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Fixed point theorems for weakly contractive mapping on generalized asymmetric metric spaces

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Abstract

In this present paper, inspired by the concept of weakly contractive mapping in metric spaces, we introduce the concept of weakly contractive mapping in generalized asymmetric metric spaces and we establish various fixed point theorems for such mappings in complete generalized metric spaces.

Keywords: Fixed point, weakly contractive, asymmetric metric spaces
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1 Introduction

Fixed point theory is one of the important research topics of functional analysis. In 1922, Banach [2] had established a remarkable fixed point theorem, known as "Banach Contraction Principle." Due to its importance, Many mathematician studied a lot of interesting extensions and generalizations, (see [1, 4, 7, 9, 8, 13, 15, 16, 17, 18]).

In 2000, for the first time generalized metric spaces were introduced by Branciari [3], in such a way that triangle inequality is replaced by the "quadrilateral inequality"

$$d(x,y) \leq d(x,z) + d(z,u) + d(u,y)$$

for all pairwise distinct points x, y, z and u. As such, any metric space is a generalized metric space but the converse is not true. Various fixed point results were established on such spaces (see [5, 6, 12]) and references therein.

Combining conditions used for definitions of asymmetric metric and generalized metric spaces, Piri *et al.* [14] announced the notions of generalized asymmetric metric space, and formulated some first fixed point theorems for $\theta-$contraction mapping in generalized asymmetric metric space.

In this paper, we introduce a new notion of weakly contractive mapping and establish various fixed point theorems for such mappings in complete generalized metric spaces.

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2 Preliminaries

Definition 2.1. [14] Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then (X, d) is called a generalized asymmetric metric space.

Definition 2.2. [14]. Let (X, d) be a generalized asymmetric metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$.

1. We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) converges to x if
$$\lim_{n\to+\infty} d(x, x_n) = \lim_{n\to+\infty} d(x_n, x) = 0.$$
2. We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) Cauchy if
$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

Example 2.3. [10] Let $X = A \cup B$, where $A = \{0, 2\}$ and $B = \{\frac{1}{n},\ n \in \mathbb{N}^*\}$ and $d : X \times X \to [0, +\infty[$ be defined by

$$\begin{cases} d(0,2) = d(2,0) = 1 \\ d\left(\frac{1}{n}, 0\right) = \frac{1}{n}, d\left(0, \frac{1}{n}\right) = 1 \\ d\left(\frac{1}{n}, 2\right) = 1, d\left(2, \frac{1}{n}\right) = \frac{1}{n} \\ d\left(\frac{1}{n}, \frac{1}{m}\right) = d\left(\frac{1}{m}, \frac{1}{n}\right) = 1 \end{cases}$$

for all $n, m \in \mathbb{N}^*$ with $n \neq m$. Then (X, d) is a generalized asymmetric metric space. However we have the following:

1. (X, d) is not a metric space, since $d\left(\frac{1}{n}, 0\right) \neq d\left(0, \frac{1}{n}\right)$ for all $n > 1$.
2. (X, d) is not a asymmetric metric space, since $d(2, 0) = 1 > \frac{1}{2} = d\left(2, \frac{1}{4}\right) + d\left(\frac{1}{4}, 0\right)$.
3. (X, d) is not a rectangular metric space, since $d\left(\frac{1}{n}, 2\right) \neq d\left(2, \frac{1}{n}\right)$, for all $n > 1$.

Remark 2.4. [10] Let (X, d) be as in Example 2.3, $\{\frac{1}{n}\}_{n\in\mathbb{N}^*}$ be a sequence in X. However, we have the following:

1. $\lim\limits_{n\to+\infty} d\left(\frac{1}{n}, 0\right) = 0$, $\lim\limits_{n\to+\infty} d\left(\frac{1}{n}, 2\right) = 1$ and $\lim\limits_{n\to+\infty} d\left(0, \frac{1}{n}\right) = 1$, $\lim\limits_{n\to+\infty} d\left(2, \frac{1}{n}\right) = 0$. Then the sequence $\{\frac{1}{n}\}$ forward converges to 2 and backward converges to 0. So the limit is not unique.
2. $\lim\limits_{n\to+\infty} d\left(\frac{1}{m}, \frac{1}{n}\right) = \lim\limits_{n\to+\infty} d\left(\frac{1}{m}, \frac{1}{n}\right) = 1$. So forward (backward) convergence does not imply forward (backward) Cauchy.

