Int. J. Nonlinear Anal. Appl. In Press, 1–11
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.32933.4898

Novel concepts of connectivity in vague incidence graphs with application

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(Communicated by Reza Saadati)

Abstract

Vague sets (VSs) being the most advanced form of fuzzy sets has more capacity to analyze the network state more intelligently. It is proven that VS is more useful to solve many real life problems having uncertainties. Fuzzy graphs (FGs) are efficient mathematical models for analyzing many problems of daily life. One of the most widely used types of FG is vague incidence graph (VIG). VIGs play an important role in various fields such as computer science, pcychology, medicine, and political sciences and are used to find effective people in an organization or social institution. They can be used to describe the problems which cannot be handled through FGs and VGs. So, in this paper, specific ideas analogous to vague cut vertices and vague bridges in VGs, vague incidence cut-vertices, and vague incidence bridges are explored. The notion of vague incidence gain and vague incidence less for vague incidence paths and pairs of vertices is also initiated.

Keywords: Vague set, vague graph, vague incidence graph, cut-vertices, bridges
2020 MSC: 05C99, 03E72

1 Introduction

Fuzzy graphs models are beneficial mathematical tools for addressing the combinatorial problems in various fields involving algebra, computing, environmental science, and optimization. In 1965 [36], fuzzy set theory was first introduced by Zadeh. Fuzzy set theory is a highly influential mathematical tool for solving approximate reasoning related problems. Gua and Buehrer [13] presented and structured the vague set theory. The vague sets describe more possibilities than fuzzy sets. Rosenfeld [30] has defined the concept of a fuzzy graph. Brualdi and Massey [8] introduced the concept of an incidence graph. The fuzzification of the incidence graphs has been proposed by Dinesh [9, 10]. Mordeson et al. [14, 15, 16, 17] defined new concepts in fuzzy incidence graphs. Ramakrishna [23] recommended the vague graph notion and evaluated some of its features. Akram et al. [1, 3, 2] presented new definitions of FGs. Borzooei and Rashmanlou [6, 5, 7, 4] investigated different results on VG. Poulik et al. [11, 21, 22] studied randic index and connectivity concepts in bipolar fuzzy graphs. Samanta et al. [31, 32] defined fuzzy competition graphs and some properties of bipolar fuzzy graphs. Rashmanlou et al. [26, 29, 25, 28, 24] studied new concepts in VGs. Talebi et al. [33, 34, 35] investigated isomorphism on VGs and defined interval-valued intuitionistic fuzzy competition graph. Some properties of interval-valued intuitionistic fuzzy graphs have been given by Borzooei et al. [27].

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Received: January 2024 *Accepted:* March 2024

Connectivity in fuzzy graphs has many application to several fields. Connectivity arises in facility location problems, where the number of facilities (e.g., hospital, fire stations) is fixed and one attempts to minimize the distance that a person needs to travel to the closest facility. Fang et al. [12] presented the connectivity index and wiener index in fuzzy incidence graphs. Nazeer et al. [18, 19] given the notion of order, size, domination, and strong pair domination in fuzzy incidence graphs. Hence, in this paper, norel concepts of vague incidence graphs such as vague incidence gain path, vague incidence loss path, vague incidence balanced path, vague incidence optimal path, vague incidence walk, vague incidence bridges, and vague incidence cut-vertices are introduced.

2 Preliminaries

A FG is the form $y = (\tau, \nu)$ which is a pair of mapping $\tau : V \to [0, 1]$ and $\nu : V \times V \to [0, 1]$ which is defined as

$$\nu(p, q) \leq \tau(p) \wedge \tau(q), \quad \forall p, q \in \nu,$$

and ν is a symmetric fuzzy relation on τ and $\wedge$ denotes minimum.

Definition 2.1. [13] A VS z is a pair (t_z, f_z) on set V where t_z and f_z are taken as real valued functions defined on $V \to [0, 1]$, so that $t_z(p) + f_z(p) \leq 1$, for all $p \in V$.

Definition 2.2. [23] $G = (z, w)$ is named a VG on G^*, which $z = (t_z, f_z)$ is a VS on $E \subseteq V \times V$, so that $t_w(pq) \leq \min\Big(t_z(p), t_z(q)\Big)$ and $f_w(pq) \geq \max\Big(f_z(p), f_z(q)\Big)$, $\forall pq \in E$.

