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On the action of Toeplits operators into new BMOA type spaces in the unit disk

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Abstract

We provide new sharp results on the action of Toeplitz operators from Triebel and Besov spaces to new BMOA-type function spaces on the unit disk. In this paper, we consider $s \geq 1$ case in previous papers $s < 1$ was covered for $BMOA_{s,q}$ and $BMOA_s^p$ spaces. We modify a little our previously known proofs.

Keywords: Toeplitz operators, Besov type spaces, Lizorkin-Triebel type spaces, analytic functions
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1 Introduction

In this note we will extend our previously known sharp theorems on the action of Toeplits operators into BMOA type function spaces in the unit disk (we consider $s \geq 1$ here). More precisely we provide new sharp results on the action of Toeplits operators from mixed norm analytic function spaces into new BMOA type classes in the unit disk. For that reason we modify the previously known proof, provided earlier by first author in classical function spaces.

Proofs of our sharp results and proofs of [10] are based mainly on similar type ideas. We introduce new BMOA type spaces in the unit disk as follows.

$$BMOA_{s,q}(U) = \left\{ f \in H^s(U) : ||f||_{BMOA_{s,q}} == \sup_{z\in U} \left(\int_T \frac{|f(\xi) - f(z)|^s}{|1-\xi\bar{z}|^q} dm(\xi)(1-|z|^2) \right)^{1/s}, 0 < q < \infty, 1 \leq s < \infty \right\};$$

$$BMOA_s^p(U) = \left\{ f \in H^s(U) : ||f||_{BMOA_s^p} == \sup_{z\in U} \left(\int_T \frac{|f(\xi) - f(z)|^s}{|1-\xi\bar{z}|^2} dm(\xi)(1-|z|^2)^p \right)^{1/s}, 0 < p < \infty, 1 \leq s < \infty \right\}$$

see definitions of all objects below. It is easy to see that in particular values of parameters quazinorms of these analytic spaces in the unit disk coincide with the so -called Garsia norm in BMOA (see [2, 5]-[8]).

The intention of this short paper to show new sharp results on the action of T_φ Toeplitz operators in some new BMOA type spaces in the unit disk. We provide a necessary and sufficient condition. on the symbol of T_φ operator.

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Note such type results have various applications. Various results on BMOA type function spaces can be seen in papers [2, 5]-[8]. Various results on the action of T_φ Toeplitz operators can been seen in recent papers [1, 2, 5, 8] on various new and classical analytic function spaces in the unit disk. We refer to [2, 4, 5, 8] for some applications of such type results in analytic function spaces.

Let further $U = \{z \in C, |z| < 1\}$ or D be the unit disk on a complex plane C, T be the unit circle on C. Let also further $I = (0, 1)$. Let further H(U) be the space of all analytic functions in U .

In this paper, we as usual denote by $\mathcal{D}^\alpha$ for any real α the fractional derivative of analytic f function in the unit disk,

$$\mathcal{D}^\alpha f(z) = \sum_{k=0}^{\infty}(k+1)^\alpha a_k z^k, z \in U$$

for any analytic f function, $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k, \alpha > -1, \alpha \in \mathbb{R}$, see [4]. Note if $f \in H(U)$ then for any $s \in \mathbb{R}$, $D^s f \in H(U)$. We define Lusin cone in a usual manner as follows (see [5, 6]).

$$\Gamma_\alpha(\xi) = \{z \in U,\ |1 - z\xi| < \alpha(1 - |z|)\}, \text{where } \alpha > 1, \xi \in T.$$

We refer to [1], [5], [6] for further details on this object. The Hardy spaces, denoted by $H^p(U)$ $(0 < p \le \infty)$, are defined as usual (see [9]) by

$$H^p(U) = \left\{ f \in H(U)\colon \sup_{0<r<1} M_p(f, r) < \infty \right\},$$

where

$$M_p^p(f, r) = \int_T |f(r\xi)|^p \, dm_1(\xi), \ \ M_\infty(f, r) = \max_{\xi \in T} |f(r\xi)|, r \in (0, 1), f \in H(U).$$

