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An algebraic method to obtain analytical solutions for a class of fractional partial differential equations

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Abstract

In this study, we apply an algebraic approach to solve a class of fractional partial differential equations (FPDEs) by utilizing conformable fractional derivatives. This method has been successfully utilized to study and achieve analytical solutions for Drinfeld-Sokolov-Wilson equations. In this approach, we use a suitable fractional transformation and the principles of fractional calculus to simplify fractional partial differential equations into ordinary differential equations.

Keywords: fractional differential equations, Drinfeld-Sokolov-Wilson equations, conformable fractional derivative, analytical solution
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1 Introduction

In recent decades, fractional differential equations(FDEs) have been used widely to model physical phenomena, especially uncommon phenomena and complex natural processes that cannot be efficiently described by classical calculus [15, 18]. These equations have applications in various fields, such as mathematical biology, fluid mechanics, nonlinear optics, image processing, plasma physics and so on. We need to find solutions for these equations in order research and describe these phenomena. It is difficult to solve FDEs. Hence, numerous researchers have developed and implemented numerical and analytical techniques to solve these equations in recent years. Some of these techniques are such as, Finite difference method[1], Adomian decomposition method [3], differential transform method [16], variational iteration method [5, 8], $\frac{G'}{G^2}-$expansion method [11], Fan-sub equation method [12], Lie group method [9], exp-expansion method [4], first integral method [2] and so on. In this paper, we study a class of fractional partial differential equations, namely Drinfeld-Sokolov-Wilson(DSW) in the following form [10]:

$$D_t^{\alpha} u + n\, v D_x^{\beta} v = 0, \tag{1.1}$$
$$D_t^{\alpha} v + q D_x^{3\alpha} v + ru D_x^{\beta} v + sv D_x^{\beta} u = 0. \tag{1.2}$$

where $p, q, r,$ and s are non-zero parameters. Also, D_t^{α} and D_x^{β} represent fractional partial derivative operators, as described in section 2. When $\alpha = \beta = 1$, equations (1.1) and (1.2) are called classical DSW equations. These equations play an important role in modeling shallow water flow and fluid dynamics. Equations (1.1) and (1.2) were

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first introduced by Drinfeld and Sokolov and then, studied by Wilson. Researchers have utilized some techniques to study this equation such as, Galerkin method [20], F-expansion method [6], simplest equation method[14] and so on. Our aim in this paper is to find analytical solutions for the DSW equations using an algebraic method, as described in section 3.

The rest of our work is organized as follows. In section 2, we present definition of the conformable derivative with its properties. Description of method and its applications to the space-time fractional differential equations are described in section 3. Then the mentioned method is applied to the DSW equations in section 4. Graphical presentation of some solutions are shown in section 5 Discussion and conclusions are presented in section 6.

2 Definition of the conformable derivative with its properties

In this section, we illustrate the definition of the conformable fractional derivative(CFD) and some its important properties of order γ with respect to the independent variable ζ as follows[13].

For a function $h : [0, \infty] \to \mathbb{R}$, theCFD of h of order α is defined by

$$D^{\gamma} h(\zeta) = \lim_{\eta \to 0} \frac{h(\zeta + \eta\zeta^{1-\gamma}) - h(\zeta)}{\eta}. \tag{2.1}$$

Some well-known properties to this newly defined fractional derivative are as follows. If g and $h \neq 0$ be two functions γ-differentiable, $\gamma \in (0, 1]$ and $a, b \in \mathbb{R}$. Then, we have

$$\begin{aligned}
&(1) \quad D^{\gamma}\{ag(\zeta) + b\, h(\zeta)\} = a\, D^{\gamma} g(\zeta) + b\, D^{\gamma} h(\zeta),\\
&(2) \quad D^{\gamma}\{g(\zeta)\, h(\zeta)\} = g(\zeta)\, D^{\gamma} h(\zeta) + h(\zeta)\, D^{\gamma} g(\zeta),\\
&(3) \quad D^{\gamma}\{\frac{g(\zeta)}{h(\zeta)}\} = \frac{h(\zeta)\, D^{\gamma} g(\zeta) - g(\zeta) \quad D^{\gamma} h(\zeta)}{h^2(\zeta)},\\
&(4) \quad D^{\alpha} C = 0, \quad \text{for all constant functions } f(z) = C,\\
&(5) \quad D^{\gamma}(g)(\zeta) = \zeta^{1-\gamma} \frac{dg}{d\zeta}.
\end{aligned}$$

