Jensen's inequality for (p-q)-convex functions and related results

Gholamreza Zabandan, Farideh Tahmasbnia

Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

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Abstract

In this paper, we establish Jensen's inequality for (p-q)-convex functions. By using Jensen's inequality we obtain some Hermite-Hadamard type inequality and several sharp inequalities. Some examples are given.

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1 Introduction

Let μ be a positive measure on X such that $\mu(X) = 1$. If f is a real-valued function in $L^1(\mu), a < f(x) < b$ for all $x \in X$ and φ is convex on (a, b), then

$$\varphi(\int_X f d\mu) \leq \int_X (\varphi \circ f) d\mu. \tag{1.1}$$

The inequality (1.1) is known as Jensen's inequality. Let $f : [a, b] \to \mathbb{R}$ be a convex function, then the inequality

$$\varphi(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b \varphi(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2} \tag{1.2}$$

which is known in the literature as Hermite-Hadamard inequality (H-H inequality). Jensen's inequality is one of the important inequalities which has been possible with the help of convexity. This inequality has preserved some important structure and also there are lot of inequalities which are the direct consequences of Jensen's inequality, such as Holder, Hermite-Hadamard and Young's inequalities etc. Recently, many of mathematicians investigated generalization of convex functions, such as MN-convex (M, N=A, G, H), p-convex, r-convex, and obtained the H-H inequality for these convexities [1, 3, 6, 9, 12, 8, 10, 11, 14, 15, 16, 20, 22, 34]. In [43, 41] the author obtained Jensen's inequality for GG-convex and HH-convex functions. In this paper, via the defining of (p-q)-convex functions, we obtain the Jensen's and H-H inequalities for (p-q)-convex function and we show that a lot of convexity, such as MN-convexity (M,N=A,G,H), p-convexity and r-convexity are special cases of (p-q)-convexity.

For the statement of the main results we introduce some notations and terminologies.

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*Corresponding author
Email addresses:* `zabandan@khu.ac.ir` (Gholamreza Zabandan), `f.tahmasb@khu.ac.ir` (Farideh Tahmasbnia)

Definition 1.1. A function $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ is called a mean if

1. $M(x, y) = M(y, x)$,
2. $M(x, x) = x$,
3. $x < M(x, y) < y, wherever\ x < y$,
4. $M(\alpha x, \alpha y) = \alpha M(x, y),\ for\ all\ \alpha > 0$.

Example 1.2.

1. The Arithmetic Mean $A(x, y) = \dfrac{x + y}{2}$,
2. The Geometric Mean $G(x, y) = \sqrt{xy}$,
3. The Harmonic Mean $H(x, y) = \dfrac{1}{A(\dfrac{1}{x}, \dfrac{1}{y})}$,
4. The Power Mean $M_p(x, y) = \begin{cases} (\dfrac{x^p + y^p}{2})^{\frac{1}{p}} & p \neq 0 \\ \sqrt{xy} & p = 0 \end{cases}$ $(p \in \mathbb{R})$.

Definition 1.3. Let M, N be two means on real interval $I \subset (0, \infty)$ and $\varphi : I \to (0, \infty)$ be a continuous function. φ is called MN-convex if for all $x, y \in I$,
$$\varphi(M(x, y)) \leq N(f(x), f(y)).$$

Note that this definition reduces to usual convexity when $M(x, y) = N(x, y) = \dfrac{x + y}{2}$. Since φ is continuous, this the MN-convexity of φ that is
$$\varphi(M(x, y); 1 - t, t) \leq N(\varphi(x), \varphi(y); 1 - t, t)$$
for all $x, y \in I, t \in [0, 1]$ (see [26]). For example

1. φ is AG-convex, if for every $x, y \in I$ and $t \in [0, 1]$,
$$\varphi(tx + (1 - t)y) \leq \varphi^t(x)\varphi^{1-t}(y)$$

2. φ is GH-convex, if for every $x, y \in I$ and $t \in [0, 1]$,
$$\varphi(x^t y^{1-t}) \leq \frac{\varphi(x)\varphi(y)}{(1 - t)\varphi(x) + t\varphi(y)}$$

Definition 1.4. Let $I \subset (0, \infty)$ be interval and $p, q \in \mathbb{R}$. A continuous function $\varphi : I \to (0, \infty)$ is said to be $M_p M_q$-convex or $(p - q)$-convex if the following inequality holds

$$\varphi(tx^p + (1 - t)y^p)^{\frac{1}{p}} \leq (t\varphi^q(x) + (1 - t)\varphi^q(y))^{\frac{1}{q}} \qquad\qquad (p \neq 0, q \neq 0)$$
$$\varphi(x^t y^{1-t}) \leq (t\varphi^q(x) + (1 - t)\varphi^q(y))^{\frac{1}{q}} \qquad\qquad (p = 0, q \neq 0)$$
$$\varphi(tx^p + (1 - t)y^p)^{\frac{1}{p}} \leq \varphi^t(x)\varphi^{1-t}(y) \qquad\qquad (p \neq 0, q = 0)$$
$$\varphi(x^t y^{1-t}) \leq \varphi^t(x)\varphi^{1-t}(y) \qquad\qquad (p = 0, q = 0)$$

wherever $x, y \in I$ and $t \in [0, 1]$. It can be easily seen that for $p = q = 1$, $p = q = 0$, $p = q = -1$ and $p = 1, q = r$, $(p - q)$-convexity reduced respectively to classical convexity, GG-convexity, HH-convexity and r-convexity. In [45] the authers proved that for $0 \leq t \leq 1, 0 \leq p \leq q$ and $x, y \geq 0$ the following inequalities hold

$$x^t y^{1-t} \leq (tx^p + (1 - t)y^p)^{\frac{1}{p}} \leq (tx^q + (1 - t)y^q)^{\frac{1}{q}} \qquad\qquad (1.3)$$

By easy calculations we see that for $p \leq q \leq 0$, the inequalities

$$(tx^p + (1 - t)y^p)^{\frac{1}{p}} \leq (tx^q + (1 - t)y^q)^{\frac{1}{q}} \leq x^t y^{1-t} \qquad\qquad (1.4)$$

hold. Inequalities (1.3) and (1.4) show that $\varphi(x) = x$ is $(p-q)$-convex if $p < q$, $(0-q)$-convex if $q > 0$ and $(p-0)$-convex if $p < 0$.

2 Main results

Theorem 2.1. Let φ be a $(p-q)$-convex function on (a, b) and let $0 < a < s < t < u < b$.