Definition 2.3. [9] Let $G = (V, E)$ be a graph then, $G^* = (V, E, I)$ is called an incidence graph, so that $I \subseteq V \times E$. If $V = \{p, q\}$, $E = \{pq\}$ and $I = \{(p, pq)\}$, then (V, E, I) is an incidence graph even though $(q, pq) \notin I$. If $(p, pq), (q, pq), (q, qz), (z, qz) \in I$ then pq or qz are called adjacent edges.

Definition 2.4. [10] Let $G^* = (V, E, I)$ be an incidence graph and σ be a fuzzy subset of V and μ, a fuzzy subset of E.
Let ψ be a fuzzy subset of I. If $\psi(v, pq) \leq \sigma(\nu) \wedge \mu(pq)$, for all $v \in V$ and $pq \in E$, then ψ is called a fuzzy incidence of graph G^* and $G = (\sigma, \mu, \psi)$ is called a fuzzy incidence graph (FIG) of G^*.

3 Vague incidence graph

Definition 3.1. $G = (M, N, P)$ is called a VIG of underlying crisp graph $G^* = (V, E, I)$, if

$$\begin{aligned} M =& \{\langle t_M(v), f_M(v)\rangle \mid v \in V\}, \\ N =& \{\langle t_N(pq), f_N(pq)\rangle \mid pq \in E\}, \\ P =& \{\langle t_P(v, pq), f_P(v, pq)\rangle \mid (v, pq) \in I\}, \end{aligned}$$

such that

$$\begin{aligned} t_N(pq) &\leq t_M(p) \wedge t_M(q), f_N(pq) \geq f_M(p) \vee f_M(q), \\ t_P(v, pq) &\leq t_M(v) \wedge t_N(pq), \\ f_P(v, pq) &\geq f_M(v) \vee f_N(pq), \quad \forall v \in V, pq \in E, \end{aligned}$$

and

$$0 \leq t_M(v) + f_M(v) \leq 1, \quad 0 \leq t_N(pq) + f_N(pq) \leq 1, \quad 0 \leq t_p(v, pq) + f_p(v, pq) \leq 1.$$

Example 3.2. Consider an incidence graph $G^* = (V, E, I)$ so that $V = \{p, q, r, s\}$, $E = \{pq, qr, qs, rs, ps\}$ and $I = \{(p, pq), (q, pq), (q, qr), (r, qr), (q, qs), (s, qs), (r, rs), (s, rs), (s, ps), (p, ps)\}$ as shown in Figure 1. It is easy to show that $G = (M, N, P)$ is a VIG of G^*, as shown in Figure 2, where:

$$\begin{aligned} M =& \left\{\frac{p}{(0.2, 0.4)}, \frac{q}{(0.3, 0.5)}, \frac{r}{(0.4, 0.6)}, \frac{s}{(0.5, 0.5)}\right\}, \\ N =& \left\{\frac{pq}{(0.2, 0.6)}, \frac{qr}{(0.3, 0.6)}, \frac{qs}{(0.2, 0.7)}, \frac{rs}{(0.4, 0.6)}, \frac{ps}{(0.2, 0.8)}\right\}, \\ P =& \left\{\frac{(p, pq)}{(0.2, 0.6)}, \frac{(q, qp)}{(0.2, 0.7)}, \frac{(q, qr)}{(0.2, 0.6)}, \frac{(r, qr)}{(0.3, 0.7)}, \frac{(q, qs)}{(0.2, 0.8)}, \frac{(s, qs)}{(0.2, 0.7)}, \frac{(r, rs)}{(0.3, 0.7)}, \frac{(s, rs)}{(0.4, 0.6)}, \frac{(s, ps)}{(0.2, 0.8)}, \frac{(p, ps)}{(0.2, 0.8)}\right\}. \end{aligned}$$

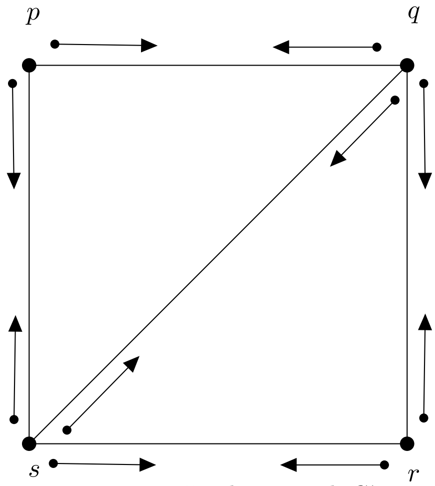

Figure 1: Incidence graph G^*

Definition 3.3. The support of a VIG $G = (M, N, P)$ is defined as $G^* = (M^*, N^*, p^*)$, where:

$$M^* = \text{Support of } \ M = \{v \in V : t_M(v) > 0, f_M(v) > 0\}$$
$$N^* = \text{Support of } \ N = \{vw \in E : t_N(vw) > 0, f_N(vw) > 0\}$$
$$P^* = \text{Support of } \ P = \{(v, vw) \in I : t_P(v, vw) > 0, f_p(v, vw) > 0\}.$$