For $\alpha > -1, 0 < p < \infty$, recall that the weighted Bergman space $A^p_\alpha(U)$ consists of all holomorphic functions on the unit disk satisfying the condition

$$\|f\|^p_{A^p_\alpha} = \int_U |f(z)|^p \,(1 - |z|^2)^\alpha dm_2(z) < \infty \ \ \text{(see [4, 6, 7, 8])}.$$

Let further $H(U)$ be the space of all analytic functions in U. Let further also (see [6, 7])

$$F^{p,q}_\alpha(U) = \left\{ f \in H(U) : \|f\|^p_{F^{p,q}_\alpha} = \int_T \left(\int_I |D^m f(r\xi)|^q (1-r)^{(m-\alpha)q-1} dr \right)^{\frac{p}{q}} d\xi < \infty \right\},$$

where $0 < p,\ q < \infty$, $m > \alpha, \alpha \in \mathbb{R}$, be the holomorphic Lizorkin-Triebel space, (see, for example, [6, 7]). Let

$$F^{p,q}_{\alpha,k}(U) = \left\{ f \in H(U) : \|D^k f\|_{F^{p,q}_\alpha} < \infty \right\}, 0 < p, q, \alpha < \infty, k \in \mathbb{N}.$$

Note that we can easily show $F^{p,q}_\alpha$ general mixed norm analytic function spaces in the unit disk are Banach spaces for all values of p and q, if $\min(p, q) > 1$ and they are complete metric spaces for all other values of p and q.

Note (see [1, 2, 6, 7]) for particular case $p = q$ we have Bergman classical class, for $q = 2$ we have so-called Hardy-Lizorkin space H^p_β for some β that is, $H^p_\beta = \{f \in H(U) : D^\beta f \in H^p\}$, $0 < p \le \infty$, $\beta > 0$, where D^β is a fractional derivative of analytic f function in U. Note (see definitions bellow) for this particular cases the action of $T\varphi$ classical Toeplitz operator is well-studied in unit disk, unit ball, unit polydisk and unit disk. We study T_φ operators in more general $F^{p,q}_\alpha$ type spaces in the unit disk. Our main sharp result provide some criteria for symbol of $T\varphi$ to obtain boundednes of T_φ in mentioned type analytic spaces.

Various sharp results on action of Teoplitz and other operators can be seen in papers of various authors in various functional spaces in the unit ball, polydisk and unit disk. We mention, for example, the following papers [4] and [8], where such type sharp results can be seen for various cases of $F^{p,q}_\alpha$ spaces namely in Bergman type and in Hardy type spaces in the unit ball, polydisk and in the unit disk. We also note similar type results in for particular values of parameters are well-known also in other domains (see, for example, [8]). Such type sharp result on boundedness of Toeplitz operators also have various applications (see, for example [4], [8]). We remind the reader the standard definition of Toeplitz T_h operators in the unit disk. Let $h \in L^1(T)$. Then we define Toeplitz T_h operator as an integral operator

$$(T_h f)(z) = \frac{1}{(2\pi)} \int_T \frac{f(\xi)h(\xi)}{(1-\bar{\xi}z)} dm(\xi),$$

$z \in U$. We stress that behavior of the operators in the unit polydisk is substantially different from the action of $T\varphi$ operators in the unit ball in $\mathbb{C}^n$ (see [4] for example). Our intention to set criteria for the action of Toeplitz T_φ operators from $F^{p,q}_{\alpha,k}(U)$ into BMOA type spaces in the unit disk, under the assumption that φ is holomorphic, $\varphi \in H(U)$ (with some restriction on symbol of Toeplitz operator). We define some new function spaces in the unit disk for formulation of our main result in the unit disk.

$$A^s_{\alpha,m}(U) = \left\{ f \in H(U) : \|f\|^s_{A_{\alpha,m}} = \int_U |(D^m f)(z)|^s (1-|z|)^{\alpha-1} dm_2(z) < \infty \right\},$$

$m \in \mathbb{N},\ 0 < s, \alpha < \infty$ (Bergman-Sobolev space). Let further

$$H^s_m(U) = \{ f \in H(U) : \|D^m f\|_{H^s} < \infty,\ m \in \mathbb{R},\ 0 < s < \infty \}$$

be analytic Hardy-Lizorkin space in the unit disk U.