Lemma 2.5. [14]. Let (X, d) be a generalized asymmetric metric space and $\{x_n\}_n$ be a forward (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then x = y.

Definition 2.6. [14]. Let (X, d) be a generalized asymmetric metric space. Then X is said to be forward (backward) complete if every forward (backward) Cauchy sequence $\{x_n\}_n$ in X forward (backward) converges to $x \in X$.

Definition 2.7. [11] A function $\psi : [0, \infty[\to [0, \infty[$ is said to be an altering distance function if it satisfies thefollowing conditions:

(a) is continuous and nondecreasing;

(b) $\psi(t) = 0$ if and only if $t = 0$.

Example 2.8. Define ψ_1, ψ_2, ψ_3: $[0, \infty[\to [0, +\infty[$ by $\psi_1(t) = t$, $\psi_t(t) = 3t$ and $\psi_3(t) = t^3$. Then they are altering distance functions.

Definition 2.9. A function $f : X \to \mathbb{R}^+$, where X is generalized asymmetric metric space is called lower semicontinuous if for all $x \in X$ and $x_n \in X$ with $lim_{n\to\infty} x_n = x$, we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Example 2.10.

$$\phi(t) = \begin{cases} \dfrac{t}{18} & if\ t \in [0, 1] \\ \dfrac{t}{9} & if\ t > 1. \end{cases}$$

Then ϕ is a lower semicontinuous function.

Definition 2.11. Let Δ be the family of function $\phi : [0, \infty[\to [0, \infty[$ which satisfy the following:

1. ϕ is lower semicontinous;
2. $\phi^n(t)_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t > 0$;
3. $\phi(t) < t$ for any $t > 0$.

3 Main results

Theorem 3.1. Let (X, d) be a generalized asymmetric metric space and $T : X \to X$ be a mapping. Assume that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \psi[d(Tx, Ty)] \leq \psi[M(x, y)] - \phi[M(x, y)] \tag{3.1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

$$d(y, x) \leq d(T^2y, x).$$

Here ψ is an altering distance function and ϕ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof . Let $x_0 \in X$ be an arbitrary point in X. Then we define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ or $d(x_{n_0+1}, x_{n_0}) = 0$, then x_{n_0} is a fixed point of T. Then we assume that $d(x_n, x_{n+1}) > 0$ and $d(x_{n+1}, x_n) > 0$.

Step 1. We prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Applying (3.1) with $x = x_n$ and $y = x_{n+1}$, we obtain

$$\psi(d(Tx_{n-1}, Tx_n)) = \psi(d(x_n, x_{n+1})) \leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)), \tag{3.2}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then we have

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})).$$

That is, $\phi(d(x_n, x_{n+1})) = 0$, i.e., $x_n = x_{n+1}$, which is a contradiction. Hence $M(x, x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Thus

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)). \tag{3.3}$$

So

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n)).$$

Since ψ is a nonincreasing and continuous function, we deduced that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{3.4}$$

Now, applying (3.1) with $x = x_n$ and $y = x_{n-1}$, we obtain

$$\psi(d(Tx_n, Tx_{n-1})) = \psi(d(x_{n+1}, x_n)) \leq \psi(M(x_n, x_{n-1})) - \phi(M(x_n, x_{n-1})),$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}. \end{aligned}$$

Suppose that $d(x_n, x_{n-1}) \leq d(x_{n+1}, x_n)$ for some $n \in \mathbb{N}$.