Definition 3.4. A partial vague subgraph is called a partial vague incidence subgraph (PVIS) $H = (M', N', P')$ of a VIG $G = (M, N, P)$ if $t_{P'}(v_i, v_iw_j) \le t_P(v_i, v_iw_j)$ and $f_{P'}(v_i, v_iw_j) \ge f_P(v_i, v_iw_j)$, for all $(v_i, v_iw_j) \in P^*$.

Definition 3.5. A vague subgraph is called a vague incidence subgraph (VIS) $H = (M', N', P')$ of a VIG $G = (M, N, P)$ if $t_{P'}(v_i, v_iw_j) = t_P(v_i, v_iw_j)$ and $f_{P'}(v_i, v_iw_j) = f_P(v_i, v_iw_j)$, for all (v_i, v_iw_j) is the set of incidence pair of H.

Example 3.6. Figure 3 and Figure 4 are examples of PVIS and VIS of VIG given in Figure 2.

Definition 3.7. A strong VG, G is called strong VIG if $t_P(v_i, v_iw_j) = \wedge\{t_M(v_i), t_N(v_iw_j)\}$ and $f_P(v_i, v_iw_j) = \vee\{f_M(v_i), f_N(v_iw_j)\}$, for each $t_P(v_i, v_iw_j)$ and $f_P(v_i, v_iw_j)$ in P^*.

Definition 3.8. A complete VG, G is called complete VIG if $t_P(v_i, v_iw_j) = \wedge\{t_M(v_i), t_N(v_iw_j)\}$ and $f_P(v_i, v_iw_j) = \vee\{f_M(v_i), f_N(v_iw_j)\}$, for each $v_i, v_j \in M^*$.

Definition 3.9. If $vw \in N^*$ then vw is called an edge of the VIG $G = (M, N, P)$ and if $(v, vw), (w, vw) \in P^*$ then (v, vw) and (w, vw) are incidence paths (IPs) of $G = (M, N, P)$.

Definition 3.10. An vague incidence path

$$P : q_0, (q_0, q_0q_1), q_0q_1, (q_1, q_0q_1), q_1, (q_1, q_1q_2), q_1q_2, (q_2, q_1q_2), q_2, \cdots, q_{n-1}, (q_{n-1}, q_{n-1}q_n), q_{n1-}q_n, (q_n, q_{n-1}q_n), q_n$$

is a sequence of different nodes so that either one of the given below condition is satisfied:

(i) $t_P(v_i, v_iw_j) > 0$ and $f_P(v_i, v_iw_j) > 0$, for some i, j,
(ii) $t_P(v_i, v_iw_j) > 0$ and $f_P(v_i, v_iw_j) = 0$, for some i, j,
(iii) $t_P(v_i, v_iw_j) = 0$ and $f_P(v_i, v_iw_j) > 0$, for some i, j.

Definition 3.11. A sequence

$$P : q_0, (q_0, q_0q_1), q_0q_1, (q_1, q_0q_1), q_1, (q_1, q_1q_2), q_1q_2, (q_2, q_1q_2), q_2, \cdots, q_{n-1}, (q_{n-1}, q_{n-1}q_n), q_{n1-}q_n, (q_n, q_{n-1}q_n), q_n$$

in G is called vague incidence walk (VIW). If $q_0 = q_n$ then a VIW is closed.