We denote by $B(z,r)$ the Bergman ball in U (see [12], [13]). Note it can easily shown that these both scales of analytic function spaces in the unit disk are Banach spaces for all values of $s,\ s \geq 1$ and they are complete metric spaces for other values of $s,\ s > 0$. These known spaces are particular cases of larger mixed norm spaces $F^{p,q}_{\alpha,k}$ which we consider in this paper. Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of the functions or variables being discussed.

We pay special attention to places where different arguments from those we see in [10] are needed. In [10], we considered $s < 1$ case, here we show same theorems for $s \geq 1$.

2 Main Results

In this section we formulate our main results.

Theorem 2.1. Let $s \geq 1, q = 2s-1, \tau = 2(1-1/s); s \leq 2$. Then $(T_{\bar{\varphi}})$ operator is a bounded operator from $B^s_{2(1-1/s)}$;

$$||f||^s_{B^s_\tau} = \int_D |(D^k f)(z)|^s \cdot (1-|z|)^{sk+1-2s} dm_2(z)$$

to $BMOA_{s,q}(D)$ if and only if $\varphi \in H^\infty(D)$

Proof . If T_φ operator is bounded then $||T_\varphi f||_{BMOA_{s,q}} \leq C||f||_{B^s_\tau}, \tau \approx 2(1-1/s)$. We estimate $||T_\varphi f||$ from bellow and $||f||_{B^s_\tau}$ from above. First we show sufficient part. We have that following arguments from [10]

$$\lim_{R\to 1} \int_T \frac{|F(R\xi)-F(Rz)|^s (1-|w|)^2}{|1-\bar{w}\xi|^q} d\xi \leq C_1 ||\varphi||_{H^\infty} \int_T \frac{(1-|w|)}{|1-w\xi|^{q-s}} \left(\int_D \frac{|D^k f(z)|(1-|z|)^{k-1}}{|1-z\xi| \cdot |1-\bar{w}z|} dm_2(z) \right)^s,$$

where $F(R\xi) = (T_\varphi(f))(R\xi), R \in (0,1), R > R_0, R_0 \in (0,1)$. Then we have obviously

$$\left(\int_D \frac{|D^k f(z)|(1-|z|)^{k-1} dm_2(z)}{|1-z\bar{\xi}| \cdot |1-\bar{w}z|^s} \right)^s \leq C_2 \int_D \frac{|D^k f(z)|^s (1-|z|)^{(k-1)s} dm_2(z)}{|1-\bar{z}\xi|^s} \times (s \geq 1; s \leq 2) \times \left(\int_D \frac{dm_2(z)}{|1-\bar{w}z|^{s'}} \right)^{s/s'}$$

$$\leq C_3 \int_D \frac{|(D^k f)(z)|^s (1-|z|)^{(k-1)s} dm_2(z)}{|1-\bar{z}\xi|^s} \cdot (1-|w|)^{-s+2(s/s')}.$$

Hence using composition lemma 7 Fubinis theorem

$$I = \left(\lim_{r\to 1}\right)\left(\int_T |F(r\xi)-F(rw)|^s \cdot \frac{(1-|w|)^s \cdot dm_2(\xi)}{|1-\bar{w}\xi|^q}\right)^{1/s}$$

$$\leq C_4\left(\int_T \frac{(1-|w|)}{|1-\bar{w}\xi|^{q-s}} \cdot \int_D \frac{|(D^k f)(z)|^s(1-|z|)^{s(k-1)}}{|1-z\bar{\xi}|^s} \cdot (1-|w|)^{-s+2(s/s')} dm_2(z)dm(\xi)\right)^{1/s}$$