1. If $(p > 0$ and $q > 0)$ or $(p < 0$ and $q < 0)$, then

$$\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \leq \frac{\varphi^q(u) - \varphi^q(t)}{u^p - t^p} \tag{2.1}$$

2. If $(p > 0$ and $q < 0)$ or $(p < 0$ and $q > 0)$, then

$$\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \geq \frac{\varphi^q(u) - \varphi^q(t)}{u^p - t^p} \tag{2.2}$$

3. If $p = 0$ and $q > 0$, then

$$\frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \leq \frac{\varphi^q(u) - \varphi^q(t)}{\ln u - \ln t} \tag{2.3}$$

4. If $p = 0$ and $q < 0$, then

$$\frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \geq \frac{\varphi^q(u) - \varphi^q(t)}{\ln u - \ln t} \tag{2.4}$$

5. If $p > 0$ and $q = 0$, then

$$\frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \leq \frac{\ln \varphi(u) - \ln \varphi(t)}{u^p - t^p} \tag{2.5}$$

6. If $p < 0$ and $q = 0$, then

$$\frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \geq \frac{\ln \varphi(u) - \ln \varphi(t)}{u^p - t^p} \tag{2.6}$$

7. If $p = q = 0$, then

$$\frac{\ln \varphi(t) - \ln \varphi(s)}{\ln t - \ln s} \leq \frac{\ln \varphi(u) - \ln \varphi(t)}{\ln u - \ln t} \tag{2.7}$$

Proof .

1. Let $p > 0, q > 0$, and $r = \dfrac{u^p - t^p}{u^p - s^p}$, then $0 < r < 1$ and $t = (rs^p + (1-r)u^p)^{\frac{1}{p}}$. By the $(p-q)$-convexity of φ we have

$$\begin{aligned} &\varphi(t) \leq (r\varphi^q(s) + (1-r)\varphi^q(u))^{\frac{1}{q}} \\ \Rightarrow\ &\varphi^q(t) \leq r\varphi^q(s) + (1-r)\varphi^q(u) \quad (q > 0) \\ \Rightarrow\ &r\varphi^q(t) + (1-r)\varphi^q(t) \leq r\varphi^q(s) + (1-r)\varphi^q(u) \\ \Rightarrow\ &r(\varphi^q(t) - \varphi^q(s)) \leq (1-r)(\varphi^q(u) - \varphi^q(t)) \\ \Rightarrow\ &\frac{u^p - t^p}{u^p - s^p}(\varphi^q(t) - \varphi^q(s)) \leq \frac{t^p - s^p}{u^p - s^p}(\varphi^q(u) - \varphi^q(t)) \\ \Rightarrow\ &\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \leq \frac{\varphi^q(u) - \varphi^q(t)}{u^p - t^p}. \end{aligned}$$

Now let $p < 0$ and $q < 0$. Then $s^p > t^p > u^p$, $0 < r = \dfrac{u^p - t^p}{u^p - s^p} < 1$ and $t = (rs^p + (1-r)u^p)^{\frac{1}{p}}$. Since φ is $(p-q)$-convex,

$$\begin{aligned} &\varphi(t) \leq (r\varphi^q(s) + (1-r)\varphi^q(u))^{\frac{1}{q}} \\ \Rightarrow\ &\varphi^q(t) \geq (r\varphi^q(s) + (1-r)\varphi^q(u)) \quad (q < 0) \\ \Rightarrow\ &r\varphi^q(t) + (1-r)\varphi^q(t) \geq r\varphi^q(s) + (1-r)\varphi^q(u) \\ \Rightarrow\ &r(\varphi^q(t) - \varphi^q(s)) \geq (1-r)(\varphi^q(u) - \varphi^q(t)) \\ \Rightarrow\ &\frac{u^p - t^p}{u^p - s^p}(\varphi^q(t) - \varphi^q(s)) \geq \frac{t^p - s^p}{u^p - s^p}(\varphi^q(u) - \varphi^q(t)) \end{aligned}$$

Since $\dfrac{u^p - s^p}{(u^p - t^p)(t^p - s^p)} < 0$, we have

$$\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \le \frac{\varphi^q(u) - \varphi^q(t)}{u^p - t^p}.$$

2. Follows by similar way and the details are omitted.

3. Let $p = 0, q > 0$ and $0 < r = \dfrac{\ln u - \ln t}{\ln u - \ln s} < 1$. Then $t = u^{1-r} s^r$. By the $(p-q)$-convexity of φ we have

$$\varphi(t) \le (r\varphi^q(s) + (1-r)\varphi^q(u))^{\frac{1}{q}}.$$

Since $q > 0$,

$$\varphi^q(t) \le r\varphi^q(s) + (1-r)\varphi^q(u)$$
$$\Rightarrow \quad r\varphi^q(t) + (1-r)\varphi^q(t) \le r\varphi^q(s) + (1-r)\varphi^q(u)$$
$$\Rightarrow \quad r(\varphi^q(t) - \varphi^q(s)) \le (1-r)(\varphi^q(u) - \varphi^q(t))$$
$$\Rightarrow \quad \frac{\ln u - \ln t}{\ln u - \ln s}(\varphi^q(t) - \varphi^q(s)) \le \frac{\ln t - \ln s}{\ln u - \ln s}(\varphi^q(u) - \varphi^q(t))$$
$$\Rightarrow \quad \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \le \frac{\varphi^q(u) - \varphi^q(t)}{\ln u - \ln t}$$

4. If $p = 0, q < 0$, by similar way we obtain

$$\varphi^q(t) \ge r\varphi^q(s) + (1-r)\varphi^q(u)$$
$$\Rightarrow \quad \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \ge \frac{\varphi^q(u) - \varphi^q(t)}{\ln u - \ln t}$$

5. Let $p > 0, q = 0$ and $0 < r = \dfrac{u^p - t^p}{u^p - s^p} < 1$. Then $t = ((1-r)u^p + rs^p)^{\frac{1}{p}}$. By the $(p-0)$-convexity of φ we have

$$\varphi(t) \le \varphi^r(s)\varphi^{1-r}(u)$$
$$\Rightarrow \quad \ln \varphi(t) \le r \ln \varphi(s) + (1-r) \ln \varphi(u)$$
$$\Rightarrow \quad r \ln \varphi(t) + (1-r) \ln \varphi(t) \le r \ln \varphi(s) + (1-r) \ln \varphi(u)$$
$$\Rightarrow \quad r(\ln \varphi(t) - \ln \varphi(s)) \le (1-r)(\ln \varphi(u) - \ln \varphi(t))$$
$$\Rightarrow \quad \frac{u^p - t^p}{u^p - s^p}(\ln \varphi(t) - \ln \varphi(s)) \le \frac{t^p - s^p}{u^p - s^p}(\ln \varphi(u) - \ln \varphi(t))$$
$$\Rightarrow \quad \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \le \frac{\ln \varphi(u) - \ln \varphi(t)}{u^p - t^p}$$

6. Let $p < 0$ and $q = 0$. Then $s^p > t^p > u^p$ and $0 < r = \dfrac{u^p - t^p}{u^p - s^p} < 1$. So

$$t = (rs^p + (1-r)u^p)^{\frac{1}{p}}$$
$$\Rightarrow \quad \varphi(t) \le \varphi^r(s)\varphi^{1-r}(u)$$
$$\Rightarrow \quad \ln \varphi(t) \le r \ln \varphi(s) + (1-r) \ln \varphi(u)$$
$$\Rightarrow \quad \frac{u^p - t^p}{u^p - s^p}(\ln \varphi(t) - \ln \varphi(s)) \le \frac{t^p - s^p}{u^p - s^p}(\ln \varphi(u) - \ln \varphi(t))$$
$$\Rightarrow \quad \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \ge \frac{\ln \varphi(u) - \ln \varphi(t)}{u^p - t^p}$$

7. If $p = q = 0$, then φ is GG-convex. For details see [43] .

$\square$

In the following theorem we obtain the Jensen's inequality for $(p-q)$-convex functions.