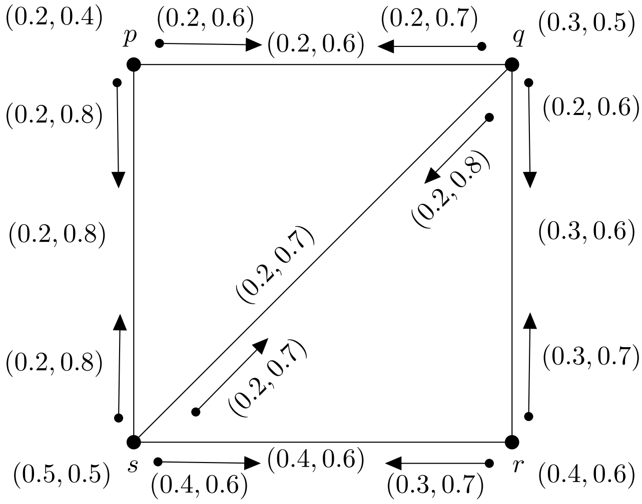

Figure 2: VIG G

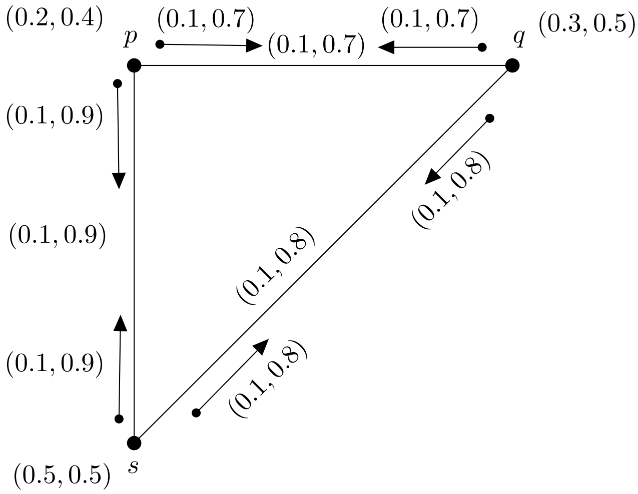

Figure 3: PVIS of VIG provided in Figure 2

Example 3.12. Let $G = (M, N, P)$ be a VIG provided in Figure 5 where

$$\begin{aligned} M =&\{(p, 0.2, 0.4), (q, 0.5, 0.2), (r, 0.2, 0.2), (s, 0.1, 0.4)\}, \\ N =&\{(pq, 0.2, 0.6), (pr, 0.2, 0.5), (ps, 0.1, 0.7), (qr, 0.2, 0.3), (rs, 0.2, 0.8)\}, \\ P =&\Big\{ \Big((p, pq), 0.2, 0.8\Big), \Big((q, pq), 0.2, 0.6\Big), \Big((p, pr), 0.1, 0.5\Big), \Big((r, pr), 0.2, 0.6\Big), \Big((p, ps), 0.1, 0.7\Big), \\ &\Big((s, ps), 0.1, 0.8\Big), \Big((q, qr), 0.1, 0.4\Big), \Big((r, qr), 0.2, 0.4\Big), \Big((r, rs), 0.1, 0.8\Big), \Big((s, rs), 0.1, 0.8\Big) \Big\}. \end{aligned}$$

A VIW $P_1 : p, (p, pr), pr(r, pr), r(r, rs), rs, (s, rs), s, (s, sp), sp(p, sp), p$ in Figure 5 is closed because its begining and final node is similar but it is not a VIP because all nodes are not different.