$$\leq C_5 \int_D |(D^k f)(z)|^s (1-|z|)^{s(k-2)+1} dm_2(z)$$

$$\leq C_6 \begin{cases} ||f||_{A^{p,q}_{\tilde{\tau}}} \\ ||f||_{F^{p,q}_{\tilde{\tau}}} \end{cases}$$

for some $\tilde{\tau}$ (see lemmas). Let us show the reverse in our Theorem. We simply repeat same arguments from [10] to get the following estimates, based on Lemmas 3.4, 3.5 and on estimates

$$||(f_r)(z)||^s_{B^s_\tau} \leq C_7(1-r)^{\frac{2-q-s}{s}+\gamma} \approx (1-r)^{\tilde{\tau}_0}, r<1, r>r_0.$$

where $(f_r)(z) = \frac{(1-r)^\gamma}{(1-rz)}$, $r \in (1/2, 1)$, $z \in D; \gamma > \gamma_0$, $\tau = 2(1-1/s)$, and $q = (2s-1)$. We also have (see [10])

$$||T_\varphi fr||_{BMOA_{q,s}} = |\varphi(r)|||f_r||_{BMOA_{q,s}} \frac{r}{2\pi}, ||f_r||_{BMOA_{q,s}} \geq C_8(1-r)^{\tilde{\tau}_0}$$

for some τ_0. Note also (see [10]) $||f_r||_{BMOA_{q,s}} \geq C_9(1-r)^{\tilde{\tau}_0}$. Note that from our arguments(proof of sufficiency) it is easy to see that Toeplits operator acts into Hardy space H^s. We omit here easy details. Theorem is completely proved now by same argument as in [10]. □

Theorem 2.1 can be reformed similarly also for more general cases of $A^{p,q}_\tau, F^{p,q}_\tau$ and Herz type spaces similarly (see remarks below). Consider now the case of $s \geq 1$ but for $BMOA^q_s$ spaces. Note

$$(BMOA^q_s)(D) = \left\{ f \in (H^s)(D) : ||f||_{BMOA^q_s} = \left(\sup_{z\in D}\right)\left(\int_T \frac{|f(\xi)-f(z)|^s dm(\xi)}{|1-\bar{\xi}z|^2} \cdot (1-|z|)^q\right)^{1/q} \right\};$$

$0 < q < \infty; 1 \leq s < \infty$. We have that following the proof above, arguments from [10], and Lemmas.

$$I(F) = \left(\lim_{r\to 1}\right)\left(\int_T |F(r\xi)-F(rw)|^s \frac{(1-|w|^2)^q}{|1-\bar{w}\xi|^2} dm(\xi)\right)^{1/s}$$

$$\leq C_{10}\left(\int_T\int_D |(D^k f)(z)|^s \cdot \frac{(1-|z|)^{s(k-1)}}{|1-z\bar{\xi}|^s} \cdot (1-|w|)^{-s+2(s/s')} \cdot \frac{(1-|w|)^q}{|1-\bar{w}\xi|^{2-s}} dm(\xi)dm_2(z)\right)^{1/s}$$

$$\leq C_{11} \int_D |D^k f(z)|^s (1-|z|)^{s(k-1)+1-s} dm_2(z).$$

where $1 \leq s \leq 2$, $q = 4-2s$ and $q \in [0,2]$. We show the reverse now. We have (see also arguments above)

$$||f_r(z)||_{(BMOA)^q_s(D)} \geq C_{12}(1-r)^{\gamma + \frac{q-1-s}{s}}; \gamma > \gamma_0,$$

and to show this we follow arguments of [10]

$$\left(||(f_r)||_{BMOA^q_s}\right) = (1-r)^\gamma \left(\left(\sup_{z\in D}\right)\int_T \left|\frac{1}{(1-rz)} - \frac{1}{(1-r\xi)}\right|^s \times \frac{(1-|z|)^q}{(1-\xi z)^2} d\xi\right)^{1/s}$$