Also, CFD of some special functions are as follows.

$$\begin{aligned}
&(a) \quad D^{\gamma}(\zeta^r) = r\zeta^{r-\gamma}\, for\, all\, r \in \mathbb{R},\\
&(b) \quad D^{\gamma}(1) = 0,\\
&(c) \quad D^{\gamma}(e^{c\zeta}) = c\zeta^{1-\gamma} e^{c\zeta},\ c \in \mathbb{R},\\
&(d) \quad D^{\gamma}(\sin b\zeta) = b\zeta^{1-\gamma} \cos b\zeta,\ b \in \mathbb{R},\\
&(e) \quad D^{\gamma}(\cos b\zeta) = -b\zeta^{1-\gamma} \sin b\zeta,\ b \in \mathbb{R},\\
&(f) \quad D^{\gamma}(\frac{1}{\gamma}\zeta^{\gamma}) = 1.
\end{aligned}$$

The proofs of these properties can be seen in[13]. Let $\gamma \in (n, n+1]$, and h be an γ-differentiable at$t > 0$. Then the CFD of h of orderγ is defined as

$$D^{\gamma}(h(t)) = \lim_{\eta \to 0} \frac{h^{(\lceil\gamma\rceil - 1)}(t + \eta t^{(\lceil\gamma\rceil - \gamma)}) - h^{(\lceil\gamma\rceil - 1)}(t)}{\eta}. \tag{2.2}$$

where $\lceil\gamma\rceil$ is the smallest integer greater than or equal to γ.

3 Description of method and its applications to the space-time FDEs

In this section, we outline the main steps of this method for solving FDEs. For a given FDE in two variables x and t we have

$$P\left(u, u_x, u_t, D_t^{\alpha} u, D_x^{\alpha} u, \ldots\right) = 0, \quad 0 < \alpha < 1, \tag{3.1}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are CFDs of u, $u = u(x,t)$ is an unknown function and P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. We take the travelling wave transformation

$$\eta = \frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha, \tag{3.2}$$

where ω and λ are nonzero constants to be determined later. Substituting (3.2) into (3.1), we reduce (3.1) to the following ordinary differential equation

$$\tilde{N}(U, U', U'', U''', ...) = 0. \tag{3.3}$$

Here prime denotes the derivative with respect to η. Exact solutions for this equation can be constructed as a finite series

$$\Phi(\xi) = \frac{\sum_{j=0}^{\eta_1} A_j\, \Theta(\xi)}{\sum_{j=0}^{\eta_2} B_j \Theta(\xi)}, \tag{3.4}$$

where the positive constants η_1 and η_2 can be calculated by considering the homogeneous balance between the highest order derivatives and the highest nonlinear terms of $\Phi(\xi)$ in equation (3.3), and A_j $(0 \le j \le \eta_1)$, B_j $(0 \le j \le \eta_2)$ are constants to be found later and $A_{\eta_1}, B_{\eta_2} \neq 0$. Here $\Theta = \Theta(\xi)$ satisfies the following ODE

$$\Theta'(\xi) = p + \Theta^2(\xi) \tag{3.5}$$

where p is a constant and which has the following special solutions [17].

Case1: When $p < 0$,

$$\Theta_1(\xi) = -\sqrt{-p}\tanh(\sqrt{-p}\,\xi), \tag{3.6}$$
$$\Theta_2(\xi) = -\sqrt{-p}\coth(\sqrt{-p}\,\xi), \tag{3.7}$$
$$\Theta_3(\xi) = -\sqrt{-p}\tanh(2\sqrt{-p}\,\xi) \pm i\sqrt{-p}\, sech(2\sqrt{-p}\,\xi), \tag{3.8}$$
$$\Theta_4(\xi) = -\sqrt{-p}\coth(2\sqrt{-p}\,\xi) \pm \sqrt{-p}\, csch(2\sqrt{-p}\,\xi), \tag{3.9}$$
$$\Theta_5(\xi) = -\frac{1}{2}\left(\sqrt{-p}\tanh(\frac{\sqrt{-p}}{2}\xi) + \sqrt{-p}\coth(\frac{\sqrt{-p}}{2}\xi)\right). \tag{3.10}$$