***Case* 1.** If $d(x_n, x_{n-1}) \geq d(x_{n-1}, x_n)$, then we get

$$\psi(d(x_n, x_{n-1})) \leq \psi(d(x_n, x_{n-1})) - \phi(d(x_n, x_{n-1})).$$

Then

$$\phi(d(x_n, x_{n-1})) = 0.$$

This is a contradiction. Hence

$$M(x_n, x_{n-1}) = d(x_n, x_{n-1}).$$

Thus

$$\psi(d(x_{n+1}, x_n)) \leq \psi(d(x_n, x_{n-1}) - \phi(d(x_n, x_{n-1}))).$$

***Case* 2.** If $d(x_n, x_{n-1}) < d(x_{n-1}, x_n)$, then we get

$$\psi(d(x_{n+1}, x_n)) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)).$$

Since $d(x, y) \leq d(T^2 y, x)$, $d(x_{n-1}, x_n) \leq d(x_{n+1}, x_n)$, which implies that

$$\psi(d(x_n, x_{n-1})) \leq \psi(d(x_n, x_{n-1})) - \phi(d(x_n, x_{n-1})).$$

Thus

$$\phi(d(x_n, x_{n-1})) = 0$$

and so $d(x_n, x_{n-1}) = 0$. This is a contradiction. Hence

$$\psi(d(x_{n+1}, x_n)) \leq \psi(d(x_n, x_{n-1})) - \phi(d(x_n, x_{n-1})). \tag{3.5}$$

Since ψ is a nonincreasing and continuous function, we deduced that

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}). \tag{3.6}$$

From (3.4), the sequence $d(x_n, x_{n+1})_{n\in\mathbb{N}}$ is monotone nonincreasing and so bounded below. So there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \alpha. \tag{3.7}$$

Taking $\limsup_{n\to\infty}$ in (3.5) and using the above limits with the continuity of ψ and the lower semicontinuity of ϕ, we get

$$\begin{aligned}\psi(\lim_{n\to\infty}\sup d(x_{n+1},x_n)) &\le \psi(\lim_{n\to\infty}\sup d(x_n,x_{n-1})) - \lim\sup_{n\to\infty} \phi(d(x_n,x_{n-1}))\\ &\le \psi(\lim_{n\to\infty}\sup d(x_n,x_{n-1})) - \lim_{n\to\infty}\inf \phi(\lim_{n\to\infty} d(x_n,x_{n-1}).\end{aligned}$$

Thus $\psi(\alpha) \le \psi(\alpha) - \phi(\alpha)$, which implies that $\phi(\alpha) = 0$. So $\alpha = 0$ by the proprety of ϕ. Then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

From (3.6), the sequence $d(x_{n+1}, x_n)_{n\in\mathbb{N}}$ is monotone nonincreasing and so bounded below. So there exists $\lambda \ge 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lambda. \tag{3.9}$$

Taking $\limsup_{n\to\infty}$ in (3.3) and using the above limits with the continuity of ψ and the lower semicontinuity of ϕ, we get

$$\begin{aligned}\psi(\lim_{n\to\infty}\sup d(x_n,x_{n+1})) &\le \psi(\lim_{n\to\infty}\sup d(x_{n-1},x_n)) - \lim\sup_{n\to\infty} \phi(d(x_{n-1},x_n))\\ &\le \psi(\lim_{n\to\infty}\sup d(x_{n-1},x_n)) - \lim_{n\to\infty}\inf \phi(\lim_{n\to\infty} d(x_{n-1},x_n).\end{aligned}$$

Thus $\psi(\lambda) \le \psi(\alpha) - \phi(\lambda)$, which implies that $\phi(\lambda) = 0$. So $\lambda = 0$ by the property of ϕ. Then

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.10}$$

We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0 \ and \ \lim_{n\to\infty} d(x_{n+2}, x_n) = 0.$$

We assume that $x_n \ne x_m$ for all $n, m \in \mathbb{N}$ with $n \ne m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$. By (3.4), we have

$$d(x_m, x_{m+1}) = d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{3.11}$$

