(x, w, \nabla\Phi) \nabla T_k(w - \Phi)\ dx \text{ as } n \to +\infty. \tag{4.33}$$

Here, we prove that

$$\int_\Omega f_n T_k(w_n - \Phi) dx \to \int_\Omega f T_k(w - \Phi) dx. \tag{4.34}$$

and

$$\int_\Omega F_n \nabla T_k(w_n - \phi)\ dx \to \int_\Omega F \nabla T_k(w - \Phi)\ dx. \tag{4.35}$$

We get $f_n T_k(w_n - \Phi) \to f T_k(w - \Phi)$ a.e. in Ω, and $F_n \nabla T_k(w_n - \Phi) \to F \nabla T_k(w - \Phi)$ a.e. in Ω, and $|F \nabla T_k(w_n - \phi)| \leq k|F|$. Therefore Vitali's theorem, implies (4.34) and (4.35). According to (4.33), (4.34) and (4.35) we pass to the limit in (4.26), so that $\forall \Phi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, we conclude

$$\int_\Omega A(x, w, \nabla\Phi) \nabla T_k(w - \Phi)\ dx \leq \int_\Omega f T_k(w - \Phi) dx + \int_\Omega F \nabla T_k(w - \Phi)\ dx.$$

According to the idea of key Lemma, we conclude that w is a solution of the problem (0.1) in the sense of the definition 4.1.

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