Case2: When $p > 0$,

$$\Theta_6(\xi) = \sqrt{p}\tan(\sqrt{p}\,\xi), \tag{3.11}$$
$$\Theta_7(\xi) = -\sqrt{p}\cot(\sqrt{p}\,\xi), \tag{3.12}$$
$$\Theta_8(\xi) = -\sqrt{p}\tan(2\sqrt{p}\,\xi) \pm \sqrt{p}\ \sec(2\sqrt{p}\,\xi), \tag{3.13}$$
$$\Theta_9(\xi) = -\sqrt{p}\cot(2\sqrt{p}\,\xi) \pm \sqrt{p}\ \csc(2\sqrt{p}\,\xi), \tag{3.14}$$
$$\Theta_{10}(\xi) = \frac{1}{2}\left(\sqrt{p}\tan(\frac{\sqrt{p}}{2}\xi) - \sqrt{p}\cot(\frac{\sqrt{p}}{2}\xi)\right). \tag{3.15}$$

Case3: When $p = 0$,

$$\Theta_{11}(\xi) = -\frac{1}{\xi + d}. \tag{3.16}$$

where d is a constant. Now, this method for obtaining exact solutions of FDEs consists from the following two main steps:

- Step (1). By substituting (3.4) with Eq.(3.5) into (3.3) and collecting all terms with the same powers of $\Theta(\xi)$ together, the left hand side of Eq.(3.3) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of $A_j\,(j = 0,1,2,...,\eta_1), B_j\,(j = 0,1,2,...,\eta_2)$, κ and ω.
- Step (2). Solving the system of algebraic equations and then substituting the results and the general solutions (3.6)-(3.16) into (3.4), it gives travelling wave solutions of (3.3).

4 Application

In this section, we consider the DSW system as follows[10, 7]

$$D_t^{\alpha} u + nvD_x^{\beta} v = 0, \tag{4.1}$$
$$D_t^{\alpha} v + qD_x^{3\alpha} v + ruD_x^{\beta} v + svD_x^{\beta} u = 0. \tag{4.2}$$

where u and v are functions of space variable x and time variable t and $0 < \alpha,\ \beta \le 1$. For obtaining exact solutions of (4.1) and (4.2), We take the traveling wave transformation

$$u(x,t) = U(\eta), \quad v(x,t) = V(\eta),$$
$$\eta = \frac{\omega}{\beta} x^{\beta} - \frac{\lambda}{\alpha} t^{\alpha}, \tag{4.3}$$

where ω and λ are constants which should to be determined later. Then equations (4.1) and (4.2) are reduced into two nonlinear ODEs

$$-\lambda U' + n\omega V V' = 0, \tag{4.4}$$
$$-\lambda V + q\omega^3 V''' + r\omega U V' + s\omega V U' = 0, \tag{4.5}$$

By integrating of Eq.(4.4) with respect to η, we obtain

$$U = \frac{n\omega}{2\lambda} V^2. \tag{4.6}$$

Substituting equation (4.6) into equation (4.5) yields

$$q\omega^3 V''' + n\omega^2 (\frac{r+2s}{2\lambda}) V^2 V' - \lambda V' = 0. \tag{4.7}$$

By integrating of equation (4.7) with respect to η, we get

$$q\omega^3 V'' + n\omega^2 (\frac{r+2s}{6\lambda}) V^3 - \lambda V = 0. \tag{4.8}$$

Balancing V'' with V^3 in (4.8) along with (3.4), we get the below:

$$\eta_1 - \eta_2 + 2 = 3(\eta_1 - \eta_2) \Longrightarrow \eta_1 = \eta_2 + 1, \tag{4.9}$$

therefore, solution of (4.8) can be expressed as follows.