Continuing this process, we can get that

$$d(x_m, x_{n+1}) = d(x_n, x_{n+1}) < d(x_m, x_{m+1}).$$

This is a contradiction. Therefore, $d(x_n, x_m) > 0$ for all $n, m \in \mathbb{N}$ with $n \ne m$. Letting $x = x_{n-1}$ and $y = x_{n+1}$ in (3.1), we obtain

$$\psi(d(x_n, x_{n+2})) \le \psi(M(x_{n-1}, x_{n+1})) - \phi(M(x_{n-1}, x_{n+1})),$$

where

$$\begin{aligned}M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}\\ &= b\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}.\end{aligned}$$

Thus

$$\psi(d(x_n, x_{n+2})) \le \psi(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}) - \phi(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}). \tag{3.12}$$

Take $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Thus, from (3.12)

$$\psi(a_n) \le \psi(\max\{a_{n-1}, b_{n-1}\}) - \phi(\max\{a_{n-1}, b_{n-1}\}), \tag{3.13}$$

which implies that

$$a_n \le \max\{a_{n-1}, b_{n-1}\}.$$

Again, by (3.3) $b_n \leq b_{n-1}$. Therefore, $\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}$ for all $n \in \mathbb{N}$. Then the sequence $\max\{a_n, b_n\}_{n\in\mathbb{N}}$ is monotone nonincreasing and so it converges to some $t \geq 0$. By (3.8), for $t > 0$, we have

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \max\{a_n, b_n\} = \lim_{n\to\infty} \sup \max\{a_{n-1}, b_{n-1}\} = t. \tag{3.14}$$

Taking $\lim_{n\to\infty} \sup$ in (3.13) and using the properties of ψ and ϕ, we obtain

$$\begin{aligned}
\psi(t) &= \psi(\lim_{n\to\infty} \sup\ a_n) = \lim_{n\to\infty} \sup\ \psi(a_n) \\
&\leq \lim_{n\to\infty} \sup\ \psi(\max\{a_{n-1}, b_{n-1}\}) - \lim_{n\to\infty} \sup\ \psi(\max\{a_{n-1}, b_{n-1}\}) \\
&\leq \lim_{n\to\infty} \sup\ \psi(\max\{a_{n-1}, b_{n-1}\}) - \lim_{n\to\infty} \inf\ \phi(\max\{a_{n-1}, b_{n-1}\}) \\
&\leq \psi(\lim_{n\to\infty} \sup \max\{a_{n-1}, b_{n-1}\}) - \phi(t) \\
&= \psi(t) - \phi(t),
\end{aligned}$$

which implies that $\phi(t) = 0$ and so $t = 0$, which is a contradiction. Thus

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Letting $x = x_n$ and $y = x_{n-1}$ in (3.1), we obtain

$$\psi(d(x_{n+2}, x_n)) \leq \psi(M(x_{n+1}, x_{n-1})) - \phi(M(x_{n+1}, x_{n-1})),$$

where

$$\begin{aligned}
M(x_{n+1}, x_{n-1}) &= \max\{d(x_{n+1}, x_{n-1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}) \\
&= \max\{d(x_{n+1}, x_{n-1}), d(x_{n-1}, x_n)\} \\
&\leq \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_n)\}.
\end{aligned}$$

Then

$$\psi(d(x_{n+2}, x_n)) \leq \psi(\max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_n)\}) - \phi(\max\{d(x_{n+1}, x_{n-1}), d(x_{n-1}, x_n)\}). \tag{3.15}$$

Take $\lambda_n = d(x_{n+2}, x_n)$ and $\beta_n = d(x_{n+1}, x_n)$. Thus, from (3.15), we have

$$\psi(\lambda_n) = \psi(\max(\lambda_{n-1}, \beta_{n-1})) - \phi(\max(\lambda_{n-1}, \beta_{n-1})), \tag{3.16}$$

which implies that

$$\lambda_n \leq \max(\lambda_{n-1}, \beta_{n-1}). \tag{3.17}$$

Again, by (3.6) $\beta_n \leq \beta_{n-1}$. Therefore, $\max(\lambda_n, \beta_n) \leq \max(\lambda_{n-1}, \beta_{n-1})$ for all $n \in \mathbb{N}$. Then the sequence $\{\max(\lambda_n, \beta_n)\}_{n\in\mathbb{N}}$ is monotone nonincreasing, and so it converges to some $r \geq 0$. By (3.10), for $r > 0$, we have

$$\lim_{n\to\infty} \sup\ \lambda_n = lim_{n\to\infty} \sup \max(\lambda_{n-1}, \beta_{n-1}) = r. \tag{3.18}$$