Theorem 2.2. Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set X, so that $\mu(X) = 1$. If f is a real function in $L^1(\mu), 0 < a < f(x) < b$ for all $x \in X$, and if φ is a $(p-q)$-convex function on (a,b), then

1. If $p \neq 0$ and $q \neq 0$, then

$$\varphi\left(\int_X f^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_X (\varphi \circ f)^q d\mu\right)^{\frac{1}{q}}.$$

2. If $p = 0$ and $q \neq 0$, then

$$\varphi\left(e^{\int_X \ln f d\mu}\right) \leq \left(\int_X (\varphi \circ f)^q d\mu\right)^{\frac{1}{q}}.$$

3. If $p \neq 0$ and $q = 0$, then

$$\varphi\left(\int_X f^p d\mu\right)^{\frac{1}{p}} \leq e^{\int_X \ln(\varphi \circ f) d\mu}.$$

4. If $p = q = 0$, then

$$\varphi\left(e^{\int_X \ln f d\mu}\right) \leq e^{\int_X \ln(\varphi \circ f) d\mu}.$$

Proof .

1. Put $t = \left(\int_X f^p d\mu\right)^{\frac{1}{p}}$, then $a < t < b$. We distinguish two cases.
   Case 1. $(p > 0$ and $q > 0)$ or $(p < 0$ and $q < 0)$. Since φ is $(p-q)$-convex, the inequality (2.1) holds. Let

$$M = \sup_{a < s < t} \frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p}.$$

Then M is no larger than any of the quotients on the right side of (2.1), for any $u \in (t, b)$. It follows that

$$\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \leq M.$$

Now let $p > 0$ and $q > 0$, then

$$\varphi^q(t) - \varphi^q(s) \leq M(t^p - s^p)$$
$$\Rightarrow \quad \varphi^q(s) \geq \varphi^q(t) + M(s^p - t^p).$$

Hence for any $x \in X$, we have

$$\varphi^q(f(x)) \geq \varphi^q(t) + M(f^p(x) - t^p).$$

Since φ is continuous, $\varphi \circ f$ and $(\varphi \circ f)^q$ are measarable. By integrating both sides with respect to measure μ, we get

$$\int_X (g \circ f)^q d\mu \geq \varphi^q(t) + \mu\left(\int_X f^p d\mu - t^p\right).$$

Now set $t = \left(\int_X f^p d\mu\right)^{\frac{1}{p}}$. It follows that

$$\int_X (g \circ f)^q d\mu \geq \varphi^q\left(\int_X f^p d\mu\right)^{\frac{1}{p}}$$
$$\Rightarrow \quad \varphi\left(\int_X f^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_X (\varphi \circ f)^q d\mu\right)^{\frac{1}{q}}.$$

If $p < 0$ and $q < 0$, then

$$\varphi^q(t) - \varphi^q(s) \geq \mu(t^p - s^p).$$

By the similar way we obtain

$$\int_X (\varphi \circ f)^q d\mu \leq \varphi^q\left(\int_X f^p d\mu\right)^{\frac{1}{p}}$$
$$\Rightarrow \quad \varphi\left(\int_X f^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_X (\varphi \circ f)^q d\mu\right)^{\frac{1}{q}}$$

Case 2. $(p > 0$ and $q < 0)$ or $(p < 0$ and $q > 0)$. Since φ is $(p-q)$-convex, the inequality (2.2) holds. Let

$$m = \inf_{a < s < t} \frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p}.$$

Then m is no smaller than any of the quotients on the right side of (2.2), for any $u \in (t, b)$. It follows that

$$\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \geq m.$$

Now let $p > 0$ and $q < 0$, then

$$
\begin{aligned}
& \varphi^q(t) - \varphi^q(s) \geq m(t^p - s^p) \\
\Rightarrow \quad & \varphi^q(s) \leq \varphi^q(t) + m(s^p - t^p) \\
\Rightarrow \quad & \varphi^q(f(x)) \leq \varphi^q(t) + m(f^p(x) - t^p) \\
\Rightarrow \quad & \int_X (\varphi \circ f)^q d\mu \leq \varphi^q(t) + m(\int_X f^p d\mu - t^p) \quad (\mu(X) = 1).
\end{aligned}
$$

Now set $t = (\int_X f^p d\mu)^{\frac{1}{p}}$. It follows that

$$
\begin{aligned}
& \int_X (g \circ f)^q d\mu \leq \varphi^q(\int_X f^p d\mu)^{\frac{1}{p}} \\
\Rightarrow \quad & \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq (\int_X (g \circ f)^q d\mu)^{\frac{1}{q}} \quad (q < 0).
\end{aligned}
$$

If $p < 0$ and $q > 0$, then

$$\varphi^q(t) - \varphi^q(s) \leq m(t^p - s^p)$$

By the similar way we obtain

$$
\begin{aligned}
& \int_X (\varphi \circ f)^q d\mu \geq \varphi^q(\int_X f^p d\mu)^{\frac{1}{p}} \\
\Rightarrow \quad & \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}} \quad (q > 0)
\end{aligned}
$$

2. Put $t = e^{\int_X \ln f d\mu}$, then $a < t < b$. Let $q > 0$ since φ is $(p - q)$-convex, the inequality (2.3) holds. Let

$$M = \sup_{a < s < t} \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s}$$

Then

$$
\begin{aligned}
& \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \leq M \\
\Rightarrow \quad & \varphi^q(t) - \varphi^q(s) \leq M(\ln t - \ln s) \\
\Rightarrow \quad & \varphi^q(s) \geq \varphi^q(t) - M(\ln t - \ln s)
\end{aligned}
$$

So for any $x \in X$, we have

$$\varphi^q(f(x)) \geq \varphi^q(t) - M(\ln t - \ln s)$$

By integrating both sides with respect to measure μ, we obtain

$$\int_X (\varphi \circ f)^q d\mu \geq \varphi^q(t) - M(\ln t - \int_X \ln f d\mu).$$

Now set $t = e^{\int_X \ln f d\mu}$. Thus

$$
\begin{aligned}
& \int_X (\varphi \circ f)^q d\mu \geq \varphi^q(e^{\int_X \ln f d\mu}) \\
\Rightarrow \quad & \varphi(e^{\int_X \ln f d\mu}) \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}} \quad (q > 0).
\end{aligned}
$$