Type 1: $\eta_1 = 1$ and $\eta_2 = 0$,

$$V(\eta) = \frac{A_0 + A_1\, \Theta(\eta)}{B_0}, \tag{4.10}$$

Substituting (4.10) into (4.8) and making use of equation (3.5) and equating each coefficient of this polynomial to zero, we obtain a set of nonlinear algebraic equations for A_0, A_1, B_0, p, κ and ω. Solving obtained system using *Mathematica*, we obtain

$$\bullet \text{Set } 1: A_0 = 0,\ A_1 = A_1,\ B_0 = \frac{i\sqrt{np(r+2s)}\,\omega}{\sqrt{6}\,\lambda} A_1, \quad p = \frac{\lambda}{2q\omega^3}. \tag{4.11}$$

By using of the (4.10), (4.11) and cases (3.6)-(3.10) respectively, we get

$$\begin{cases} v_1(x,t) = -\frac{\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega} \tanh(\sqrt{-p}(\frac{\omega}{\beta}x^{\beta} - \frac{\lambda}{\alpha}t^{\alpha})), \\ u_1(x,t) = \frac{3\lambda}{(r+2s)\,\omega} \tanh^2(\sqrt{-p}(\frac{\omega}{\beta}x^{\beta} - \frac{\lambda}{\alpha}t^{\alpha})). \end{cases}$$

$$\begin{cases} v_2(x,t) = -\frac{\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega} \coth(\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_2(x,t) = \frac{3\lambda}{(r+2s)\,\omega} \coth^2(\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)). \end{cases}$$

$$\begin{cases} v_3(x,t) = -\frac{\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega}[-\tanh(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm isech(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_3(x,t) = \frac{3\lambda}{(r+2s)\,\omega}[-\tanh(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm isech(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_4(x,t) = -\frac{\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega}[-\coth(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm csch(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_4(x,t) = \frac{3\lambda}{(r+2s)\,\omega}[-\coth(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm csch(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_5(x,t) = -\frac{\sqrt{6}\,\lambda}{2\sqrt{n(r+2s)}\,\omega} \tanh(\frac{\sqrt{-p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) + \coth(\frac{\sqrt{-p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_5(x,t) = \frac{3\lambda}{4(r+2s)\,\omega} \tanh(\frac{\sqrt{-p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) + \coth(\frac{\sqrt{-p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))^2. \end{cases}$$

By using of the (4.10), (4.11) and cases (3.11)-(3.15) respectively, we get

$$\begin{cases} v_6(x,t) = -\frac{i\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega} \tan(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_6(x,t) = -\frac{3\lambda}{(r+2s)\,\omega} \tan^2(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)). \end{cases}$$

$$\begin{cases} v_7(x,t) = \frac{i\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega} \cot(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_7(x,t) = -\frac{3\lambda}{(r+2s)\,\omega} \cot^2(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)). \end{cases}$$

$$\begin{cases} v_8(x,t) = \frac{i\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega}[-\tan(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm \sec(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_8(x,t) = -\frac{3\lambda}{(r+2s)\,\omega}[-\tan(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm \sec(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_9(x,t) = \frac{i\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega}[-\cot(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm \csc(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_9(x,t) = -\frac{3\lambda}{(r+2s)\,\omega}[-\cot(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm \csc(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_{10}(x,t) = -\frac{i\sqrt{6}\,\lambda}{2\sqrt{n(r+2s)}\,\omega}[\tan(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) - \cot(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_{10}(x,t) = \frac{3\lambda}{4(r+2s)\,\omega}[\tan(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) - \cot(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

Type 2: $\eta_1 = 2$ and $\eta_2 = 1$,

$$V(\eta) = \frac{A_0 + A_1\,\Theta(\eta) + A_2\,\Theta^2(\eta)}{B_0 + B_1\,\Theta(\eta)}. \tag{4.12}$$

Substituting (4.12) into (4.8) and making use of equation (3.5) and equating each coefficient of this polynomial to zero, we obtain a set of nonlinear algebraic equations for $A_0, A_1, A_2, B_0, B_1, p, \kappa$ and ω. Solving obtained system using *Mathematica*, we obtain

$$\bullet \text{ Set } 1: A_0 = A_0,\ A_1 = 0,\ A_2 = \frac{1}{p}A_0,\ B_0 = 0,\ B_1 = \frac{\sqrt{n(r+2s)}\,\omega}{\sqrt{3p}\,\lambda}A_0,\ p = -\frac{\lambda}{4q\omega^3}. \tag{4.13}$$