$$= ((1-r)^\gamma)\left(\sup_{z\in D}\right)\int_T \frac{(dm(\xi))\cdot(1-|z|)^q}{|1-r\xi|^s|1-\xi z|^{2-s}|1-rz|^s}$$

$$\geq C_{13}(1-r)^{\frac{\gamma+q-s-1}{s}} = (1-r)^{\tau_0},$$

for $s \geq 1$. Using that $(\sup_{z\in D}) \geq (\sup_{z=r})$ and lemmas also

$$||f_r||_{B^s_{2(1-\frac{1}{s})}} \leq C_{14}(1-r)^{\tau_0}.$$

We finish the proof using same the arguments as in [10] or in previous Theorem 2.1.

Theorem 2.2. Let $(s) \geq 1, q = 4-2s, q \in [0,2]; s \leq 2.$. Then $(T_{\bar{\varphi}})$ is a bounded operator from $B^s_{2(1-1/s)}$ to $BMOA^q_s(D)$ if and only if $\varphi \in H^\infty(D)$

The formulation for mixed norm A^{pq}_α spaces is the following.

Theorem 2.3. Let $(s) \geq 1, q = 2s-1, \tau = 2(1-1/s); s \leq 2$. Then (T_φ) is a bounded operator from $(D^k A^{p,\tilde{q}}_\alpha)$ or $D^k F^{p\tilde{q}}_\alpha$ to $BMOA_{s,q}$ if and only if $\varphi \in H^\infty(D)$, for $(\max)(p,\tilde{q}) \leq s < \infty, p = s/q$, $\alpha = k - 1/s, k \geq 1/s$.

The proof of these general result follow from $p = \tilde{q}$ case and embedding in lemmas we leave this task to readers. We add some words for Herz spaces. Note that the following embeddings hold

$$\int_D |f(z)|^p(1-|z|)^\alpha dm_2(z) \leq C_1 \sum_{k\geq 0} \left(\int_{B(a_k,r)} |f(z)|^p(1-|z|)^\alpha dm_2(z) \right)^{q/p}$$

$p \in (0,\infty), q \leq p, \alpha > -1;$

$$\int_D |f(z)|^p(1-|z|)^\alpha dm_2(z) \leq C_2 \int_U \left(\int_{B(v,r)} |f(z)|^q(1-|z|)^{\alpha+2} dm_2(z) \right)^{q/p} dv;$$

$\alpha > -1, 0 < p, q < \infty$. these are quasinorms of Herz spaces. Since similar sharp result is valid for $s \geq 1$, $BMOA^s_q$, $A^{p,q}_\alpha$, $F^{p,q}_\alpha$ triple. This gives us a chance to easily extend our Theorems 2.1 and 2.2 also to Herz type function analytic spaces in the Unit disk U. Consider other spaces with following quasinorms

$$\left(\sup_{|z|<1}\right) \left|(D^k f)(z)\right| \cdot (1-|z|)^{k-s} < \infty \qquad (A)$$

or

$$\int_0^1 \left(M_\infty(D^k f, r)\right)^p \cdot (1-r)^{\tilde{k}-\alpha} dr < \infty; \qquad (B)$$

$0 < p < \infty, \alpha > -1, k > s, \tilde{k} > \alpha$, these are another analytic Besov spaces, or

$$\left(\sup_{r<1}\right) \left(\int_T |(D^k f)(r\xi)|^p d\xi \right)^{1/p} \cdot (1-r)^{k-s} < \infty; \qquad (C)$$

$k > s, 0 < p < \infty$. Denote them by X, We wish to find sharp conditions of φ, so that (T_φ) acting from X to $(BMOA_{s,q})(D)$ or $(BMOA^q_s)(D)$ as bounded operator. We can modify the proof we suggested above for these type of analytic Besov type spaces also.