If $p = 0$ and $q < 0$, by the similar way we get

$$\frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \geq m$$

where

$$m = \inf_{a<s<t} \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s}.$$

Hence

$$\varphi^q(t) - \varphi^q(s) \geq m(\ln t - \ln s)$$
$$\Rightarrow \quad \varphi^q(s) \leq \varphi^q(t) - m(\ln t - \ln s)$$
$$\Rightarrow \quad \int_X (\varphi \circ f)^q d\mu \leq \varphi^q(t) - m(\ln t - \int_X \ln f d\mu).$$

Now set $t = e^{\int_X \ln f d\mu}$. Thus

$$\int_X (\varphi \circ f)^q d\mu \leq \varphi^q(e^{\int_X \ln f d\mu})$$
$$\Rightarrow \quad \varphi(e^{\int_X \ln f d\mu}) \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}} \quad (q < 0)$$

3. Put $t = (\int_X f^p d\mu)^{\frac{1}{p}}$, then $a < t < b$. Let $p > 0$. By the $(p - q)$-convexity of φ the inequality (2.5) holds. Let

$$M = \sup_{a<s<t} \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p}$$
$$\Rightarrow \quad \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \leq M$$
$$\Rightarrow \quad \ln \varphi(t) - \ln \varphi(s) \leq M(t^p - s^p)$$
$$\Rightarrow \quad \ln \varphi(s) \geq \ln \varphi(t) - M(t^p - s^p)$$
$$\Rightarrow \quad \ln \varphi(f(x)) \geq \ln \varphi(t) - M(t^p - f^p(x)) \quad (\forall x \in X)$$
$$\Rightarrow \quad \int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(t) - M(t^p - \int_X f^p d\mu)$$

Since $t = (\int_X f^p d\mu)^{\frac{1}{p}}$, we obtain

$$\int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(\int_X f^p d\mu)^{\frac{1}{p}}$$
$$\Rightarrow \quad \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq e^{\int_X \ln(\varphi \circ f) d\mu}$$

If $p < 0$ and $q = 0$, put $t = (\int_X f^p d\mu)^{\frac{1}{p}}$. By the similar way we obtain

$$\frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \geq m \quad (m = \inf_{a<s<t} \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p})$$
$$\Rightarrow \quad \ln \varphi(t) - \ln \varphi(s) \leq m(t^p - s^p)$$
$$\Rightarrow \quad \ln \varphi(s) \geq \ln \varphi(t) - m(t^p - s^p)$$
$$\Rightarrow \quad \ln \varphi(f(x)) \geq \ln \varphi(t) - m(t^p - f^p(x))$$
$$\Rightarrow \quad \int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(t) - m(t^p - \int_X f^p d\mu)$$
$$\Rightarrow \quad \int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(\int_X f^p d\mu)^{\frac{1}{p}}$$
$$\Rightarrow \quad \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq e^{\int_X \ln(\varphi \circ f) d\mu}$$

4. For the proof of (4) see [43].

$\square$

Corollary 2.3. Let $X = \{x_1, x_2, \cdots, x_n\}$, $\mu(\{x_i\}) = \frac{1}{n}$ and $f(x_i) = a_i > 0$. If φ is a $(p - q)$-convex function on J which includes the image of f, then

1. If $p \neq 0$ and $q \neq 0$, then

$$\varphi\left(\frac{1}{n}\sum_{i=0}^{n} a_i^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{n}\sum_{i=0}^{n}\varphi^q(a_i)\right)^{\frac{1}{q}}$$

2. If $p = 0$ and $q \neq 0$, then

$$\varphi\left(\sqrt[n]{a_1 a_2 \cdots a_n}\right) \leq \left(\frac{1}{n}\sum_{i=0}^{n}\varphi^q(a_i)\right)^{\frac{1}{q}}$$

3. If $p \neq 0$ and $q = 0$, then

$$\varphi\left(\frac{1}{n}\sum_{i=0}^{n} a_i^p\right)^{\frac{1}{p}} \leq \sqrt[n]{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)}$$

4. If $p = q = 0$, then

$$\varphi\left(\sqrt[n]{a_1 a_2 \cdots a_n}\right) \leq \sqrt[n]{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)}$$

Proof is obvious by theorem 2.2 .

Corollary 2.4. Let $f : [a, b] \to (0, \infty)$ $(b > a > 0)$ be a continuous function and $\varphi : J \to (0, \infty)$ be a $(p-q)$-convex function on an interval J which includes the image of f. Then

1. If $p \neq 0$ and $q \neq 0$, then

$$\varphi\left(\frac{p}{b^p - a^p}\int_a^b \frac{f^p(x)}{x^{1-p}}dx\right)^{\frac{1}{p}} \leq \left(\frac{p}{b^p - a^p}\int_a^b \frac{\varphi^q(f(x))}{x^{1-p}}dx\right)^{\frac{1}{q}}$$

2. If $p = 0$ and $q \neq 0$, then

$$\varphi\left(e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x}dx}\right) \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\varphi^q(x)}{x}dx\right)^{\frac{1}{q}}$$

3. If $p \neq 0$ and $q = 0$, then

$$\varphi\left(\frac{p}{b^p - a^p}\int_a^b \frac{f^p(x)}{x^{1-p}}dx\right)^{\frac{1}{p}} \leq e^{\frac{p}{b^p - a^p}\int_a^b \frac{\ln\varphi(f(x))}{x^{1-p}}dx}$$

4. If $p = q = 0$, then

$$\varphi\left(e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x}dx}\right) \leq e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln\varphi(f(x))}{x}dx}$$

Proof . For the proof of (1) and (3) put $X = [a, b]$ and $d\mu = \dfrac{pdx}{x^{1-p}(b^p - a^p)}$ in the theorem 2.2 . For the proof of (2) and (4) put $X = [a, b]$ and $d\mu = \dfrac{dx}{(\ln b - \ln a)x}$ in theorem2.2 . $\square$

In the following theorem we obtain the Hermite-Hadamard inequality for $(p-q)$-convex functions.

Theorem 2.5. Let $\varphi : [a, b] \to (0, \infty)$ $(b > a > 0)$ be a $(p-q)$-convex function. Then

1. If $p \neq 0$ and $q \neq 0$, then

$$\varphi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \leq \left(\frac{p}{b^p - a^p}\int_a^b \frac{\varphi^q(x)}{x^{1-p}}dx\right)^{\frac{1}{q}} \leq \left(\frac{\varphi^q(a) + \varphi^q(b)}{2}\right)^{\frac{1}{q}}$$

2. If $p = 0$ and $q \neq 0$, then

$$\varphi(\sqrt{ab}) \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\varphi^q(x)}{x}dx\right)^{\frac{1}{q}} \leq \left(\frac{\varphi^q(a) + \varphi^q(b)}{2}\right)^{\frac{1}{q}}$$

3. If $p \neq 0$ and $q = 0$, then

$$\varphi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \leq e^{\frac{p}{b^p - a^p}\int_a^b \frac{\ln\varphi(x)}{x^{1-p}}dx} \leq \sqrt{\varphi(a)\varphi(b)}$$

4. If $p = q = 0$, then

$$\varphi(\sqrt{ab}) \le e^{\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln \varphi(x)}{x} dx} \le \sqrt{\varphi(a)\varphi(b)}$$

Proof .