By using of the (4.12), (4.13) and cases (3.11)-(3.15) respectively, we get

$$\begin{cases} v_1(x,t) = \frac{\sqrt{3}\,\lambda}{\sqrt{n(r+2s)}\,\omega}[\tan(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) + \cot(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_1(x,t) = \frac{3\lambda}{2(r+2s)\,\omega}[\tan(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) + \cot(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_2(x,t) = -\frac{\sqrt{3}\,\lambda}{\sqrt{n(r+2s)}\,\omega}[\tan(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) + \cot(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_2(x,t) = \frac{3\lambda}{2(r+2s)\,\omega}[\tan(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) + \cot(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_3(x,t) = \frac{\sqrt{3}\,\lambda}{\sqrt{n(r+2s)}\,\omega}\left(\frac{1+[-\tan(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))\pm\sec(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]^2}{[-\tan(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))\pm\sec(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]}\right), \\ u_3(x,t) = \frac{3\lambda}{2(r+2s)\,\omega}\left(\frac{1+[-\tan(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))\pm\sec(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]^2}{[-\tan(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))\pm\sec(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]}\right)^2. \end{cases}$$

$$\begin{cases} v_4(x,t) = \frac{\sqrt{3}\,\lambda}{\sqrt{n(r+2s)}\,\omega}\left(\frac{1+[-\cot(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))\pm\csc(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]^2}{[-\cot(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))\pm\csc(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]}\right), \\ u_4(x,t) = \frac{3\lambda}{2(r+2s)\,\omega}\left(\frac{1+[-\cot(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))\pm\csc(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]^2}{[-\cot(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))\pm\csc(2\sqrt{p}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]}\right)^2. \end{cases}$$

$$\begin{cases} v_5(x,t) = \frac{\sqrt{3}\,\lambda}{\sqrt{n(r+2s)}\,\omega}\left(\frac{2+[\tan(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))-\cot(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]^2}{[\tan(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))-\cot(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]}\right), \\ u_5(x,t) = \frac{3\lambda}{2(r+2s)\,\omega}\left(\frac{2+[\tan(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))-\cot(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]^2}{[\tan(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))-\cot(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta-\frac{\lambda}{\alpha}t^\alpha))]}\right)^2. \end{cases}$$

$$\begin{aligned} \bullet\text{Set } 2: \quad & A_0 = 0,\ A_1 = A_1,\ A_2 = A_2,\ B_0 = \frac{i\sqrt{np(r+2s)}\,\omega}{\sqrt{6}\,\lambda}A_1, \\ & B_1 = \frac{i\sqrt{np(r+2s)}\,\omega}{\sqrt{6}\,\lambda}A_2, p = \frac{\lambda}{2q\omega^3}. \end{aligned} \tag{4.14}$$

By using of the (4.12), (4.14) and cases (3.6)-(3.10) respectively, we get

$$\begin{cases} v_1(x,t) = -\frac{\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega}\tanh(\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_1(x,t) = \frac{3\lambda}{(r+2s)\,\omega}\tanh^2(\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)). \end{cases}$$

$$\begin{cases} v_2(x,t) = -\frac{\sqrt{6}\,\lambda}{\sqrt{n(r+2s)}\,\omega}\coth(\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_2(x,t) = \frac{3\lambda}{(r+2s)\,\omega}\coth^2(\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)). \end{cases}$$

$$\begin{cases} v_3(x,t) = \frac{\sqrt{6}\,\lambda}{\sqrt{n(r+2s)\,\omega}}[-\tanh(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm isech(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_3(x,t) = \frac{3\lambda}{(r+2s)\,\omega}[-\tanh(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm isech(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_4(x,t) = \frac{\sqrt{6}\,\lambda}{\sqrt{n(r+2s)\,\omega}}[-\coth(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm csch(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_4(x,t) = \frac{3\lambda}{(r+2s)\,\omega}[-\coth(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm csch(2\sqrt{-p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_5(x,t) = -\frac{\sqrt{6}\,\lambda}{2\sqrt{n(r+2s)\,\omega}}[\tanh(\frac{\sqrt{-p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) + \coth(\frac{\sqrt{-p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_5(x,t) = \frac{3\lambda}{4(r+2s)\,\omega}[\tanh(\frac{\sqrt{-p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) + \coth(\frac{\sqrt{-p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