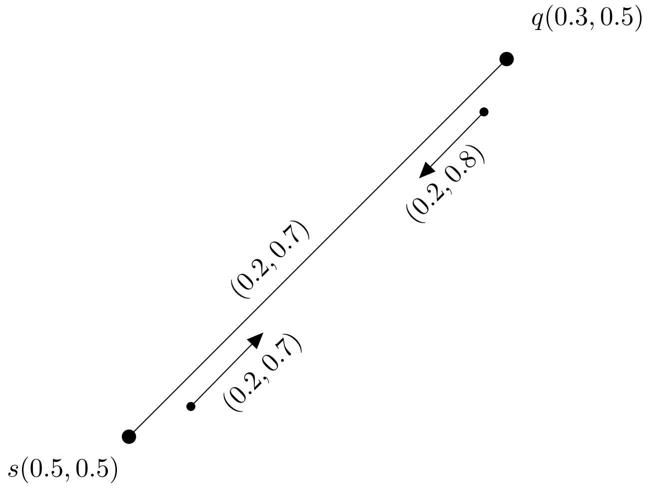

Figure 4: VIS of VIG given in Figure 2

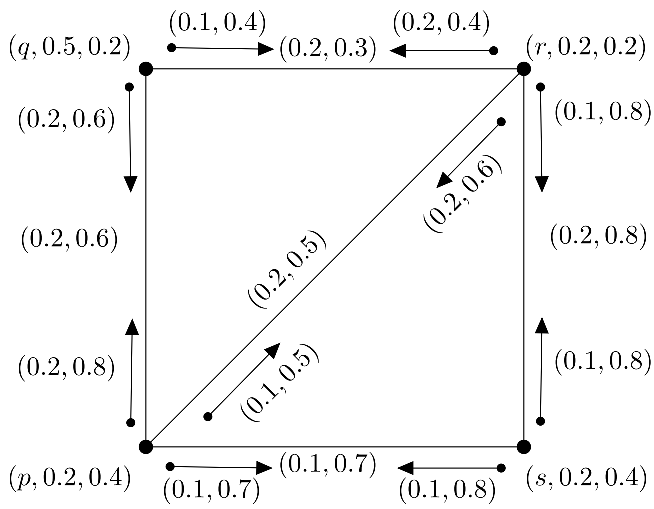

Figure 5: A VIG G

Definition 3.13. Let $G = (M, N, P)$ be a VIG for any q-r VIP

$$P : q = q_1, (q_1, q_1q_2), q_1q_2, (q_2, q_1q_2), q_2, \cdots, q_{n-1}, (q_{n-1}, q_{n-1}q_n), q_{n1-}q_n, (q_n, q_{n-1}q_n), q_n = r$$

in G, we define

$$\wedge\Big\{t_p(q_1, q_1q_2), t_p(q_2, q_1q_2), t_p(q_2, q_2q_3), t_p(q_3, q_2q_3), \cdots, t_p(q_{n-1}, q_{n-1}q_n), t_p(q_n, q_{n-1}q_n)\Big\}$$

as the vague incidence gain (VIG) of P. VIG of P is expressed by $G_{VI}(p)$ and

$$\vee\Big\{f_p(q_1, q_1q_2), f_p(q_2, q_1q_2), f_p(q_2, q_2q_3), f_p(q_3, q_2q_3), \cdots, f_p(q_{n-1}, q_{n-1}q_n), f_p(q_n, q_{n-1}q_n)\Big\}$$

as the vague incidence loss (VIL) of P. VIL of P is denoted by $L_{VI}(p)$.

Example 3.14. In Figure 5, for a VIP $P = psr$, $G_{VI}(p) = \wedge(0.1, 0.1, 0.1, 0.1) = 0.1$ and $L_{VI}(p) = \vee(0.7, 0.8, 0.8, 0.8) = 0.8$.

Definition 3.15. A VIP, P is called a vague incidence gain path (VIGP) if $G_{VI}(p) > L_{VI}(p)$ and vague incidence loss path (VILP) if $G_{VI}(p) < L_{VI}(p)$.

Example 3.16. In Figure 5, $P : psr$ is a VILP because $G_{VI}(p) = 0.1 < L_{VI}(p) = 0.8$.

Definition 3.17. A VIG is named as connected if there exists a VIP between each pair of nodes.

Definition 3.18. Let r and s be nodes in a connected VIG G. Among all r-s VIP in G, a VIP whose VIG is greater than or equal to of any other r-s VIP in G is called a maximum r-s VIGP. It is shown by $\max(r - s)_{VIGP}$. Similarly, a r-s VIP whose VIL is smaller than or equal to that of any other r-s VIP in G is known as minimum r-s VILP. It is expressed by $\min(r - s)_{VILP}$. That is a VIP, P is $\max(r - s)_{VIGP}$ if $G_{VI}(p) > G_{VI}(p^\diamondsuit)$ and is a $\min(r - s)_{VILP}$ if $L_{VI}(p) \leq L_{VI}(p^\diamondsuit)$, where $p^\diamondsuit$ is any r-s VIP in G.

Example 3.19. Table 1 shows the vague incidence max gain and vague incidence min loos between each pair of nodes of a VIG given in Figure 5.

Table 1: Name of employees in a hospital and their services.

Vertices	**Max-VIG**	**Max-VIGP**	**Min-VIL**	**Min-VILP**
p-q	0.2	pq	0.6	prq
p-r	0.1	Any path	0.6	pr
p-s	0.1	Any path	0.8	Any path
q-r	0.1	Any path	0.4	qr
q-s	0.1	Any path	0.8	Any path
r-s	0.1	Any path	0.8	Any path

Definition 3.20. In a VIG, a r-s VIP, P is called a vague incidence balanced path (VIBP) if $G_{VI}(p) = L_{VI}(p)$ and p is called vague incidence optimal path if P is a $\max(r - s)_{VIGP}$ and $\min(r - s)_{VILP}$.

Example 3.21. Figure 5 does not have any VIBP because for each VIP $G_{VI}(p) \neq L_{VI}(p)$ but $r-s : r, (r, rs), rs, (s, rs), s$ is a VIOP.

Definition 3.22. Assume $G = (M, N, P)$ is a VIG and let r and s be any two nodes of G. The VIG of r and s is defined as VIG of $\max(r-s)_{VIGP}$. It is presented by $G_{VI}(r, s)$. In the same way, the VIL of r and s is defined as VIL of $\min(r - s)_{VILP}$. It is denoted by $L_{VI}(r, s)$.