1. Put $f(x) = x$ in corollary 2.4 (1). Hence

$$\varphi\left(\frac{p}{b^p - a^p} \int_a^b \frac{x^p}{x^{1-p}} dx\right) \le \left(\frac{p}{b^p - a^p} \int_a^b \frac{\varphi^q(x)}{x^{1-p}} dx\right)^{\frac{1}{q}}$$

$$\Rightarrow \quad \varphi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \le \left(\frac{p}{b^p - a^p} \int_a^b \frac{\varphi^q(x)}{x^{1-p}} dx\right)^{\frac{1}{q}}$$

This proves the left side. For the proof of the right side, by the change of variable $x = ((1-t)a^p + tb^p)^{\frac{1}{p}}$ and the $(p-q)$-convexity of φ we have

$$\left(\frac{p}{b^p - a^p} \int_a^b \frac{\varphi^q(x)}{x^{1-p}} dx\right)^{\frac{1}{q}} = \left(\frac{p}{b^p - a^p} \int_0^1 \frac{\varphi^q((1-t)a^p + tb^p)^{\frac{1}{p}}}{((1-t)a^p + tb^p)^{\frac{1}{p}-1}} \frac{1}{p} ((1-t)a^p + tb^p)^{\frac{1}{p}-1}(b^p - a^p) dt\right)^{\frac{1}{q}}$$

$$\le \left(\int_0^1 (1-t)\varphi^q(a) + t\varphi^q(b) dt\right)^{\frac{1}{q}} = \left(\frac{\varphi^q(a) + \varphi^q(b)}{2}\right)^{\frac{1}{q}}$$

2. Put $f(x) = x$ in corollary 2.4 (2). Hence

$$\varphi\left(e^{\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln x}{x} dx}\right) \le \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\varphi^q(x)}{x} dx\right)^{\frac{1}{q}}$$

$$\Rightarrow \quad \varphi(\sqrt{ab}) \le \left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\varphi^q(x)}{x} dx\right)^{\frac{1}{q}}$$

This proves the left side. For the proof of the right side by change of variable $x = a^{1-t}b^t$ and $(p-q)$-convexity of φ we have

$$\left(\frac{1}{\ln b - \ln a} \int_a^b \frac{\varphi^q(x)}{x} dx\right)^{\frac{1}{q}} = \left(\frac{1}{\ln b - \ln a} \int_0^1 \frac{\varphi^q(a^{1-t}b^t)}{a^{1-t}b^t} a^{1-t}b^t (\ln b - \ln a) dt\right)^{\frac{1}{q}}$$

$$\le \left(\int_0^1 ((1-t)\varphi^q(a) + t\varphi^q(b)) dt\right)^{\frac{1}{q}} = \left(\frac{\varphi^q(a) + \varphi^q(b)}{2}\right)^{\frac{1}{q}}$$

3. Put $f(x) = x$ in corollary 2.4 (3). Hence

$$\varphi\left(\frac{p}{b^p - a^p} \int_a^b \frac{x^p}{x^{1-p}} dx\right)^{\frac{1}{p}} \le e^{\frac{p}{b^p - a^p} \int_a^b \frac{\ln \varphi(x)}{x^{1-p}} dx}$$

$$\Rightarrow \quad \varphi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \le e^{\frac{p}{b^p - a^p} \int_a^b \frac{\ln \varphi(x)}{x^{1-p}} dx}$$

For the proof of the right side by change of variable $x = ((1-t)a^p + tb^p)^{\frac{1}{p}}$ and $(p-q)$-convexity of φ we have

$$\frac{p}{b^p - a^p} \int_a^b \frac{\ln \varphi(x)}{x^{1-p}} dx = \frac{p}{b^p - a^p} \int_0^1 \frac{\ln \varphi((1-t)a^p + tb^p)^{\frac{1}{p}}}{((1-t)a^p + tb^p)^{\frac{1}{p}-1}} \frac{1}{p}(b^p - a^p)((1-t)a^p + tb^p)^{\frac{1}{p}-1} dt$$

$$\le \int_0^1 \ln \varphi^{1-t}(a)\varphi^t(b) dt = \frac{\ln \varphi(a) + \ln \varphi(b)}{2} = \ln \sqrt{\varphi(a)\varphi(b)}$$

So

$$e^{\frac{p}{b^p - a^p} \int_a^b \frac{\ln \varphi(x)}{x^{1-p}} dx} \le e^{\ln \sqrt{\varphi(a)\varphi(b)}} = \sqrt{\varphi(a)\varphi(b)}$$

4. Put $f(x) = x$ in corollary 2.4 (4). See also [43].

□

Theorem 2.6. (i) If φ is convex (concave) and increasing on (a, b), then φ is $(p - q)$-convex (concave) on (a, b), whenever $p \leq 1 \leq q$ $(p \geq 1 \geq q)$

(ii) If φ is $(p - q)$-convex (concave) and decreasing on (a, b), then φ is convex (concave) on (a, b), whenever $p \leq 1 \leq q$ $(p \geq 1 \geq q)$

Proof . (i) Let φ be convex, increasing and $p \leq 1 \leq q$. We have

$$\varphi(tx^p + (1-t)y^p)^{\frac{1}{p}} \leq \varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$$
$$\leq (t\varphi^q(x) + (1-t)\varphi^q(y))^{\frac{1}{q}} \quad (p \neq 0)$$
$$\varphi(x^t y^{1-t}) \leq \varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$$
$$\leq (t\varphi^q(x) + (1-t)\varphi^q(y))^{\frac{1}{q}} \quad (p = 0)$$

Hence φ is $(p - q)$-convex. The proof of (ii) is simillar and can be omitted. □

Theorem 2.7. Let $\varphi : [a, b] \to (0, \infty)$ $(b > a > 0)$, $q > 0$ and $g(x) = \varphi^q(x^{\frac{1}{p}})$ $(p \neq 0)$, $g(x) = \varphi^q(e^x)$ $(p = 0)$. Then φ is $(p - q)$-convex on (a, b) if and only if g is convex on (a^p, b^p) $(p > 0)$ or (b^p, a^p) $(p < 0)$ or $(\ln a, \ln b)$ $(p = 0)$.