Taking $\lim_{n\to\infty} \sup$ in (3.15) and using the properties of ψ and ϕ, we obtain

$$\begin{aligned}
\psi(r) &= \psi(\lim_{n\to\infty} \sup\ \lambda_n) \\
&= \lim_{n\to\infty} \psi(\lambda_n) \\
&\leq \lim_{n\to\infty} \sup \psi(\max(\lambda_{n-1}, \beta_{n-1})) - \lim_{n\to\infty} \sup\ \phi(\max(\lambda_{n-1}, \beta_{n-1})) \\
&= \lim_{n\to\infty} \sup\ \psi(\max(\lambda_{n-1}, \beta_{n-1})) - \lim_{n\to\infty} \inf\ \phi(\max(\lambda_{n-1}, \beta_{n-1})) \\
&\leq \psi(\lim_{n\to\infty} \sup \max(\lambda_{n-1}, \beta_{n-1})) - \phi(r) \\
&= \psi(r) - \phi(r),
\end{aligned}$$

which implies that $\phi(r) = 0$ and so $r = 0$, which is a contradiction. Thus

$$\lim_{n\to\infty} d(x_{n+2}, x_n) = 0. \tag{3.19}$$

Step 3. We prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Firstly, we show $\{x_n\}_{n\in\mathbb{N}}$ is a right-Cauchy sequence. Otherwise, there exist an $\varepsilon > 0$ and sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that, for all positive integers k, $(n_{(k)}) > (m_{(k)}) > k$,

$$d(m_{(k)}, n_{(k)}) \leq \varepsilon \text{ and } d(m_{(k)}, n_{(k-1)}) < \varepsilon. \tag{3.20}$$

By quadrilateral inequality, we obtain

$$\begin{aligned}\varepsilon &\leq d(x_{m_{(k)}}, x_{n_{(k)}}) + d(x_{n_{(k)}}, x_{n_{(k)}+1}) + d(x_{n_{(k)}+1}, x_{n_{(k)}}) \\ &< \varepsilon + d(x_{n_{(k)}}, x_{n_{(k)}+1}) + d(x_{n_{(k)}+1}, x_{n_{(k)}}).\end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d(x_{m_{(k)}, x_{n_{(k)}}}) = \varepsilon. \tag{3.21}$$

Now, by quadrilateral inequality, we have

$$d(x_{m_{(k)+1}, x_{n_k+1}}) \leq d(x_{m_{(k)+1}, x_{m_k}}) + d(x_{m_{(k)}, x_{n_k}}) + d(x_{n_{(k)}, x_{n_{(k)}+1}}), \tag{3.22}$$

$$d(x_{m_{(k)}, x_{n_k}}) \leq d(x_{m_{(k)}, x_{m_k+1}}) + d(x_{m_{(k)+1}, x_{n_k+1}}) + d(x_{n_{(k)+1}, x_{n_{(k)}}}). \tag{3.23}$$

Letting $k \to \infty$ in the above inequalities and using (3.20), we obtain

$$\lim_{k\to\infty} d(x_{m_{(k)+1}, x_{n_k+1}}) = \varepsilon. \tag{3.24}$$

Let $B = \dfrac{\varepsilon}{2} > 0$. By (3.24), from the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|d(x_{m_{(k)+1}, x_{n_k+1}}) - \varepsilon| \leq B, \quad \forall n \geq n_0.$$