Example 3.23. In Figure 5, $G_{VI}(p, s) = 0.1$, $L_{VI}(p, s) = 0.8$, $G_{VI}(q, r) = 0.1$, $L_{VI}(q, r) = 0.4$.

Definition 3.24. Let H be a VIS of G, r and s be any two nodes of H, then a VIG of r and s in H is the VIG of $\max(r - s)_{VIGP}$ strictly belongs to H and it is shown by $G_{VIH}(r, s)$. In a similar way, VIL of r and s in H is the VIL of $\min(r - s)_{VILP}$ strictly belongs to H and is denoted by $L_{VIH}(r, s)$. If there dose not exist $\max(r - s)_{VIGP}$ and $\min(r - s)_{VILP}$ in H then $G_{VIH}(r, s) = 0$ and $L_{VIH}(r, s) = 0$.

Example 3.25. Let $G = (M, N, P)$ be a VIG provided in Figure 6 which is a VIS of VIG shown in Figure5 which,

$$M =\Big\{(p, 0.2, 0.4), (q, 0.5, 0.2), (r, 0.2, 0.2), (s, 0.2, 0.4)\Big\}$$
$$N =\Big\{(pq, 0.2, 0.6), (pr, 0.2, 0.5)\Big\}$$
$$P =\Big\{\Big((p, pq), 0.2, 0.8\Big), \Big((q, pq), 0.2, 0.6\Big), \Big((p, pr), 0.1, 0.5\Big), \Big((r, pr), 0.2, 0.6\Big)\Big\}.$$

It can be seen that $G_{VIH}(r, s) = 0$ and $L_{VIH}(r, s) = 0$

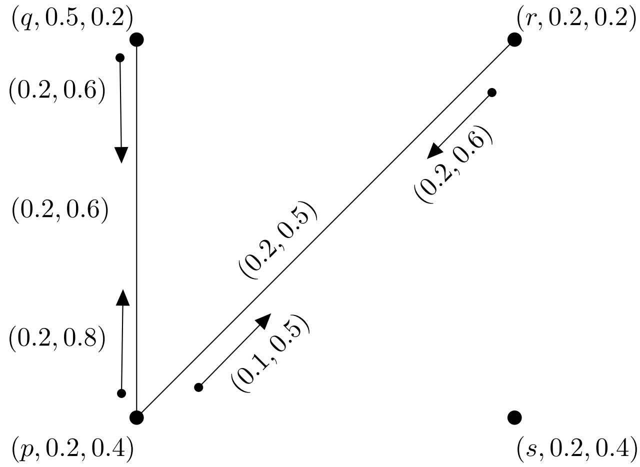

Figure 6: VIS of Figure 5

Proposition 3.26. If H is a VIS of a VIG $G = (M, N, P)$ then, $G_{VIH}(r, s) \leq G_{VI}(r, s)$ and $L_{VIH}(r, s) \leq L_{VI}(r, s)$, for each pairs of nodes r and s.

Proof . Let H be a VIS of G having same number of nodes, edges and VPs with equal membership degree and non-membership degree of nodes, edges and VPs. This implies that $G_{VIH}(r, s) = G_{VIG}(r, s)$ and $L_{VIH}(r, s) = L_{VIG}(r, s)$, for each pair of nodes r and s in H. Now, if H has less nuber of nodes, edges and VPs then membership degree and non-membership degree of nodes, edges, and VPs will less this implies $G_{VIH}(r, s) < G_{VIG}(r, s)$ and $L_{VIH}(r, s) < L_{VIG}(r, s)$ so, $G_{VIH}(r, s) \leq G_{VIG}(r, s)$ and $L_{VIH}(r, s) \leq L_{VIG}(r, s)$. $\square$

Definition 3.27. Let $G = (M, N, P)$ be a VIG with n nodes, $\{r_1, r_2, \cdots, r_n\}$. The vague incidence gain loss matrix (VIGLM) is defined as $M = [G_{VI_{i,j}}, L_{VI_{i,j}}]$ which $G_{VI_{i,j}} = G_{VI}(r_i, r_j)$ and $L_{VI_{i,j}} = L_{VI}(r_i, r_j)$, for $i \neq j$ and $G_{VI_{i,j}} = t_M(r_i)$, $L_{VI_{i,j}} = f_M(r_i)$, for $i = j$.