Proof . Let φ be $(p - q)$-convex on (a, b). So

$$\varphi(tx^p + (1-t)y^p)^{\frac{1}{p}} \leq (t\varphi^q(x) + (1-t)\varphi^q(y))^{\frac{1}{q}} \quad (p \neq 0)$$

whenever $x, y \in (a, b)$ and $t \in [0, 1]$. Put $X = x^p$ and $Y = y^p$. Then

$$\varphi(tX + (1-t)Y)^{\frac{1}{p}} \leq (t\varphi^q(X^{\frac{1}{p}}) + (1-t)\varphi^q(Y^{\frac{1}{p}}))^{\frac{1}{q}}$$
$$\Rightarrow \quad \varphi^q(tX + (1-t)Y)^{\frac{1}{p}} \leq \varphi^q(X^{\frac{1}{p}}) + (1-t)\varphi^q(Y^{\frac{1}{p}}) \quad (q > 0)$$
$$\Rightarrow \quad g(tX + (1-t)Y) \leq tg(X) + (1-t)g(Y)$$

It is obvious that $X, Y \in (a^p, b^p)$ $(p > 0)$ or $X, Y \in (b^p, a^p)$ $(p < 0)$. Now let $p = 0$ and φ be $(p - q)$-convex on (a, b), then

$$\varphi(x^t y^{1-t}) \leq (t\varphi^q(x) + (1-t)\varphi^q(y))^{\frac{1}{q}}$$
$$\Rightarrow \quad \varphi^q(x^t y^{1-t}) \leq t\varphi^q(x) + (1-t)\varphi^q(y) \quad (q > 0)$$

Put $x = e^X$ and $y = e^Y$. So

$$\varphi^q(e^{tX + (1-t)Y}) \leq t\varphi^q(e^X) + (1-t)\varphi^q(e^Y)$$
$$\Rightarrow \quad g(tX + (1-t)Y) \leq tg(X) + (1-t)g(Y) \quad (\ln a \leq X, Y \leq \ln b).$$

□

Theorem 2.8. Let φ be a $(p - q)$-convex function on $[a, b]$ and $q \geq 1$, $p \neq 0$. Then the following inequality holds

$$\frac{1}{b-a} \int_a^b \varphi(x)dx \leq \left[\frac{\varphi^q(a)(pb^{p+1} + a^{p+1} - (p+1)ab^p) + \varphi^q(b)(pa^{p+1} + b^{p+1} - (p+1)ba^p)}{(b-a)(b^p - a^a)(p+1)} \right]^{\frac{1}{q}}.$$

Especially for $p = q = 1$, $\frac{1}{b-a} \int_a^b \varphi(x)dx \leq \frac{\varphi(a) + \varphi(b)}{2}$.

Proof . By change of variable $x = (tb^p + (1-t)a^p)^{\frac{1}{p}}$ and $(p-q)$-convexity of φ we have

$$\frac{1}{b-a}\int_a^b \varphi(x)dx = \frac{1}{b-a}\int_0^1 \varphi(tb^p + (1-t)a^p)^{\frac{1}{p}}\frac{b^p - a^p}{p}(tb^p + (1-t)a^p)^{\frac{1}{p}-1}dt$$

$$\leq \frac{b^p - a^p}{p(b-a)}\int_0^1 (t\varphi^q(b) + (1-t)\varphi^q(a))^{\frac{1}{q}}(tb^p + (1-t)a^p)^{\frac{1}{p}-1}dt.$$

Since $\dfrac{b^p - a^p}{p(b-a)}\int_0^1 (tb^p + (1-t)a^p)^{\frac{1}{p}-1}dt = 1$ and $\dfrac{1}{q} \leq 1$, by the concavity of $h(x) = x^{\frac{1}{q}}$ and classical Jensen's inequality (1.1) $(d\mu = \dfrac{b^p - a^p}{p(b-a)}(tb^p + (1-t)a^p)^{\frac{1}{p}-1}dt)$ we obtain

$$\leq \left(\frac{b^p - a^p}{p(b-a)}\int_0^1 (t\varphi^q(b) + (1-t)\varphi^q(a))(tb^p + (1-t)a^p)^{\frac{1}{p}-1}dt\right)^{\frac{1}{q}}$$

$$= \left(\frac{b^p - a^p}{p(b-a)}\int_0^1 (t(\varphi^q b) - \varphi^q(a)) + \varphi^q(a))(t(b^p - a^a) + a^p)^{\frac{1}{p}-1}dt\right)^{\frac{1}{q}}$$

Again by change of variable $t(b^p - a^p) + a^p = x^p$ and easy calculations we get

$$= \left(\frac{b^p - a^p}{p(b-a)}\int_a^b [\frac{x^p - a^p}{b^p - a^p}(\varphi^q(b) - \varphi^q(a)) + \varphi^q(a)]\frac{p}{b^p - a^a}dx\right)^{\frac{1}{q}}$$

$$= \left[\frac{\varphi^q(a)(pb^{p+1} + a^{p+1} - (p+1)ab^p) + \varphi^q(b)(pa^{p+1} + b^{p+1} - (p+1)ba^p)}{(b-a)(b^p - a^a)(p+1)}\right]^{\frac{1}{q}}$$

□

3 Examples

1. $\varphi(x) = x$ is $(p-q)$-convex $(p < q)$ on $[a,b]$ $(b > a > 0)$. Hence by theorem 2.5 we have

$$\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \leq \left(\frac{p}{b^p - a^p}\int_a^b \frac{x^q}{x^{1-p}}dx\right)^{\frac{1}{q}} \leq \left(\frac{a^q + b^q}{2}\right)^{\frac{1}{q}} \quad (0 \neq p < q)$$

$$\sqrt{ab} \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{x^q}{x}dx\right)^{\frac{1}{q}} \leq \left(\frac{a^q + b^q}{2}\right)^{\frac{1}{q}} \quad (p = 0 < q)$$

$$\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \leq e^{\frac{p}{b^p - a^p}\int_a^b \frac{\ln x}{x^{1-p}}dx} \leq \sqrt{ab} \quad (p < 0 = q)$$

Thus

$$\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \leq \left(\frac{p(b^{p+q} - a^{p+q})}{(p+q)(b^p - a^p)}\right)^{\frac{1}{q}} \leq \left(\frac{a^q + b^q}{2}\right)^{\frac{1}{q}} \quad (0 \neq p < q)$$

$$\sqrt{ab} \leq \left(\frac{b^q - a^q}{q(\ln b - \ln a)}\right)^{\frac{1}{q}} \leq \left(\frac{a^q + b^q}{2}\right)^{\frac{1}{q}} \quad (0 < q)$$

$$\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \leq e^{\frac{b^p \ln b - a^p \ln a}{b^p - a^p} - \frac{1}{p}} \leq \sqrt{ab} \quad (p < 0)$$