By using of the (4.12), (4.14) and cases (3.11)-(3.15) respectively, we get

$$\begin{cases} v_6(x,t) = -\frac{i\sqrt{6}\,\lambda}{\sqrt{n(r+2s)\,\omega}}\tan(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_6(x,t) = -\frac{3\lambda}{(r+2s)\,\omega}\tan^2(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)). \end{cases}$$

$$\begin{cases} v_7(x,t) = -\frac{i\sqrt{6}\,\lambda}{\sqrt{n(r+2s)\,\omega}}\cot(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_7(x,t) = -\frac{3\lambda}{(r+2s)\,\omega}\cot^2(\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)). \end{cases}$$

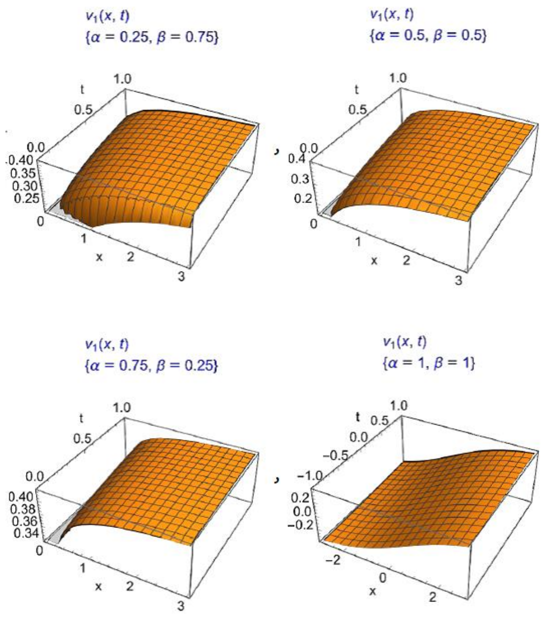
$$\begin{cases} v_8(x,t) = -\frac{i\sqrt{6}\,\lambda}{\sqrt{n(r+2s)\,\omega}}[-\tan(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm \sec(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_8(x,t) = -\frac{3\lambda}{(r+2s)\,\omega}[-\tan(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm \sec(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

$$\begin{cases} v_9(x,t) = -\frac{i\sqrt{6}\,\lambda}{\sqrt{n(r+2s)\,\omega}}[-\cot(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm \csc(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))], \\ u_9(x,t) = -\frac{3\lambda}{(r+2s)\,\omega}[-\cot(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) \pm \csc(2\sqrt{p}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))]^2. \end{cases}$$

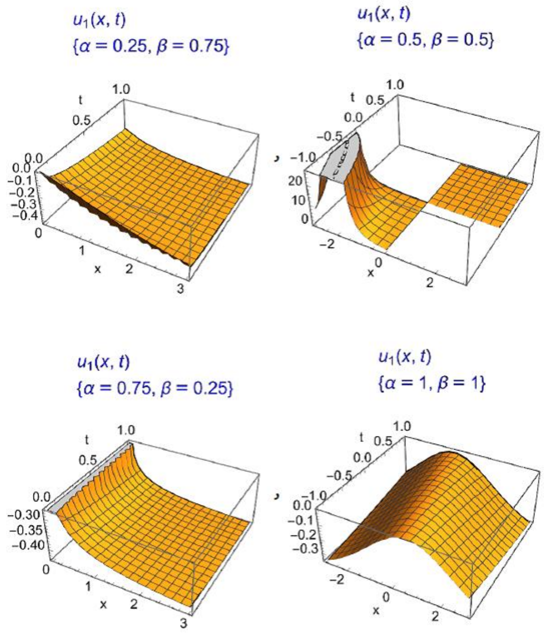
$$\begin{cases} v_{10}(x,t) = -\frac{i\sqrt{6}\,\lambda}{2\sqrt{n(r+2s)\,\omega}}\tan(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) - \cot(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)), \\ u_{10}(x,t) = -\frac{3\lambda}{4(r+2s)\,\omega}\tan(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha)) - \cot(\frac{\sqrt{p}}{2}(\frac{\omega}{\beta}x^\beta - \frac{\lambda}{\alpha}t^\alpha))^2. \end{cases}$$