Example 3.28. VIGLM of G shown in Figure5 is given as below.

$$\begin{bmatrix} (0.2, 0.4) & (0.2, 0.6) & (0.1, 0.6) & (0.1, 0.8) \\ (0.2, 0.6) & (0.5, 0.2) & (0.1, 0.4) & (0.1, 0.8) \\ (0.1, 0.6) & (0.1, 0.4) & (0.2, 0.2) & (0.1, 0.8) \\ (0.1, 0.8) & (0.1, 0.8) & (0.1, 0.8) & (0.2, 0.4) \end{bmatrix}$$

It is clear from the matrix that VIGLM of a VIG is a symmetric matrix.

Definition 3.29. Suppose that $G = (M, N, P)$ is a VIG. A node $q \in M^*$ is called as a vague incidence cut-node (C_{VI}) if there exists two nodes $r, s \in M^*$ of the type $q \neq r \neq s$ so that $G_{VI_{(G-q)}}(r, s) < G_{VI_G}(r, s)$ and $L_{VI_{(G-q)}}(r, s) > L_{VI_{(G-q)}}(r, s)$. A node q is called a vague incidence gain cut-node (C_{VIL}) if $L_{VI_{(G-q)}}(r, s) < G_{VI_G}(r, s)$ and is called a vague incidence loss cut-node (C_{VIL}) if $L_{VI_{(G-q)}}(r, s) > L_{VI_G}(r, s)$.

Example 3.30. In Figure7 , p is a C_{VIL} because $L_{VI_{(G-q)}}(q, r) = 0.9 > L_{VI_G}(q, r) = 0.84$ and r is a C_{VIG} because $G_{VI_{(G-r)}}(q, s) = 0 < G_{VI_G}(q, s) = 0.01$.

Theorem 3.31. A node q in a VIG $G = (M, N, P)$ is a C_{VI} if and only if q is a node in each $\max(r, s)_{VIGP}$ and in each $\min(r, s)_{VILP}$, for some r and s in M^*.

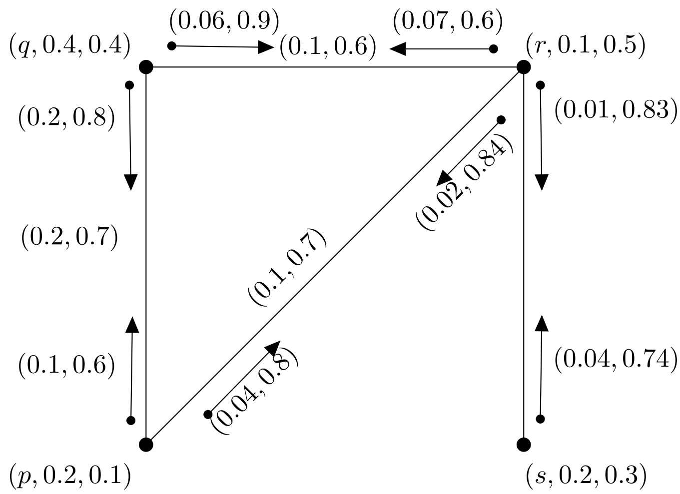

Figure 7: VIG having

Proof . Let $G = (M, N, P)$ be a VIG. Suppose that q is a C_{VI}. According to the definition of C_{VI}, there exists nodes r and s in G so that $q \neq r \neq s$ and $(i) : G_{VI_{(G-q)}}(r, s) < G_{VI_G}(r, s)$ and $(ii) : L_{VI_{(G-q)}}(r, s) > L_{VI_G}(r, s)$. From (i) it is clear that the deleting of q from G deletes all $\max(r, s)_{VIGP}$ and from (ii) it can be seen that the deletion of q deletes each $\min(r, s)_{VILP}$. So, q is in each $\max(r, s)_{VIGP}$ and $\min(r, s)_{VILP}$.

Conversely, suppose that q is in each $\max(r, s)_{VIGP}$ and in each $\min(r, s)_{VILP}$. Then, the deletion of q from G results in the deletion of all $\max(r, s)_{VIGPs}$ and $\min(r, s)_{VILPs}$. This shows the G_{VI} will lessen and L_{VI} will enhance between r and s. So, $G_{VI_{(G-q)}}(r, s) < G_{VI_G}(r, s)$ and $L_{VI_{(G-q)}}(r, s) > L_{VI_G}(r, s)$. Therefor, q is a C_{VI}. □

Definition 3.32. Let $G = (M, N, P)$ be a VIG. An edge $e = rs$ in G is called a vague incidence bridge (B_{VI}) if $G_{VI_{(G-e)}}(p, q) < G_{VI_G}(p, q)$ and $L_{VI_{(G-e)}}(p, q) > L_{VI_G}(p, q)$, for some $p, q \in M^*$.

