

On some anisotropic elliptic problem with measure data

Ouidad Azraibi^a, Abdelkarim Derham^b, Badr El Haji^{b,*}

^aLaboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, Sidi Mohammed Ben Abdallah University, PB 1796 Fez-Atlas, Fez, Morocco

^bLaboratory LaR2A, Departement of Mathematics, Faculty of Sciences Tetouan, Abdelmalek Essaadi University, BP 2121, Tetouan, Morocco

(Communicated by Abdolrahman Razani)

Abstract

We prove optimal existence results for entropy solutions to some anisotropic boundary value problems like

$$\begin{cases} -\sum_{i=1}^N D^i A_i(x, w, \nabla w) = f - \operatorname{div} F(w) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (0.1)$$

where $f \in L^1(\Omega)$, $F = (F_1, \dots, F_N)$ satisfies $F \in (C^0(\mathbb{R}))^N$. and Ω is a bounded, open subset of \mathbb{R}^N , $N \geq 2$, and the function $A_i(x, s, \xi)$ verify the large monotonicity condition. The construction of the proof of our theorem is done by using Minty's Lemma in its modified version.

Keywords: Entropy solutions, nonlinear elliptic equations, anisotropic Sobolev spaces, entropy solutions
2020 MSC: Primary 35J15; Secondary 35J62

1 Introduction

The study of anisotropic elliptic equations on bounded domain has been intensively studied by large number of scientists and researchers, this study is motived by the fact that this type of equations can intimate connections with some application in elasticity, in the process of image restoration and Stochastic Processes with constraints (see for instance [40, 12], and references therein).

In order to fix the ideas let us consider the strongly anisotropic elliptic problems as

$$\begin{cases} -\sum_{i=1}^N D^i A_i(x, w, \nabla w) = f - \operatorname{div} F(w) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $f \in L^1(\Omega)$, $F \in (C^0(\mathbb{R}))^N$, Ω is a bounded domain of \mathbb{R}^N , $N \geq 2$.

In large recent researches, existence result with some qualitative properties and regularity of nonlinear anisotropic elliptic equations where the data belongs to L^1 — have been proved see the references [16] when Badr EL HAJI et al have been shown the existence result of entropy solution in weighted-Orlicz spaces, other works found by Youssef

*Corresponding author

Email addresses: ouidadazraibi@gmail.com (Ouidad Azraibi), abdederham@gmail.com (Abdelkarim Derham), b.elhaji@uae.ac.ma (Badr El Haji)

AKDIM et al. in their paper [1] devoted to study a degenerated problem (0.1) via Minty's Lemma in weighted Orlicz-Sobolev space, in the similar direction faria et al (see [22]) have been treated the similar problem as (0.1) where the solution u of the elliptic problem studied depend on the gradient.

On the other hand, by using as main tool an L^1 version of Minty's lemma EL HAJI et al (see [2]) extending the main result under studies to the Musielak-orlicz spaces by giving an existence result for an entropy solutions of elliptic problem as

$$\begin{cases} L(w) = g(x) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

where $g \in L^1(\Omega)$ and $L(w) = -\operatorname{div} l(x, w, \nabla w)$.

The mathematical researches dealing the existence of solutions to some problems parabolic and elliptic under a different assumptions is massive; we refer the reader to [7, 15, 20, 3, 4, 5, 8, 17, 10, 19, 9, 18, 14, 13, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] and the references therein.

Our goal in this paper is to solve the problem (0.1)(existence results) where the function $A_i(x, s, \xi)$ satisfy the large monotonicity condition and without adopting the almost everywhere convergence of the gradients, and in order to overcome this difficulties, we exploit the technique of Minty's lemma for proving the existence of an entropy solutions, However the approach that we used in the proof differs from that adopted by A. Benkirane et al used in [6]

The outline of this note is as follows. After giving the definition and some auxiliary results on anisotropic Sobolev space, we recall in Section 3 some essential assumptions which are necessary to have an existence solution, finally section 4 will be devoted to give our main results and their proofs.

2 Preliminaries

Let Ω be a bounded open subset of \mathbb{R}^N ($N \geq 2$). Let p_1, \dots, p_N be N real constants numbers, with $\infty > p_i > 1$ for $i = 1, \dots, N$. We set

$$\vec{p} = (p_1, \dots, p_N), \underline{p} = p^-, p_0 = p^+ \quad D^0 w = w \quad \text{and} \quad D^i w = \frac{\partial w}{\partial x_i} \quad \text{for } i = 1, \dots, N,$$

and we set

$$\underline{p} = \min\{p_1, p_2, \dots, p_N\} \quad \text{and} \quad p_0 = \max\{p_1, p_2, \dots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1, \vec{p}}(\Omega)$ like :

$$W^{1, \vec{p}}(\Omega) = \{w \in W^{1,1}(\Omega) \text{ such that } D^i w \in L^{p_i}(\Omega) \text{ for } i = 1, 2, \dots, N\},$$

endowed with the norm

$$\|w\|_{\vec{p}} = \|w\|_{L^1(\Omega)} + \sum_{i=1}^N \|D^i w\|_{L^{p_i}(\Omega)}. \quad (2.1)$$

The space $(W^{1, \vec{p}}(\Omega), \|w\|_{1, \vec{p}})$ is a reflexive Banach (separable) space (cf [25]). We denote by $W_0^{1, \vec{p}}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1, \vec{p}}(\Omega)$ with respect to (2.1).