We turn to other function spaces (A), (B), (C). We give partial answers in Theorem 2.4. For $s \leq 1$ case we arrived similarly (see above) and [10]..

$$||T_\varphi f||_{BMOA_{s,q}} \leq C_1 \int_T \left(\int_D \frac{|D^k f(z)|^s (1-|z|)^{ks+s-2}(1-|w|) dm_2(z)}{|1-z\bar{\xi}|^s |1-\bar{w}\xi|^s |1-\bar{w}\xi|^{q-s}} \right)^{1/s},$$

$w \in D; BMOA_{s,q}$ space. For $s \geq 1$ case we arrived to the estimate (see the above).

$$||T_\varphi f||_{BMOA_{s,q}} \leq C_2 \int\limits_T \left(\int\limits_D \frac{|D^k f(z)|^s (1-|z|)^{s(k-1)} (1-|w|)^{2s/s'-s+1} (dm(\xi))(dm_2(z))}{|1-\xi\bar{z}|^s |1-\bar{w}\xi|^s |1-\bar{w}\xi|^{q-s}} \right)^{1/s'}$$

for $BMOA_{s,q}$ space. Estimating further accurately we have the following chain of estimates for $s \geq 1$ case separately for $BMOA_{s,q}$ and $BMOA_s^q$ spaces. We have

$$\int\limits_T \int\limits_D \frac{|D^k f(z)|^s (1-|z|)^{s(k-1)} (1-|w|)^{2s/s'-s+1} dm_2(z) dm(\xi)}{|1-\bar{\xi}z|^s |1-\bar{w}\xi|^{q-s}} \leq C_3 \int\limits_0^1 \left(M_\infty (D^k f(z))^s \cdot (1-|z|) \right)^{s(k-1)+1-s} dr = M(f),$$

$(q-s) > 1; (q-s) - 1 = 2s/s' - s + 1, q = 2s, s \geq 1$. The reverse implication. Note that for our test function f_r

$$M(f)^{\frac{1}{s}} \leq C_4 (1-r)^\gamma \cdot (1-r)^{-(k+1)s+2-s+s(k-1)} = C_5 (1-r)^{\gamma + \frac{2-3s}{s}},$$

and

$$||f_r||_{BMOA_{s,q}} \geq (1-r)^{\gamma + \frac{2-3s}{s}}.$$

And following the proof of previous case we arrive at new sharp result for T_φ. A version of this theorem for $(BMOA_q^s)$ spaces can be also formulated.

Theorem 2.4. Let $s > 1, q = 2s, s \leq 2$. Then (T_φ) is a bounded operator from a Besov space with quazinorm

$$K(f) = \int\limits_0^1 \left(M_\infty (D^k f, |z|)^s \right) \cdot (1-|z|)^{s(k-1)+(1-s)} dr < \infty,$$

to $(BMOA_{s,q})(D)$ if and only if $\varphi \in H^\infty(D)$.

3 Lemmas

We collect some interesting facts and lemmas in this section. They are taken from [10] and important for this paper. We define analytic Besov type and Lizorkin-Triebel type spaces in the unit disk as follows.

$$A_s^{p,\tilde{q}} = \left\{ f \in H(U) : \int\limits_I \left(\int\limits_T |D^k f(r\xi)|^p d\xi \right)^{\tilde{q}/p} (1-r)^{\tilde{q}(k-s)-1} dr < \infty \right\},$$

$k > s, 0 < p, \tilde{q} < \infty, s \in \mathbb{R}$; and $F_s^{p,\tilde{q}}$ defined similarly based on definition of $F_{\alpha,k}^{p,q}$ spaces above changing the order of integration (see the above). Indeed similar results are valid for Herz spaces (not only $A_\alpha^{p,q}, F_\alpha^{p,q}$, spaces) For $0 < p, q < \infty, \alpha > -1$ we define analytic Herz spaces as follows. Let $B(z,r)$ be Bergman ball in U. These are spaces with quasinorms:

$$\int\limits_U \left(\int\limits_{B(z,r)} |D^k f(\tilde{z})|^p \cdot (1-|\tilde{z}|)^\alpha \cdot dm_2(\tilde{z}) \right)^{q/p} dm_2(z)$$