Example 3.33. Let $G = (M, N, P)$ be a VIG shown in Figure8 which,

$$
\begin{aligned}
M =& \Big\{(p, 0.2, 0.5), (q, 0.3, 0.3), (r, 0.1, 0.6), (s, 0.4, 0.1)\Big\},\\
N =& \Big\{(pq, 0.04, 0.73), (pr, 0.07, 0.63), (qr, 0.1, 0.7), (qs, 0.2, 0.4), (rs, 0.1, 0.8)\Big\},\\
P =& \Big\{\Big((p, pq), 0.03, 0.79\Big), \Big((q, pq), 0.02, 0.76\Big), \Big((p, pr), 0.03, 0.65\Big), \Big((r, pr), 0.04, 0.67\Big), \Big((q, qr), 0.1, 0.83\Big),\\
& \Big((r, qr), 0.1, 0.7\Big), \Big((q, qs), 0.1, 0.4\Big), \Big((s, qs), 0.07, 0.5\Big), \Big((r, rs), 0.06, 0.87\Big), \Big((s, rs), 0.04, 0.84\Big)\Big\}.
\end{aligned}
$$

In Figure 8, edges qr and qs are B_{VI}.

Theorem 3.34. An edge $e \in N^*$ of a VIG $G = (M, N, P)$ is a B_{VI} if and only if it is in each $\max(r, s)_{VIGP}$ and in each $\min(r, s)_{VILP}$ for some r and $s \in M^*$.

Proof . Let $G = (M, N, P)$ be a VIG. Suppose that $e = rs$ is a B_{VI}. So, $\exists$ nodes q and z in G so that $(i) : G_{VI_{G-e}}(q, z) < G_{VI_G}(q, z)$, and $(ii) : L_{VI_{G-e}}(q, z) > L_{VI_G}(q, z)$. From (i) it can be observe that the deleting of e from G deletes each $\max(q, z)_{VILP}$ and from (ii) it can be seen that the deletion of e deletes each $\min(q, z)_{VILP}$. So, e is in each $\max(q, z)_{VIGP}$ and $\min(q, z)_{VILP}$.

Conversely, suppose that e is in each $\max(q, z)_{VIGP}$ and in each $\min(q, z)_{VILP}$. Then, the deletion of e from G results in the deletion of each $\max(q, z)_{VIGPs}$ and $\min(q, z)_{VILPs}$. It shows that G_{VI} will lessen and L_{VI} will enhance between q and z. Therefore, $G_{VI_{G-e}}(q, z) < G_{VI_G}(q, z)$ and $L_{VI_{G-e}}(q, z) > L_{VI_G}(q, z)$. This proves that $e = rs$ is a B_{VI}. □

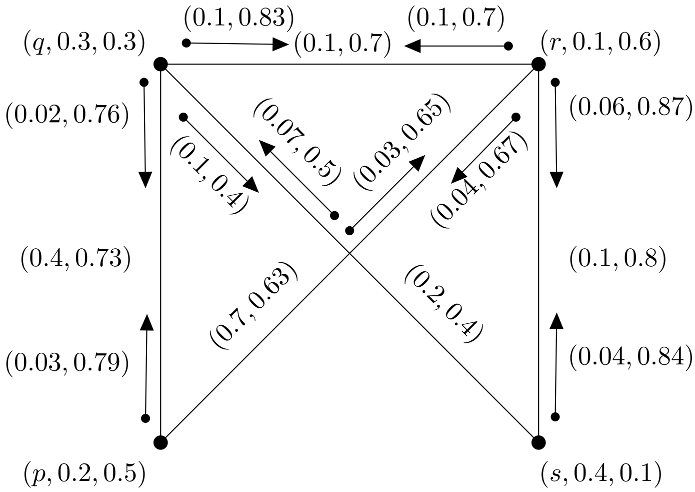

Figure 8: VIG having B_{VI}

4 Conclusion

Vague incidence graph, belonging to the FG family has good capabilities when facing with problems that cannot be expressed by FGs. It has various applications in modern science and technology, especially in the fields of computer science, decision making, and operation research. So, VIG and VISG are defined and their properties are investigated by several examples. Also, specific ideas analogous to vague cut vertices and vague bridges in VGs, vague incidence cut-vertices, and vague incidence bridges are explored. Finally, new concepts of VIGs such as VIGP, VILP, VIBP, VIGLM, and B_{VI} are introduced. One may extend this study to introduce more generalized forms of the connectivity index in VIGs such as Randic index, Zagreb index, Hyper Zagreb index, Wiener index etc. This research can be extended towards spherical and T-spherical fuzzy sets.

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