Proposition 2.1. (see. [21, 26]) Let $w \in W_0^{1, \vec{p}}(\Omega)$, we have

(i) : there exists $C_p > 0$, such that

$$\|w\|_{L^{p_i}(\Omega)} \leq C_p \sum_{i=1}^N \|D^i w\|_{L^{p_i}(\Omega)} \quad \text{for any } i = 1, \dots, N.$$

(ii) : there exists $C_s > 0$, such that

$$\|w\|_{L^q(\Omega)} \leq \frac{C_s}{N} \sum_{i=1}^N \left\| \frac{\partial w}{\partial x_i} \right\|_{L^{p_i}(\Omega)},$$

where

$$\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i} \quad \text{and} \quad \begin{cases} q = \bar{p}^* = \frac{N\bar{p}}{N - \bar{p}} & \text{if } \bar{p} < N \\ q \in [1, +\infty[& \text{if } \bar{p} \geq N \end{cases}$$

Lemma 2.2. Let Ω be a bounded open set in \mathbb{R}^N ($N \geq 2$), we set

$$s = \max(q, \max_{1 \leq i \leq N} p_i),$$

therefore, the embedding listed below holds :

- if $\bar{p} < N$ so $W_0^{1,\bar{p}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,
- if $\bar{p} = N$ therefore $W_0^{1,\bar{p}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty[$,
- if $\bar{p} > N$ we have $W_0^{1,\bar{p}}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^0(\bar{\Omega})$ is compact.

The proof of the above result (lemma 2.2) depends to the Proposition 2.1.

Definition 2.3. For $k > 0$, we give the following truncation $T_k(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$, that will be used latter

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,\bar{p}}(\Omega) := \{w : \Omega \rightarrow \mathbb{R} \text{ measurable} / T_k(w) \in W_0^{1,\bar{p}}(\Omega) \text{ for any } k > 0\}.$$

Lemma 2.4. Let $w \in \mathcal{T}_0^{1,\bar{p}}(\Omega)$, there exists one function $v_i : \Omega \rightarrow \mathbb{R}$ measurable with $i \in \{1, \dots, N\}$, such that

$$\forall k > 0 \quad D^i T_k(w) = v_i \cdot \chi_{\{|w| < k\}} \quad \text{a.e. } x \in \Omega,$$

with χ_A be a characteristic function of a measurable set A . v_i define the weak partial derivatives of w denoted by $D^i w$. Therefore, if $w \in W_0^{1,1}(\Omega)$, then $(v_i = D^i w)$.

Lemma 2.5. (see [24], Theorem 13.47) Let $(w_n)_n$ be a sequence in $L^1(\Omega)$ and $w \in L^1(\Omega)$ such that

- (i) $w_n \rightarrow w$ a.e. in Ω ,
- (ii) $w_n \geq 0$ and $w \geq 0$ a.e. in Ω ,
- (iii) $\int_{\Omega} w_n dx \rightarrow \int_{\Omega} w dx$,

then $w_n \rightarrow w$ in $L^1(\Omega)$.

Lemma 2.6. Let Ω be a bounded open subset of \mathbb{R}^N with the segment property. If $w \in (W_0^{1,\bar{p}}(\Omega))^N$ then $\int_{\Omega} \operatorname{div}(w) dx = 0$.

Proof . Fix a vector $w = (w^1, \dots, w^N) \in (W_0^{1,\bar{p}}(\Omega))^N$. since $W_0^1 L_\varphi(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_0^{1,\bar{p}}(\Omega)$, then each term w^i can be approximated by a suitable sequence $w_k^i \in D(\Omega)$ such that w_k^i converges to w^i in $W_0^{1,\bar{p}}(\Omega)$. Moreover, due to the fact that $w_k^i \in C_0^\infty(\Omega)$, then the Green formula gives

$$\int_{\Omega} \frac{\partial w_k^i}{\partial x_i} = \int_{\partial\Omega} w_k^i \vec{n} ds = 0. \quad (2.2)$$