This implies that

$$d(x_{m_{(k)+1}, x_{n_k+1}}) \geq B \qquad \forall n \geq n_0.$$

Letting $x = x_{n_{(k)}}$ and $y = x_{m_{(k)}}$ in (3.1), we have

$$\psi(d(x_{m_{(k)}}, x_{m_{(k)}})) \leq \psi(M(x_{m_{(k)}}, x_{n_{(k)}})) - \phi(M(x_{m_{(k)}}, x_{n_{(k)}})), \tag{3.25}$$

where

$$M(x_{n_{(k)}}, x_{m_{(k)}}) = \max\{d(x_{n_{(k)}}, x_{m_{(k)}}, d(x_{n_{(k)}}, x_{n_{(k)}+1}), d(x_{m_{(k)}}, x_{m_{(k)}+1})\}.$$

Therefore by (3.8) and (3.21), we get that

$$\lim_{k\to\infty} M(x_{m_{(k)}}, x_{m_{(k)}}) = \varepsilon.$$

Letting $k \to \infty$ in (3.25), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon),$$

which is a contradiction by virtue of a property of ϕ. Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a right-Cauchy sequence in (X, d).

Secondly, we prove that $\{x_n\}_{n\in\mathbb{N}}$ is a left-Cauchy sequence. Otherwise, there exist an $\varepsilon > 0$ and sequences $(n_k)_k$ and $(m_{(k)})_k$ such that for all positive integers k, $(n_k) > (m_k) > k$,

$$d(n_k, m_k) \leq \varepsilon \textit{ and } d(n_{(k)-1}, m_{(k)}) < \varepsilon \tag{3.26}$$

By quadrilateral inequality, we obtain

$$\begin{aligned}\varepsilon \leq d(x_{n_{(k)}}, x_{n_{(k)}}) &\leq d(x_{n_{(k)}}, x_{n_{(k)}+1}) + d(x_{n_{(k)}+1}, x_{n_{(k)}-1}) + d(x_{n_{(k)}-1}, x_{m_{(k)}}) \\ &< \varepsilon + d(x_{n_{(k)}+1}, x_{n_{(k)}-11}) + d(x_{n_{(k)}-1}, x_{m_{(k)}}).\end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d(x_{n_{(k)}}, x_{m_{(k)}}) = \varepsilon. \tag{3.27}$$

Now, by quadrilateral inequality, we have

$$d((x_{n_{(k)}+1}, x_{m_{(k)}+1}) \leq d((x_{n_{(k)}+1}, x_{n_{(k)}}) + d((x_{n_{(k)}}, x_{m_{(k)}}) + d((x_{m_{(k)}}, x_{m_{(k)}+1}). \tag{3.28}$$

$$d((x_{n_{(k)}}, x_{m_{(k)}}) \leq d((x_{n_{(k)}}, x_{n_{(k)}+1}) + d((x_{n_{(k)}+1}, x_{n_{(k)}+1}) + d((x_{m_{(k)}+1}, x_{m_{(k)}}). \tag{3.29}$$

Letting $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} d(x_{n_{(k)+1}, x_{m_k+1}}) = \varepsilon. \tag{3.30}$$

Let $A = \dfrac{\varepsilon}{2} > 0$. By (3.30), from the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$|d(x_{n_{(k)+1}, x_{m_k+1}}) - \varepsilon| \leq A, \qquad \forall n \geq n_1.$$

This implies that

$$d(x_{n_{(k)+1}, x_{m_k+1}}) \geq A, \qquad \forall n \geq n_1.$$

Letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (3.1), we have

$$\psi(d(x_{m_{(k)}}, x_{n_{(k)}})) \leq \psi(M(x_{m_{(k)}}, x_{n_{(k)}})) - \phi(M(x_{n_{(k)}}, x_{m_{(k)}})), \tag{3.31}$$

where

$$M(x_{m_{(k)}}, x_{n_{(k)}}) = \max\{d(x_{m_{(k)}}, x_{n_{(k)}}, d(x_{m_{(k)}}, x_{m_{(k)}+1}), d(x_{n_{(k)}}, x_{n_{(k)}+1})\}. \tag{3.32}$$