and

$$\sum_{k \geq 0} \left(\int\limits_{B(a_k,r)} |D^k f(\tilde{z})|^p \cdot (1-|\tilde{z}|)^\alpha dm_2(\tilde{z}) \right)^{q/p},$$

where $\{a_k\}$ is an $r-$lattice in U. Let

$$\tilde{A}_\alpha^{p,q} = \left\{ f \in H(U) : \int\limits_I \left(\int\limits_T |f(r\xi)|^p d\xi \right)^{q/p} (1-r)^{\alpha q - 1} dr < \infty \right\},$$

and

$$\tilde{F}_{\alpha}^{p,q} = \left\{ f \in H(U) : \int\limits_{T} \left(\int\limits_{I} |f(r\xi)|^q (1-r)^{\alpha q - 1} dr \right)^{p/q} d\xi < \infty \right\},$$

$0 < p, q < \infty, \alpha \in (0;\infty)$. We formulate now several lemmas which are needed for proofs of our main results.

Lemma 3.1. (see [8, 9] for mixed norm spaces) Let $0 < max(p,q) \leq \tilde{s} < \infty$. Then we have that

$$\left(\int\limits_{U} |f(z)|^{\tilde{s}} \cdot (1-|z|)^{\tilde{s}(\alpha + \frac{1}{p}) - 2} \cdot dm_2(z) \right)^{(1/\tilde{s})} \leq C_1 ||f||_{\tilde{F}_{\alpha}^{p,q}};$$

and

$$\int\limits_{U} |f(z)|^{\tilde{s}} \cdot (1-|z|)^{\tilde{s}(\alpha + \frac{1}{p}) - 2} \cdot dm_2(z) \leq C_2 ||f||^{\tilde{s}}_{\tilde{A}_{\alpha}^{p,q}}.$$

As corollaries of Lemma 3.1, we have Lemmas 3.2 and 3.3, for some positive constants C and C_1.

Lemma 3.2. (see [6, 8, 9]) Let $0 < \max(p, \tilde{q}) \leq \tilde{s} < \infty$. Then we have that

$$J = \left(\int\limits_{U} |(D^k f)(z)|^{\tilde{s}} \cdot (1-|z|)^{k\tilde{s} - q} \cdot dm_2(z) \right)^{1/\tilde{s}} \leq C_3 \left(\int\limits_{I} \left(\int\limits_{T} |(D^k f)(r\xi)|^p d\xi \right)^{\tilde{q}/p} (1-r)^{\alpha \tilde{q} - 1} dr \right)^{1/\tilde{q}};$$

$$\alpha = (\frac{1}{s})(k\tilde{s} - q); p = \frac{\tilde{s}}{2}; k > k_0, k_0 = \frac{q}{\tilde{s}};$$

and

$$J^p \leq C_4 \int\limits_{T} \left(\int\limits_{I} |(D^k f)(r\xi)|^{\tilde{q}} \cdot (1-r)^{\alpha \tilde{q} - 1}\ dr \right)^{p/\tilde{q}} d\xi.$$

Lemma 3.3. (see [6, 8, 9] Let $0 < \max(p, \tilde{q}) \leq \tilde{s} < \infty$. Then we have that

$$J = \left(\int\limits_{U} |D^k f(z)|^{\tilde{s}} \cdot (1-|z|)^{k\tilde{s} + q - 3} \cdot dm_2(z) \right)^{q/\tilde{s}} \leq C_5 ||D^k f||_{\tilde{A}_{\alpha}^{p,\tilde{q}}} = C_6 \left(\int\limits_{0}^{1} \left(\int\limits_{T} |(D^k f)(r\xi)|^p d\xi \right)^{\tilde{q}/p} (1-r)^{\alpha \tilde{q} - 1} dr \right)^{1/\tilde{q}}$$

where $p = \frac{\tilde{s}}{q}, \alpha = k - \frac{1}{2}, k > \frac{1}{\tilde{s}}$, and similarly for spaces $F_{\alpha}^{p,\tilde{q}}$.