So if $\gamma < 0 < \alpha < \beta$, then

$$\left(\frac{a^\gamma + b^\gamma}{2}\right)^{\frac{1}{\gamma}} \leq e^{\frac{b^\gamma \ln b - a^\gamma \ln a}{b^\gamma - a^\gamma} - \frac{1}{\gamma}} \leq \sqrt{ab} \leq \left(\frac{b^\alpha - a^\alpha}{\alpha(\ln b - \ln a)}\right)^{\frac{1}{\alpha}}$$

$$\leq \left(\frac{\alpha(b^{\alpha+\beta} - a^{\alpha+\beta})}{(\alpha+\beta)(b^\alpha - a^\alpha)}\right)^{\frac{1}{\beta}} \leq \left(\frac{a^\beta + b^\beta}{2}\right)^{\frac{1}{\beta}}$$

In the other words by means notations we have

$$M_\gamma(a,b) \leq e^{\frac{b^\gamma \ln b - a^\gamma \ln a}{b^\gamma - a^\gamma} - \frac{1}{\gamma}} \leq G(a,b) \leq T_\alpha(a,b)$$
$$\leq L_{\alpha,\beta}(a,b) \leq M_\beta(a,b)$$

2. $\varphi(x) = \dfrac{1}{1+x^2}$ is not convex on $(0,\infty)$, but it is $(p-q)$-convex on $(0,\infty)$, where $p=2$, $q=-2$. Because by theorem 2.7, $g(x) = \varphi^{-2}(x^{\frac{1}{2}}) = (1+x)^2$ is convex on $(0,\infty)$.

3. $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ is concave and nondecreasing on $(0,\infty)$ [13]. So by theorem 2.6 (i) ψ is $(p-q)$-concave on $(0,\infty)$, when $p \geq 1 \geq q$. Hence by theorem 2.5 (1) for $b > a > 0$, we have

$$\psi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \geq \left(\frac{p}{b^p - a^p}\int_a^b \frac{\psi^q(x)}{x^{1-p}}\right)^{\frac{1}{q}} \geq \left(\frac{\psi^q(a) + \psi^q(b)}{2}\right)^{\frac{1}{q}}$$

We investigate these inequalities in two special cases.

(i) For $p = q = 1$ we have

$$\frac{\psi(a) + \psi(b)}{2} \leq \frac{1}{b-a}\int_a^b \psi(x)dx \leq \psi\left(\frac{a+b}{2}\right)$$
$$\Rightarrow \quad \frac{\psi(a) + \psi(b)}{2} \leq \frac{\ln\Gamma(b) - \ln\Gamma(a)}{b-a} \leq \psi\left(\frac{a+b}{2}\right)$$
$$\Rightarrow \quad e^{(b-a)\frac{\psi(a)+\psi(b)}{2}} \leq \frac{\Gamma(b)}{\Gamma(a)} \leq e^{(b-a)\psi(\frac{a+b}{2})}$$

Especially for $b = a+1$ we get

$$e^{\frac{\psi(a)+\psi(a+1)}{2}} \leq \frac{\Gamma(a+1)}{\Gamma(a)} \leq e^{\psi(\frac{2a+1}{2})}$$

Finally since $\Gamma(a+1) = a\Gamma(a)$ and $\psi(a+1) = \psi(a) + \dfrac{1}{a}$ $(a > 0)$ we obtain

$$e^{\psi(a)+\frac{1}{2a}} \leq a \leq e^{\psi(a+\frac{1}{2})}$$

(ii) For $p = 2$ and $q = 1$ we have

$$\frac{\psi(a) + \psi(b)}{2} \leq \frac{2}{b^2 - a^2}\int_a^b x\psi(x)dx \leq \psi\left(\frac{a^2 + b^2}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow \quad \frac{\psi(a) + \psi(b)}{2} \leq \frac{2}{b^2 - a^2}\left[x\ln\Gamma(x)\big|_a^b - \int_a^b \ln\Gamma(x)dx\right] \leq \psi\left(\frac{a^2 + b^2}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow \quad \frac{\psi(a) + \psi(b)}{2} \leq \frac{2}{b^2 - a^2}\left[b\ln\Gamma(b) - a\ln\Gamma(a) - \int_a^b \ln\Gamma(x)dx\right] \leq \psi\left(\frac{a^2 + b^2}{2}\right)^{\frac{1}{2}}$$

Especially for $b = a+1$ and considering $\int_a^{a+1} \ln\Gamma(x)dx = -a + a\ln a + \sqrt{2\pi}$ [40] we obtain

$$\frac{\psi(a) + \psi(a+1)}{2} \leq \frac{2}{2a+1}\left[(a+1)\ln\Gamma(a+1) - a\ln\Gamma(a) + a - a\ln a - \sqrt{2\pi}\right]$$
$$\leq \psi\left(a^2 + a + \frac{1}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow \quad \psi(a) + \frac{1}{2a} \leq \frac{2}{2a+1}\left[\ln a + \ln\Gamma(a) + a - \sqrt{2\pi}\right] \leq \psi\left(a^2 + a + \frac{1}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow \quad (a + \frac{1}{2})(\psi(a) + \frac{1}{2a}) \leq \ln\Gamma(a+1) + a - \sqrt{2\pi} \leq (a + \frac{1}{2})\psi\left(a^2 + a + \frac{1}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow \quad e^{\sqrt{2\pi} - a + (a+\frac{1}{2})(\psi(a)+\frac{1}{2a})} \leq \Gamma(a+1) \leq e^{\sqrt{2\pi} - a + (a+\frac{1}{2})\psi(a^2+a+\frac{1}{2})^{\frac{1}{2}}}$$

References

[1] I. Abbas Baloch, A.A. Mughal, Y.M. Chu, A.U. Haq, and M.D.L. Sen, *A variant of Jensen-type inequality and related results for harmonic convex functions*, AIMS Math. **5** (2020), 6404–6418.

[2] M. Adil Khan, J. Khan, and J. Pečarić, *Generalization of Jensen's and Jensen-Steffensen's inequalities by generalized majorization theorem*, J. Math. Inequal. **11** (2017), no. 4, 1049–1074.

[3] G.D. Anderson, M.K. Vamanamurtby, and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl. **335** (2007), 1294–1308.

[4] H. Barsam, *Some new Hermite-Hadamard type inequalities for convex functions*, Wavelets Linear Algebra **7** (2020), no. 2, 11–22.

[5] H. Barsam, A.R. Sattarzadeh, *Some results on Hermite-Hadamard inequalities*, Mahani Math. Res. Cent. **9** (2020), no. 2, 79–86.

[6] C. Chen, M.S. Sallem, and M.S. Zahoor, *Some inequalities of generalized p-convex functions concerning Raina's fractional integral operators*, J. Math. **2021** (2021), Article ID 3089553.

[7] D. Choi, M. Krnic, and J. Pecaric, *More accurate classes of Jensen-type inequalities for convex and operator convex functions*, Math. Inequal. Appl. **21** (2018), no. 2, 301–321.