On the other hand, $\frac{\partial w_k^i}{\partial x_i} \rightarrow \frac{\partial w^i}{\partial x_i}$ in $L_\varphi(\Omega)$. Thus $\frac{\partial w_k^i}{\partial x_i} \rightarrow \frac{\partial w^i}{\partial x_i}$ in $L^1(\Omega)$, which gives in view of (2.2) that

$$\int_{\Omega} \operatorname{div}(w) dx = 0.$$

□

3 Essential assumptions

We consider a Leray-Lions operator $\mathbb{A} : W_0^{1,\vec{p}}(\Omega) \mapsto W^{-1,\vec{p}'}(\Omega)$ modeled by

$$\mathbb{A}w = - \sum_{i=1}^N D^i A_i(x, w, \nabla w)$$

where $A_i : \Omega \times \mathbf{R} \times \mathbf{R}^N \mapsto \mathbf{R}$ are Carathéodory functions, for $i = 1, \dots, N$, which satisfy the hypothesis listed below as follows:

$$|A_i(x, s, \xi)| \leq \beta \left(R_i(x) + |s|^{p_i-1} + |\xi|^{p_i-1} \right) \quad \text{for } i = 1, \dots, N, \quad (3.1)$$

$$A_i(x, s, \xi) \xi_i \geq \alpha |\xi_i|^{p_i} \quad \text{for } i = 1, \dots, N, \quad (3.2)$$

$$(A_i(x, s, \xi) - A_i(x, s, \xi')) (\xi_i - \xi'_i) > 0 \quad \text{for } \xi_i \neq \xi'_i, \quad (3.3)$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$, where $R_i(x) \in L^{p'_i}(\Omega)$ and $p_i - 1 > q_i > 0$ for $i = 1, \dots, N$, where $R_i(x), \alpha, \beta > 0$.

$$f \in L^1(\Omega) \quad (3.4)$$

and $F = (F_1, \dots, F_N)$ satisfies

$$F \in (C^0(\mathbb{R}))^N. \quad (3.5)$$

The following section devoted to stating our Main results and their proofs

4 Main results

The approach used by Boccardo [11] of entropy solution is given by the following notion.

Definition 4.1. A function w (measurable) is named an entropy solution of (0.1) if $T_k(w) \in W_0^{1,\vec{p}}(\Omega)$ and satisfy

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla w) D^i T_k(w - \Phi) dx \leq \int_{\Omega} f T_k(w - \Phi) dx + \int_{\Omega} F(w) \nabla T_k(w - \Phi) dx$$

for any $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.2. Suppose that (3.1)-(3.4) are holds, then the problem (0.1) admit one entropy solution w .

4.1 The key Lemma

Lemma 4.3. Let w (measurable function) such that $T_k(w) \in W_0^{1,\vec{p}}(\Omega)$ for every $k > 0$. Then

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla w) D^i T_k(w - \Phi) dx \leq \int_{\Omega} f T_k(w - \Phi) dx + \int_{\Omega} F(w) \nabla T_k(w - \Phi) dx \quad (4.1)$$

is equivalent to

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla w) D^i T_k(w - \Phi) dx = \int_{\Omega} f T_k(w - \Phi) dx + \int_{\Omega} F(w) \nabla T_k(w - \Phi) dx \quad (4.2)$$

for every $\Phi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$.

4.2 Proof of The key lemma

It's clear that The equation (4.2) implies (4.1). Now, by adding and subtracting

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla w) D^i T_k(w - \Phi) dx,$$

therefore by using assumption (3.2), we can prove that (4.1) implies (4.2). Let $h, k > 0$, let $\lambda \in]-1, 1[$ and $\Theta \in W_0^{1, \bar{p}}(\Omega) \cap L^\infty(\Omega)$. We take, $\Phi = T_h(w - \lambda T_k(w - \Theta)) \in W_0^{1, \bar{p}}(\Omega) \cap L^\infty(\Omega)$ in (4.1), we get:

$$E_{hk} \leq F_{hk}, \quad (4.3)$$

with

$$E_{hk} = \sum_{i=1}^N \int_{\Omega} A_i(x, w, D^i T_h(w - \lambda T_k(w - \Theta))) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx,$$

and

$$F_{hk} = \int_{\Omega} f T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx + \int_{\Omega} F(w) \nabla T_k(w - T_h(w - \lambda T_k(w - \Theta))).$$

Put

$$S_{hk} = \{x \in \Omega, |w - T_h(w - \lambda T_k(w - \Theta))| \leq k\},$$

and

$$T_{hk} = \{x \in \Omega, |w - \lambda T_k(w - \Theta)| \leq h\}.$$

Then, we obtain

$$\begin{aligned} E_{hk} &= \sum_{i=1}^N \int_{S_{kh} \cap T_{hk}} A_i(x, w, D^i T_h(w - \lambda T_k(w - \Theta))) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx \\ &\quad + \sum_{i=1}^N \int_{S_{kh} \cap T_{hk}^C} A_i(x, w, D^i T_h(w - \lambda T_k(w - \Theta))) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx \\ &\quad + \sum_{i=1}^N \int_{S_{kh}^C} A_i(x, w, D^i T_h(w - \lambda T_k(w - \Theta))) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx. \end{aligned}$$