Lemma 3.4. (see [6, 8, 9]) Let $(f_r)(z) = \frac{(1-r)^{\gamma}}{1-rz}, \gamma > \gamma_0, \gamma_0 = \gamma_0(p, \tilde{q}, s), r \in (1/2, 1)$, then we have that

$$||f_r(z)||_{A^{p,\tilde{q}}_{q/\tilde{s}}} \leq C_8 (1-r)^{\gamma + (2-q-s)/s}; \quad ||f_r(z)||_{F^{p,\tilde{q}}_{q/\tilde{s}}} \leq C_9 (1-r)^{\gamma + (2-q-s)/s}.$$

The proof is standard. It uses classical estimates of function theory in the unit disk.

Lemma 3.5. (see [8, 9]) Let $F \in H(U), p \leq 1, \beta \in (-1, \infty); w, w_1 \in U; 0 \leq q, q_1 < \infty$, then we have that

$$\left(\int\limits_{U} \frac{|F(z)|(1-|z|)^{\beta} dm_2(z)}{|1-zw_1|^{q_1} |1-zw|^q} \right)^p \leq C_{10} \int\limits_{U} \frac{|F(z)|^p (1-|z|)^{\beta p + 2p - 2} dm_2(z)}{|1-zw_1|^{q_1 p} |1-zw|^{qp}}, w, w_1 \in U.$$

Note in particular cases, if $q = 0$ or $q = 1$ this result is well-known. Note this result is also well-known in the ball, polydisk. And similar result is valid for $p > 1$ (see the above).

Lemma 3.6. (see [4, 6, 7, 8]) Let

$$(D^m f)(z) = \sum_{k\geq 0} \frac{\Gamma(m+k+1))a_k z^k}{\Gamma(m+1)\Gamma(k+1)},$$

$m \geq 0, f \in H(U), f(z) = \sum_{k\geq 0} a_k z^k, z \in U$. Then $D^m f \in H(U)$, if $f \in H(U), f(z) = \sum_{k\geq 0} a_k z^k$ and in addition, we have

$$\left(\frac{1}{2\pi}\right) \int_T f(rt)\overline{g(rt)}dm(t) = \left(\frac{2^{m-1}}{r^2(m+1)\pi}\right) \int_0^r \int_T (f(r\xi))\, \overline{D^m g(r\xi)}(1-r)^{m-1} r dr dm(\xi), m \geq 0, r \in (0,1).$$

Lemma 3.7. (see [7]) We have the following estimates

$$\int_U \frac{(1-|\xi|^2)^s dm_2(\xi)}{|1-\bar{\xi}z|^r |1-\bar{\xi}w|^v} \leq \frac{C_{11}}{|1-\bar{z}w|^{r+v-s-2}};$$

$z, w \in U, s > -1; r, v \geq 0, r+v-s > 2, r-s < 2, v-s < 2$

$$\int_U \frac{(1-|\xi|^2)^s dm_2(\xi)}{|1-\bar{\xi}z|^r |1-\bar{\xi}w|^v} \leq \frac{C_{12}}{(1-|z|^2)^{r-s-2}|1-\xi w|^v};$$

$z, w \in U, s > -1; r, v \geq 0, r+v-s > 2, v-s < 2 < r-s;$

$$\int_T \frac{d\xi}{|1-\bar{\xi}z|^s |1-\bar{\xi}w|^{q-s}} \leq \frac{C_{13}}{(|1-\bar{z}w|^{q-1}};$$

$z, w \in U, 1 < q < 1+s, 0 < s < 1; r, v \geq 0, r+v-s > 2, v-s < 2 < r-s;$

$$\int_T \frac{d\xi}{|1-\xi z|^s |1-\xi w|^{q-s}} \leq \frac{C_{14}(1-|w|)^{s+1-q}}{(|1-zw|^s};$$

$z, w \in U, 1+2s < q < \infty, 0 < s < 1.$

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