Therefore, by (3.8) and (3.30), we get that

$$\lim_{k\to\infty} M(x_{m_{(k)}}, x_{m_{(k)}}) = \varepsilon. \tag{3.33}$$

Letting $k \to \infty$ in (3.31) and using (3.33), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon), \tag{3.34}$$

which is a contradiction by virtue of the property of ϕ. Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a left-Cauchy sequence in (X, d). Hence, by completeness of (X, d), there exist $z, u \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(u, x_n) = 0. \tag{3.35}$$

So, from Lemma 2.5, we get $z = u$ and hence

$$\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(z, x_n) = 0. \tag{3.36}$$

***Step* 4.** We prove that $z = Tz$, i.e., $d(Tz, z) = 0$ and $d(z, Tz) = 0$. Arguing by contradiction, we assume that

$$d(Tz, z) > 0 \ \textit{or} \ d(z, Tz) > 0.$$

First assume that $d(Tz, z) > 0$. By quadrilateral inequality, we get

$$d(Tx_n, Tz) \leq d(Tx_n, x_n) + d(x_n, z) + d(z, Tz) \tag{3.37}$$

and

$$d(z,Tz) \leq d(z,x_n) + d(x_n,Tx_n) + d(Tx_n,Tz). \tag{3.38}$$

It follows from (3.37) and (3.38) that

$$\lim_{n\to\infty} d(Tx_n,Tz) = d(z,Tz). \tag{3.39}$$

So there exists $n_0 \in \mathbb{N}$ such that

$$d(Tx_n,Tz) \geq d(z,Tz) > 0, \quad \forall n \geq n_0.$$

Letting $x = Tx_n$ and $y = Tz$ in (3.1), we obtain

$$\psi(d(Tx_n,Tz)) \leq \psi(M(x_n,z)) - \phi(M(x_n,z)), \tag{3.40}$$

where

$$M(x_n,z) = \max\{d(x_n,Tx_n), d(z,Tz), d(x_n,z)\}.$$

Since $\lim_{n\to\infty} d(x_n,x_{n+1}) = d(x_n,z) = 0$, we obtain that

$$\lim_{n\to\infty} M(x_n,z) = d(z,Tz).$$

Taking the limit as $n \to \infty$ in (3.40) and using the properties of ψ and ϕ, we obtain

$$\begin{aligned}\psi(d(Tx_n,Tz)) &\leq \psi(\lim_{n\to\infty} M(x_n,z)) - \lim_{n\to\infty}\sup\ \phi(M(x_n,z))\\ &\leq \psi(\lim_{n\to\infty} M(x_n,z)) - \lim_{n\to\infty}\inf\ \phi(M(x_n,z))\\ &\leq \psi(d(z,Tz)) - \phi(d(z,Tz)),\end{aligned}$$

which implies that $\phi(d(z,Tz)) = 0$, so and $d(z,Tz) = 0$, This is contradiction. If $d(Tz,z) > 0$, by similar method, we get $d(Tz,z) = 0$. Therefore, $d(z,Tz) = 0$ and $d(Tz,z) = 0$ and hence $z = Tz$.

***Step* 5.** (Uniqueness) Suppose that there are two distinct points $z,u \in X$ such that $Tz = z$ and $Tu = u$. Then $d(z,u) = d(Tz,Tu) = d(Tz,Tu) > 0$. Letting $x = z$ and $y = u$ in (3.1), we obtain

$$\psi(d(z,u)) \leq \psi(M(z,u)) - \phi(M(z,u)), \tag{3.41}$$

where

$$M(z,u) = max\{d(z,u), d(z,Tz), d(u,Tu)\} = d(z,u). \tag{3.42}$$

This implies that $\phi(d(z,u)) = 0$, and so $z = u$. □

Theorem 3.2. Let (X,d) be a generalized asymmetric metric space and $T : X \to X$ be a mapping. If there exists $\phi \in \Delta$ such that for all $x,y \in X$

$$d(Tx,Ty) > 0 \Rightarrow d(Tx,Ty) \leq \phi[M(x,y)], \tag{3.43}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$ and $d(y,x) \leq d(T^2y,x)$. Then T has a unique fixed point.