Since $D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) \neq 0$ on S_{kh} , we get

$$\sum_{i=1}^N \int_{S_{kh}^C} A_i(x, w, D^i T_h(w - \lambda T_k(w - \Theta))) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx = 0. \quad (4.4)$$

Therefore, if $x \in T_{hk}^C$, we can get $D^i T_h(w - \lambda T_k(w - \Theta)) = 0$ and using (3.3), we conclude the following equality,

$$\begin{aligned} &\sum_{i=1}^N \int_{S_{kh} \cap T_{hk}^C} A_i(x, w, D^i T_h(w - \lambda T_k(w - \Theta))) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx \\ &= \sum_{i=1}^N \int_{S_{kh} \cap T_{hk}^C} A_i(x, w, 0) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx = 0. \end{aligned} \quad (4.5)$$

According to (4.4) and (4.5), we get

$$E_{hk} = \sum_{i=1}^N \int_{S_{kh} \cap T_{hk}} A_i(x, w, D^i T_h(w - \lambda T_k(w - \Theta))) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx.$$

Let $h \rightarrow +\infty, |\lambda| \leq 1$, we obtain

$$S_{kh} \rightarrow \{x, |\lambda||T_k(w - \Theta)| \leq h\} = \Omega, \quad (4.6)$$

$$T_{hk} \rightarrow \Omega \quad \text{implies that} \quad S_{kh} \cap T_{hk} \rightarrow \Omega. \quad (4.7)$$

By applying the Lebesgue theorem, we obtain

$$\begin{aligned} & \lim_{h \rightarrow +\infty} \sum_{i=1}^N \int_{S_{kh} \cap T_{hk}} A_i(x, u, D^i T_h(w - \lambda T_k(w - \Theta))) D^i T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx \\ &= \lambda \sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla(w - \lambda T_k(w - \Theta))) D^i T_k(w - \Theta) dx. \end{aligned} \quad (4.8)$$

thus implies that,

$$\lim_{h \rightarrow +\infty} E_{hk} = \lambda \sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla(w - \lambda T_k(w - \Theta))) D^i T_k(w - \Theta) dx. \quad (4.9)$$

Moreover, one has

$$F_{hk} = \int_{\Omega} f T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx + \int_{\Omega} F(w) \nabla T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx.$$

Then

$$\lim_{h \rightarrow +\infty} \int_{\Omega} f T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx + \int_{\Omega} F(w) \nabla T_k(w - T_h(w - \lambda T_k(w - \Theta))) dx. \quad (4.10)$$

$$= \lambda \left(\int_{\Omega} f T_k(w - \Theta) dx + \int_{\Omega} F(w) \nabla(T_k(w - \Theta)) dx \right), \quad (4.11)$$

i.e.,

$$\lim_{h \rightarrow +\infty} F_{hk} = \lambda \left(\int_{\Omega} f T_k(w - \Theta) dx + \int_{\Omega} F(w) \nabla(T_k(w - \Theta)) dx \right). \quad (4.12)$$

Thanking to (4.9), (4.12) therefore by passing to the limit in (4.3), we can get,

$$\lambda \left(\sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla(w - \lambda T_k(w - \Theta))) D^i T_k(w - \Theta) dx \right) \leq \lambda \left(\int_{\Omega} f T_k(w - \Theta) dx + \int_{\Omega} F(w) \nabla(T_k(w - \Theta)) dx \right) \quad (4.13)$$

for every $\Theta \in W_0^{1, \vec{p}}(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$. Let us take $\lambda > 0$, dividing by λ , and $\lambda \rightarrow 0$, we have

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla w) D^i T_k(w - \Theta) dx \leq \int_{\Omega} f T_k(w - \Theta) dx + \int_{\Omega} F(w) \nabla(T_k(w - \Theta)) dx. \quad (4.14)$$

for $\lambda < 0$, dividing by λ , and $\lambda \rightarrow 0$, we have

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla w) D^i T_k(w - \Theta) dx \geq \int_{\Omega} f T_k(w - \Theta) dx + \int_{\Omega} F(w) \nabla(T_k(w - \Theta)) dx. \quad (4.15)$$

Thanking to (4.14) and (4.15), we deduce that :

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla w) D^i T_k(w - \Theta) dx = \int_{\Omega} f T_k(w - \Theta) dx + \int_{\Omega} F(w) \nabla(T_k(w - \Theta)) dx. \quad (4.16)$$

This achieve the demonstration of Lemma 4.3.