[8] S.S. Dragomir, *Inequalities of Hermite-Hadamard type for HA-convex functions*, Moroccan J. Pure Appl. Anal. **3** (2017), no. 3, 83–101.

[9] S.S. Dragomir, *Inequalities of Hermite-Hadamard type for GG-convex functions*, Indian J. Math. **60** (2018), no. 1, 1–21.

[10] S.S. Dragomir, *Inequalities of Hermite-Hadamard type for HH-convex functions*, Acta Comment. Univer. Tartuensis Math. 22 (2018), no. 2, 179–190.

[11] S.S. Dragomir, *Weighted integral inequalities for GG-convex functions*, J. Appl. Math. Inf. **36** (2018), no. 3, 155–171.

[12] S.S. Dragomir, *Inequalities of Hermite-Hadamard type for GH-convex functions*, Electron. J. Math. Anal. Appl. **7** (2019), no. 2, 244–255.

[13] S.S. Dragomir, R.P. Agarwal, and N.S. Barnett, *Inequalities for Beta and Gamma functions via some classical and new integral inequalities*, J. Inequal. Appl. **2** (1999), no. 3.

[14] Z. Eken, S. Kemali, G. Tınaztepe, and G. Adilov, *The Hermite-Hadamard inequalities for p-convex functions*, Hacettepe J. Math. Statist. **50** (2021), no. 5, 1268–1279.

[15] I. Iscan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe J. Math. Statist. **43** (2014), no. 6, 935–942.

[16] I. Iscan, *Hermite-Hadamard type inequalities for p-convex functions*, Int. J. Anal. Appl. **11** (2016), no. 2, 137–145.

[17] I. Iscan, *Ostrowski type inequalities for p-convex functions*, New Trends Math. Sci. **4** (2016), no. 3, 140–150.

[18] I. Iscan and S. Wu, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Comput. **238** (2014), 237–244.

[19] M. Kunt and I. Iscan, *Hermite-Hadamard-Fejér type inequalities for p-convex functions*, Arab. J. Math. Sci. **23** (2017), 215–230.

[20] M. Kunt and I. Iscan, *Hermite-Hadamard-Fejer type inequalities for p-convex functions via fractional integrals*, Iran. J. Sci. Technol. Trans. Sci. **42** (2018), 2079–2089.

[21] N. Mehreen and M. Anwar, *Hadamard and Fejer type inequalities for p-convex functions via Caputo fractional derivatives*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. 1, 253–266.

[22] N. Mehreen and M. Anwar, *Hermite-Hadamard type inequalities via exponentially p-convex functions and exponentially s-convex functions in second sense with applications*, J. Inequal. Appl. **2019** (2019), 92.

[23] C.P. Niculescu, *A multiplicative mean value and its applications*, Inequality Theory and Applications, Vol.1, Nova

Science Publishers, Huntington, New York, 2001, pp. 243–255.

[24] C.P. Niculescu, *Convexity according to the geometric mean*, Math. Inequal. Appl. **3** (2000), 155–167.

[25] C.P. Niculescu, *Convexity according to means*, Math. Inequal. Appl. **6** (2003), 571–579.

[26] C.P. Niculescu and L.E. Persson, *Convex Functions and Their Applications*, Springer, Monograph, 2004.

[27] M.E. Ozdemir, E. Set, and M. Alomari, *Integral inequalities via several kinds of convexity*, Creat. Math. Inf. **20** (2011), no. 1, 62–73.

[28] J. Pecaric and J. Peric, *New improvement of the converse Jensen inequality*. Math. Inequal. Appl. **21** (2018), no. 1, 217–234.

[29] F. Popovici and C.I. Spiridon, *The Jensen inequality for (M, N)-convex functions*, Ann. Univer. Craiova Math. Comput. Sci. Ser. **38** (2011), no. 4, 63–66.

[30] W. Rudin, *Real and Complex Analysis*, McGrow-Hill, New York, 1974.

[31] M. Sababheh, *Improved Jensen's inequality*. Math. Inequal. Appl. **20** (2017), no. 2, 389–403.

[32] S. Salas, Y. Erdas, T. Toplu and E. Set, *On some generalized fractional integral inequalities for p-convex functions*, Fractal Fract. **3** (2019), no. 29, 1–9.

[33] J. Sandor, *On refinements of certain inequalities for means*, Arch. Math. (Brno) **31** (1995), no. 4, 274–282.

[34] M.Z. Sarikaya and H. Budak, *On generalized Hermite-Hadamard inequality for generalized convex function*, Int. J. Nonlinear Anal. Appl. **8** (2017), no, 2, 209–222.

[35] S. Sezer, Z. Eken, and G. Tnaztepe, *Adilov, p-convex functions and their some properties*, Numer. Func. Anal. Opt. **42** (2021), no. 4, 443–459.

[36] M. Vivas-Cortez, A. Kashuri, S.I. Butt, M. Tariq, and J. Nasir, *Exponential type p-convex function with some related inequalities and their applications*, Appl. Math. Inf. Sci. **15** (2021), no. 3, 253–261.

[37] L.C. Wang, *On extensions and refinements of Hermite-Hadamard inequalities for convex functions*, Math. Inequal. Appl. **6** (2003), 659–666.

[38] W. Wang and J. Qi, *Some new estimates of Hermite-Hadamard inequalities for harmonically convex functions with applications*, Int. J. Anal. Appl. **13** (2017), no. 1, 15-21.

[39] G. Zabandan, *A new refinement of Hermite-Hadamard for convex functions*, J. Ineq. Pure Appl. Math. **10** (2009), Article, ID 45.

[40] G. Zabandan, *Note on Stirling's formula and Wallis inequality of order m*, Adv. Inequal. Apple. **2015** (2015), Article ID 4.

[41] G. Zabandan, *Jensen's inequality for HH-convex functions and related results*, Adv. inequal. Appl. **2016** (2016) Article 10.

[42] G. Zabandan, *New Inequalities of Hermite-Hadamard type for MN-convex functions*, Adv. Inequal. Appl. **2016** (2016), 7.

[43] G. Zabandan, *Jensen's inequality for GG-convex functions*, Int. J. Nonlinear Anal. Appl. **10** (2019), no. 1, 1–7.

[44] G. Zabandan, *An extension and refinement of Hermite-Hadamard inequalities and related results*, Int. J. Nonlinear Anal. Appl. **11** (2020), no. 2, 379–390.

[45] G. Zabandan, A. Bodaghi, and A. Kilicman, *The Hermite-Hadamard inequality for r-convex function*, J. Inequal. Appl. **2012** (2012), Article ID 215.

[46] G. Zabandan and A. Kilicman, *A new version of Jensen's inequality and related results*, J. Inequal. Appl. **2012** (2012), 238.

[47] K.S. Zhang and J.P. Wan, *p-convex functions and their properties*, Pure Appl. Math. **23** (2007), no. 1, 130–133.