5 Graphical Presentation of Specific Solutions

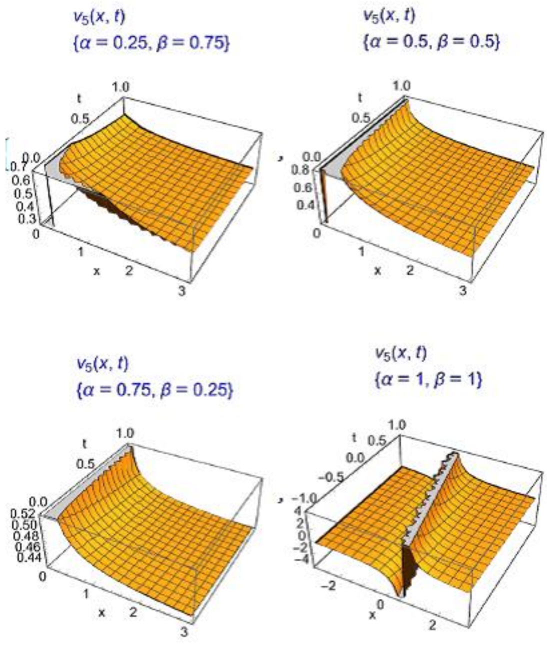
In this section, we provide a visual representation of certain solutions corresponding to equations (4.1) and (4.2), utilizing the relationships (4.10) and (4.11). The solutions are depicted in three-dimensional space, and are categorized based on the parameter p. Specifically, we present solutions for the cases where $p < 0$ ((3.6)-(3.10)), as well as those where $p > 0$ ((3.11)-(3.15)). These graphical representations aim to offer an intuitive insight into the behavior of the solutions under specific parameter conditions. The figures labeled from (1) and (2) correspond to various solution scenarios and illustrate the dynamic nature of the system.