4.3 Proof of Main results

4.3.1 Approximate problem

For $n \in \mathbb{N}$, define $f_n := T_n(f)$. Let $w_n \in W_0^{1,p}(\Omega)$ be solution of the approximate equation of the type

$$\begin{cases} A_n w_n = f_n - \operatorname{div} F_n(w_n) & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.17)$$

which exists according to ([23]). We take $T_k(w_n)$ as test in (4.17), we get

$$\sum_{i=1}^N \int_{\Omega} A_i(x, T_n(w_n), \nabla w_n) D^i T_k(w_n) dx = \int_{\Omega} f_n T_k(w_n) dx + \int_{\Omega} F_n(w_n) \nabla T_k(w_n) dx.$$

We claim that:

$$\int_{\Omega} F_n(w_n) \nabla T_k(w_n) dx = 0, \quad (4.18)$$

using $\nabla T_k(w_n) = \nabla w_n \chi_{\{|w_n| \leq k\}}$, define $\Theta(t) = F_n(t) \chi_{\{|t| \leq k\}}$, and $\tilde{\Theta}(t) = \int_0^t \Theta(\tau) d\tau$, by using the divergence lemma we can get $\tilde{\Theta}(w_n) \in (W_0^1 L^\varphi(\Omega))^N$

$$\begin{aligned} \int_{\Omega} F_n(w_n) \nabla T_k(w_n) dx &= \int_{\Omega} F_n(w_n) \chi_{\{|w_n| \leq k\}} \nabla w_n dx \\ &= \int_{\Omega} \Theta(w_n) \nabla w_n dx = \int_{\Omega} \operatorname{div}(\tilde{\Theta}(w_n)) dx = 0 \end{aligned} \quad (4.19)$$

(by lemma 2.6) which proves the claim. Now thanks to (3.3), we obtain

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla w_n) D^i T_k(w_n) dx \geq \alpha \int_{\Omega} |D^i T_k(w_n)|^{p_i} dx,$$

then

$$\int_{\Omega} |D^i T_k(w_n)|^{p_i} dx \leq k \|f\|_{L^1(\Omega)}. \quad (4.20)$$

Then

$$\int_{\Omega} |D^i T_k(w_n)|^{p_i} dx \leq C_1 k, \quad (4.21)$$

where C_1 is a constant independently of n .

4.3.2 Locally convergence of w_n in measure

Taking $\lambda |T_k(w_n)|$ in (4.17) and using (4.21), one has

$$\int_{\Omega} \lambda_1 \frac{|D^i T_k(w_n)|^{p_i}}{\lambda} dx \leq \int_{\Omega} \lambda_1 |D^i T_k(w_n)|^{p_i} dx \leq C_1 k. \quad (4.22)$$

by using (4.22), we can have

$$\begin{aligned} \operatorname{meas}\{|w_n| > k\} &\leq \frac{1}{\inf_k \frac{k}{\lambda}} \int_{\{|w_n| > k\}} \frac{|w_n(x)|^{p_i}}{\lambda} dx \\ &\leq \frac{1}{\inf_k \frac{k}{\lambda}} \int_{\Omega} \frac{1}{\lambda} |T_k(w_n)|^{p_i} dx \\ &\leq \frac{C_1 k}{\inf_k \frac{k}{\lambda}} \quad \forall n, \quad \forall k \geq 0. \end{aligned} \quad (4.23)$$

For any $\beta > 0$, we have

$$\text{meas}\{|w_n - w_m| > \beta\} \leq \text{meas}\{|w_n| > k\} + \text{meas}\{|w_m| > k\} + \text{meas}\{|T_k(w_n) - T_k(w_m)| > \beta\},$$

and so that

$$\text{meas}\{|w_n - w_m| > \beta\} \leq \frac{2C_1 k}{\inf_{x \in \Omega} \frac{k}{\lambda}} + \text{meas}\{|T_k(w_n) - T_k(w_m)| > \beta\}. \quad (4.24)$$

By Applying Poincaré inequality (proposition 2.1) and according to (4.21) we obtain the boundedness of $(T_k(w_n))$ in $W_0^{1,\vec{p}}(\Omega)$, therefore there exists $\omega_k \in W_0^{1,\vec{p}}(\Omega)$ such that $T_k(w_n) \rightharpoonup \omega_k$ weakly in $W_0^{1,\vec{p}}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω . So, we suppose that $(T_k(w_n))_n$ is a Cauchy sequence in measure in Ω . Let $\varepsilon > 0$, then by (4.24) and in view of $\frac{2C_1 k}{\inf_{x \in \Omega} \frac{k}{\lambda}} \rightarrow 0$ as $k \rightarrow +\infty$ there exists some $k = k(\varepsilon) > 0$ such that

$$\text{meas}\{|w_n - w_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

This proves that w_n is a Cauchy sequence in measure, thus, w_n converges almost everywhere to w (measurable function). Therefore, there exist a subsequence of $\{w_n\}_n$, still indexed by n , and a function $w \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} w_n \rightharpoonup w \text{ weakly in } W_0^{1,\vec{p}}(\Omega) \\ w_n \rightarrow w \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \end{cases} \quad (4.25)$$

4.3.3 An intermediate Inequality

Here, we can show that, for $\Phi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, we get

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla \Phi) D^i T_k(w - \Phi) dx \leq \int_{\Omega} f_n T_k(w_n - \Phi) dx + \int_{\Omega} F_n \nabla T_k(w_n - \Phi) dx. \quad (4.26)$$

Now, we take $T_k(w_n - \Phi)$ as test in (4.17), with Φ in $W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, we can obtain

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla \Phi) D^i T_k(w - \Phi) dx = \int_{\Omega} f_n T_k(w_n - \Phi) dx + \int_{\Omega} F_n \nabla T_k(w_n - \Phi) dx. \quad (4.27)$$

The term $\sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla \Phi) D^i T_k(w - \Phi) dx$ can be added and subtracted to the equation (4.27) we can obtain,

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla w_n) D^i T_k(w - \Phi) dx + \sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla \Phi) D^i T_k(w - \Phi) dx \\ & - \sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla \Phi) D^i T_k(w - \Phi) dx = \int_{\Omega} f_n T_k(w_n - \Phi) dx + \int_{\Omega} F_n \nabla T_k(w_n - \Phi) dx - \Phi dx. \end{aligned} \quad (4.28)$$

By (3.2) and truncation function, we can get

$$\sum_{i=1}^N \int_{\Omega} (A_i(x, w_n, \nabla w_n) - A_i(x, w_n, \nabla \Phi)) D^i T_k(w - \Phi) dx \geq 0. \quad (4.29)$$

According to (4.28) and (4.29), we get (4.26).

4.3.4 Passing to the limit

We verify that for $\Phi \in W_0^{1,\bar{p}}(\Omega) \cap L^\infty(\Omega)$, one has

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla \Phi) D^i T_k(w_n - \Phi) dx \leq \int_{\Omega} f T_k(w - \Phi) dx + \int_{\Omega} F \nabla T_k(w - \Phi) dx.$$

Now, we show that

$$\sum_{i=1}^N \int_{\Omega} A_i(x, w_n, \nabla \Phi) D^i T_k(w_n - \Phi) dx \rightarrow \sum_{i=1}^N \int_{\Omega} A_i(x, w, \nabla \Phi) D^i T_k(w - \Phi) dx \text{ as } n \rightarrow +\infty.$$

as $T_M(w_n) \rightharpoonup T_M(w)$ weakly in $W_0^{1,\bar{p}}(\Omega)$, with $M = k + \|\Phi\|_\infty$, therefore

$$T_k(w_n - \Phi) \rightharpoonup T_k(w - \Phi) \text{ in } W_0^{1,\bar{p}}(\Omega), \quad (4.30)$$

then

$$\frac{\partial T_k}{\partial x_i}(w_n - \Phi) \rightharpoonup \frac{\partial T_k}{\partial x_i}(w - \Phi) \text{ weakly in } L^{\bar{p}}(\Omega) \quad \forall i = 1, \dots, N. \quad (4.31)$$

Let us prove that

$$A_i(x, T_M(w_n), \nabla \Phi) \rightarrow A_i(x, T_M(w), \nabla \Phi) \text{ strongly in } (L^{\bar{p}}(\Omega))^N.$$

By(3.1), we get

$$|A_i(x, T_M(w_n), \nabla \Phi)| \leq \beta \left(R_i(x) + |T_M(w_n)|^{p_i-1} + |\nabla \Phi|^{p_i-1} \right),$$

with β be a positive constant. as $T_M(w_n) \rightharpoonup T_M(w)$ weakly in $W_0^{1,\bar{p}}(\Omega)$ and $W_0^{1,\bar{p}}(\Omega) \hookrightarrow L^{\bar{p}}(\Omega)$, therefore $T_M(w_n) \rightharpoonup T_M(w)$ strongly in $L^{\bar{p}}(\Omega)$ and a.e. in Ω , hence

$$|A_i(x, T_M(w_n), \nabla \Phi)| \rightarrow |A_i(x, T_M(w), \nabla \Phi)| \text{ a.e. in } \Omega.$$

and

$$\beta \left(R_i(x) + |T_M(w_n)|^{p_i-1} + |\nabla \Phi|^{p_i-1} \right) \rightarrow \beta \left(R_i(x) + |T_M(w)|^{p_i-1} + |\nabla \Phi|^{p_i-1} \right),$$

a.e. in Ω . Therefore, Vitali's theorem, implies

$$A_i(x, T_M(w_n), \nabla \Phi) \rightarrow A_i(x, T_M(w), \nabla \Phi) \text{ strongly in } (L^{\bar{p}}(\Omega))^N, \text{ as } n \rightarrow \infty. \quad (4.32)$$