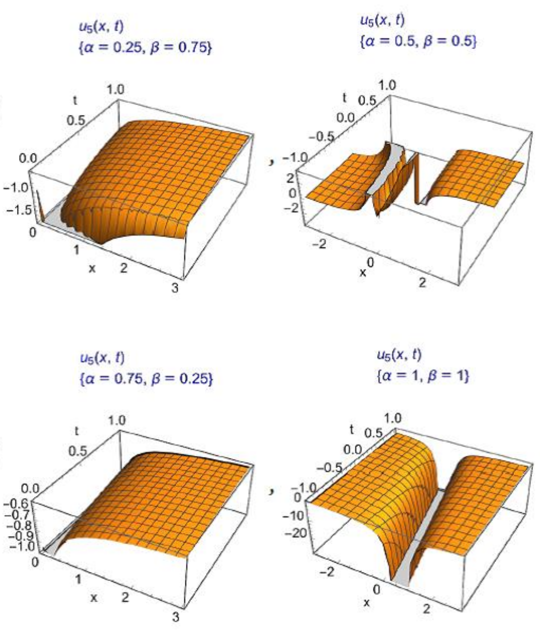
(a) $v_1(x,t)$



(b) $u_1(x,t)$

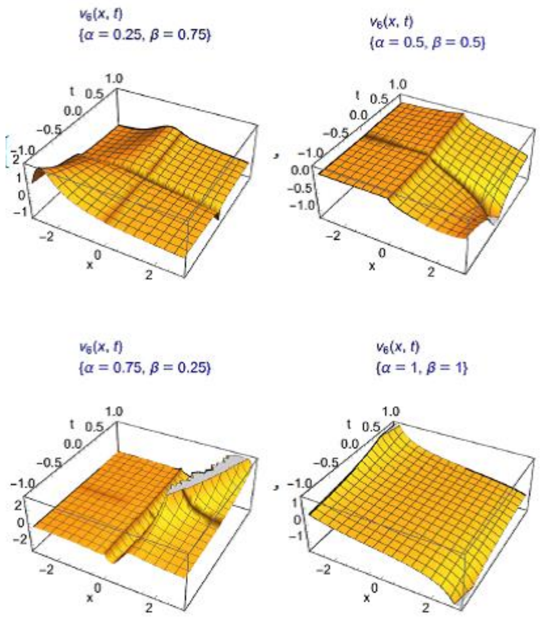


(c) $v_5(x,t)$

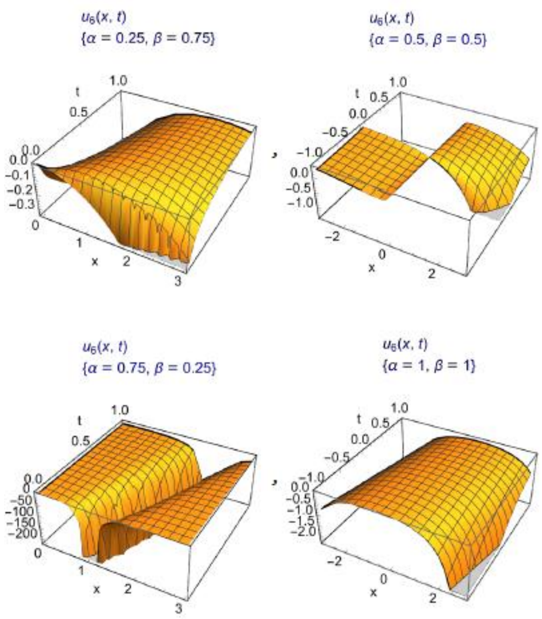


(d) $u_5(x,t)$

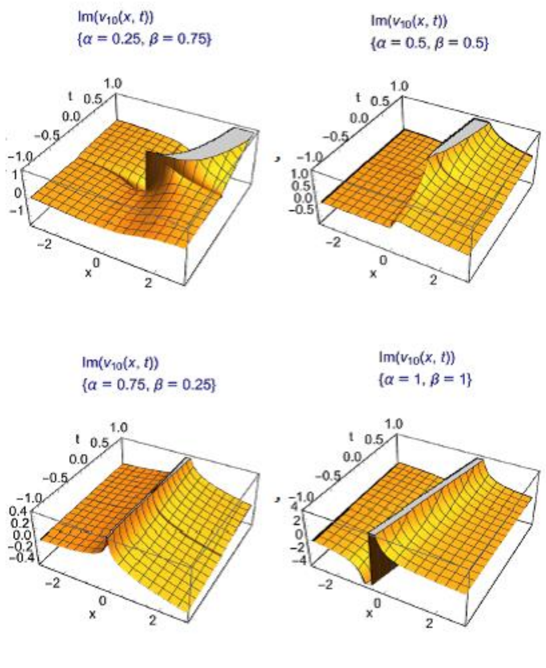
Figure 1: In cases $p = -0.6, n = 3, q = 2, r = 2, s = 1, \lambda = -0.3, \omega = 0.5$



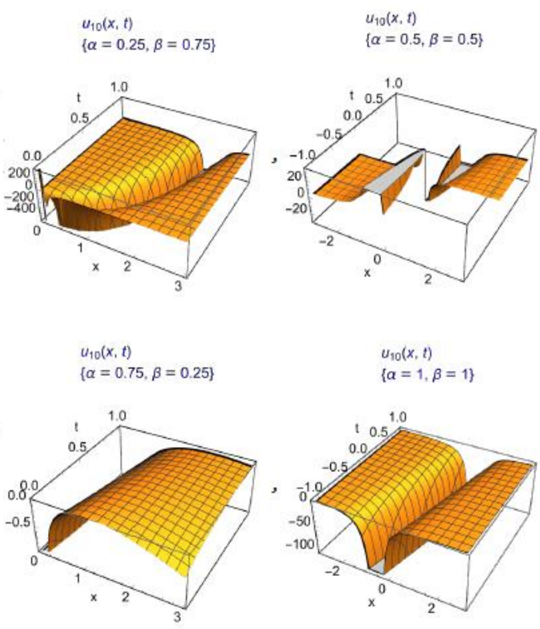
(a) $v_6(x, t)$



(b) $u_6(x, t)$



(c) $v_{10}(x, t)$



(d) $u_{10}(x, t)$

Figure 2: In cases $p = 0.6, n = 3, q = 2, r = 2, s = 1, \lambda = 0.3, \omega = 0.5$

6 Concluding Remarks

In this study, we have employed an extended algebraic method to investigate a specific class of fractional differential equations, namely, the Drinfeld-Sokolov-Wilson system. Our findings demonstrate the effectiveness of the applied method, indicating its potential applicability in solving a wide range of nonlinear evolution equations. It is noteworthy that computational and programming tasks were performed using *Mathematica* throughout this research endeavor.

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