According to (4.31) and (4.32), we can get

$$\int_{\Omega} A_i(x, w_n, \nabla \Phi) \nabla T_k(w_n - \Phi) dx \rightarrow \int_{\Omega} A_i(x, w, \nabla \Phi) \nabla T_k(w - \Phi) dx \text{ as } n \rightarrow +\infty. \quad (4.33)$$

Here, we prove that

$$\int_{\Omega} f_n T_k(w_n - \Phi) dx \rightarrow \int_{\Omega} f T_k(w - \Phi) dx. \quad (4.34)$$

and

$$\int_{\Omega} F_n \nabla T_k(w_n - \Phi) dx \rightarrow \int_{\Omega} F \nabla T_k(w - \Phi) dx. \quad (4.35)$$

We get $f_n T_k(w_n - \Phi) \rightarrow f T_k(w - \Phi)$ a.e. in Ω , and $F_n \nabla T_k(w_n - \Phi) \rightarrow F \nabla T_k(w - \Phi)$ a.e. in Ω , and $|F \nabla T_k(w_n - \Phi)| \leq k|F|$. Therefore Vitali's theorem, implies (4.34) and (4.35). According to (4.33), (4.34) and (4.35) we pass to the limit in (4.26), so that $\forall \Phi \in W_0^{1,\bar{p}}(\Omega) \cap L^\infty(\Omega)$, we conclude

$$\int_{\Omega} A(x, w, \nabla \Phi) \nabla T_k(w - \Phi) dx \leq \int_{\Omega} f T_k(w - \Phi) dx + \int_{\Omega} F \nabla T_k(w - \Phi) dx.$$

According to the idea of key Lemma, we conclude that w is a solution of the problem (0.1) in the sense of the definition 4.1.

References

- [1] Y. Akdim, E. Azroul, and M. Rhoudaf, *Existence of T -solution for degenerated problem via Minty's Lemma*, Acta Math. Sinica English Ser. **24** (2008), 431–438.
- [2] O. Azraibi, B. El Haji, and M. Mekour, *Entropy solution for nonlinear elliptic boundary value problem having large monotonicity in Musielak-Orlicz-Sobolev spaces*, Asia Pac. J. Math. **10** (2023), no. 7.
- [3] O. Azraibi, B. El Haji, and M. Mekour, *Nonlinear parabolic problem with lower order terms in Musielak-Sobolev spaces without sign condition and with Measure data*, Palestine J. Math. **11** (2022), no. 3, 474–503.
- [4] O. Azraibi, B. EL haji, and M. Mekour, *On some nonlinear elliptic problems with large monotonicity in Musielak-Orlicz-Sobolev spaces*, J. Math. Phys. Anal. Geom. **18** (2022), no. 3, 1–18.
- [5] O. Azraibi, B. EL Haji, and M. Mekour, *Strongly nonlinear unilateral anisotropic elliptic problem with data*, Asia Math. **7** (2023), no. 1, 1–20.
- [6] A. Benkirane and A. Elmahi, *Almost everywhere convergence of the gradient of solutions to elliptic equations in Orlicz spaces*, Nonlinear Anal. T.M.A. **28** (1997), no. 11, 1769–1784.
- [7] A. Benkirane, B. El Haji, and M. El Moumni, *On the existence of solution for degenerate parabolic equations with singular terms*, Pure Appl. Math. Quart. **14** (2018), no. 3-4, 591–606.
- [8] A. Benkirane, B. El Haji, and M. El Moumni, *Strongly nonlinear elliptic problem with measure data in Musielak-Orlicz spaces*, Complex Variab. Elliptic Equ. **67** (2022), no. 6, 1447–1469.
- [9] A. Benkirane, B. El Haji, and M. El Moumni, *On the existence solutions for some nonlinear elliptic problem*, Bol. Soc. Paran. Mat. (3s.) **40** (2022), 1–8.
- [10] A. Benkirane, N. El Amarty, B. El Haji, and M. El Moumni, *Existence of solutions for a class of nonlinear elliptic problems with measure data in the setting of Musielak-Orlicz-Sobolev spaces*, J. Elliptic Parabol. Equ. **9** (2023), 647–672.
- [11] L. Boccardo, *Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form*, Prog. Elliptic Parabolic Partial Differ. Equ. **350** (1996), 43–57.
- [12] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration* SIAM J. Appl. Math. **66** (2006), 1383–1406.
- [13] J. Droniou and A. Prignet, *Equivalence between entropy and renormalized solutions for parabolic equations with smooth measure data*, NoDEA Nonlinear Differ. Equ. Appl. **14** (2007) no. 1-2, 181–205.
- [14] B. El Haji and M. El Moumni, and A. Talha, *Entropy solutions for nonlinear parabolic equations in Musielak Orlicz spaces without Δ_2 -conditions*, Gulf J. Math. **9** (2020), Issue 1, 1–26.
- [15] B. El Haji and M. El Moumni, *Entropy solutions of nonlinear elliptic equations with L^1 -data and without strict monotonicity conditions in weighted Orlicz-Sobolev spaces*, J. Nonlinear Funct. Anal. **2021** (2021), Article ID 8, 1–17.
- [16] B. El Haji, M. El Moumni, and K. Kouhaila, *On a nonlinear elliptic problems having large monotonicity with L^1 -data in weighted Orlicz-Sobolev spaces*, Moroccan J. Pure Appl. Anal. **5** (2021), 104–116.
- [17] B. El Haji and M. El Moumni, and K. Kouhaila, *Existence of entropy solutions for nonlinear elliptic problem having large monotonicity in weighted Orlicz-Sobolev spaces*, LE Math. **76** (2021), no. 1, 37–61.
- [18] B. El Haji, M. El Moumni, and A. Talha, *Entropy Solutions of Nonlinear Parabolic Equations in Musielak Framework Without Sign Condition and L^1 -Data* Asian J. Math. Appl. **2021** (2021), Article ID ama0575.
- [19] N. El Amarty, B. El Haji and M. El Moumni, *Entropy solutions for unilateral parabolic problems with L^1 -data in Musielak-Orlicz-Sobolev spaces*, Palestine J. Math. **11** (2022), no. 1, 504–523.
- [20] N. El Amarty, B. El Haji, and M. El Moumni, *Existence of renormalized solution for nonlinear elliptic boundary value problem without Δ_2 -condition*, SeMA **77** (2020), 389–414.
- [21] I. Fragalà, F. Gazzola, and B. Kawohl, *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), 715–734.

- [22] L.F.O. Faria, O.H. Miyagaki, D. Motreanu, and M. Tanaka, *Existence results for nonlinear elliptic equations with Leray-Lions operator and dependence on the gradient*, *Nonlinear Anal.* **96** (2014), 154–166.
- [23] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces*, *Nonlinear Anal.* **11** (1987), no. 3, 379–392.
- [24] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, Berlin Heidelberg New York, 1965.
- [25] M. Mihailescu, P. Pucci, and V. Radulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, *J. Math. Anal. Appl.* **340** (2008), 687–698.
- [26] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, *Ricerche Mat.* **18** (1969), 3–24.
- [27] F. Petitta, A.C. Ponce, and A. Porretta, *Diffuse measures and nonlinear parabolic equations*, *J. Evol. Equ.* **11** (2011), no. 4, 861–905.
- [28] F. Petitta, *Asymptotic behavior of solutions for parabolic operators of Leray-Lions type and measure data*, *Adv. Differ. Equ.* **12** (2007), no. 8, 867–891.
- [29] A. Prignet, *Existence and uniqueness of "entropy", solutions of parabolic problems with L^1 data*, *Nonlinear Anal.* **28** (1997) no. 12, 1943–1954 .
- [30] A. Porretta and S. Segura de León; *Nonlinear elliptic equations having a gradient term with natural growth*, *J. Math. Pures Appl.* **85** (2006), no. 3, 465–492.
- [31] A. Razani, *Nonstandard competing anisotropic $(p; q)$ -Laplacians with convolution*, *Boundary Value Prob.* **2022** (2022), 87.
- [32] A. Razani, *Entire weak solutions for an anisotropic equation in the Heisenberg group*, *Proc. Amer. Math. Soc.* **151** (2023), no. 11, 47714779.
- [33] A. Razani, G.S. Costa, and G. M. Figueiredo, *A positive solution for a weighted anisotropic p -Laplace equation involving vanishing potential*, *Mediterr. J. Math.* **21** (2024), no. 2, 59.
- [34] A. Razani and G.M. Figueiredo, *Existence of infinitely many solutions for an anisotropic equation using genus theory*, *Math. Meth. Appl. Sci.* **45** (2022), no. 12, 7591–7606.
- [35] A. Razani and G.M. Figueiredo, *Degenerated and competing anisotropic $(p; q)$ Laplacians with weights*, *Appl. Anal.* **102** (2023), no. 16, 4471–4488.
- [36] A. Razani and G.M. Figueiredo, *A positive solution for an anisotropic $p\&q$ -Laplacian*, *Discrete Continuous Dyn. Syst. Ser. S* **16** (2023), no. 6, 16291643.
- [37] A. Razani and G.M. Figueiredo, *Positive solutions for a semipositone anisotropic p -Laplacian problem*, *Boundary Value Prob.* **2024** (2024), no. 1, 34.
- [38] T. Soltani and A. Razani, *Solutions for an anisotropic elliptic problem involving nonlinear terms*, *Q. Math.* **47** (2024), no. 1, 93–112.
- [39] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1974.
- [40] V. Zhikov, *Averaging of functionals in the calculus of variations and elasticity*, *Math. USSR Izv.* **29** (1987), 33–66.