Proof . Let $x_0 \in X$ be an arbitrary point in X. Then we define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0},x_{n_0+1}) = 0$ or $d(x_{n_0+1},x_{n_0}) = 0$, then x_{n_0} is a fixed point of T. Then we assume that $d(x_n,x_{n+1}) > 0$ and $d(x_{n+1},x_n) > 0$.

***Step* 1.** We prove that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \lim_{n\to\infty} d(x_{n+1},x_n) = 0.$$

Letting $x = x_n$ and $y = x_{n+1}$ in (3.43), we obtain

$$d(Tx_{n-1},Tx_n) = d(x_n,x_{n+1}) \leq \phi(M(x_{n-1},x_n)) < M(x_{n-1},x_n),$$

where

$$\begin{aligned}M(x,x) &= \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1})\}\\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}.\end{aligned}$$

If $M(x_{n-1},x_n) = d(x_n,x_{n+1})$, then we have

$$d(x_n, x_{n+1}) \leq \phi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}).$$

This is a contradiction. Hence $M(x, x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Thus

$$d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n)). \tag{3.44}$$

So

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{3.45}$$

Letting $x = x_{n+1}$ and $y = x_n$ in (3.43), we obtain

$$d(x_{n+1}, x_m) = d(x_{n+1}, x_n) \leq \phi(M(x_n, x_{n-1})),$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}. \end{aligned}$$

Suppose that $d(x_n, x_{n-1}) \leq d(x_{n+1}, x_n)$ for some $n \in \mathbb{N}$.

Case 1. If $d(x_n, x_{n-1}) \geq d(x_{n-1}, x_n)$, then we get

$$d(x_n, x_{n-1}) \leq \phi(d(x_n, x_{n-1})) < d(x_n, x_{n-1}).$$

This is a contradiction.

Case 2. If $d(x_n, x_{n-1}) < d(x_{n-1}, x_n)$, then we get

$$d(x_{n+1}, x_n)) \leq \phi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n).$$

Since $d(y, x) \leq d(T^2 y, x)$, $d(x_{n-1}, x_n) \leq d(x_{n+1}, x_n)$, which implies that

$$d(x_n, x_{n-1}) \leq \phi(d(x_n, x_{n-1})) < d(x_n, x_{n-1}),$$

which is a contradiction. Therefore,

$$d(x_{n+1}, x_n) \leq \phi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n). \tag{3.46}$$

From (3.45), the sequence $d(x_n, x_{n+1})_{n\in\mathbb{N}}$ is monotone nonincreasing and so bounded below. So there exists $\mu \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \mu.$$

By induction, (3.44) yields

$$d(x_n, x_{n+1}) \leq \phi^n d(x_0, x_1), \quad \forall n \in \mathbb{N}.$$

By the property of ϕ, it is evident that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.47}$$

From (3.46), the sequence $\{d(x_{n+1}, x_n)\}_{n\in\mathbb{N}}$ is monotone nonincreasing and so bounded below. So there exists $\delta \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta$. By induction, (3.46) yields

$$d(x_{n+1}, x_n) \leq \phi^n d(x_1, x_0), \quad \forall n \in \mathbb{N}.$$

By the property of ϕ, it is evident that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.48}$$

***Step* 2.** We prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = \lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. Letting $x = x_n$ and $y = x_{n+2}$ in (3.43), we obtain

$$d(x_n, x_{n+2}) \leq \phi(M(x_{n-1}, x_{n+1})), \tag{3.49}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\} \\ &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Thus

$$d(x_n, x_{n+2}) \leq \phi(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}). \tag{3.50}$$

Take $\gamma_n = d(x_n, x_{n+2})$ and $\delta_n = d(x_n, x_{n+1})$. Then, from (3.49), we have

$$\gamma_n) \leq \phi(\max\{\gamma_{n-1}, \delta_{n-1}\}), \tag{3.51}$$

which implies that

$$\gamma_n \leq \max\{\gamma_{n-1}, \delta_{n-1}\}. \tag{3.52}$$

Again, by (3.47), $\delta_n \leq \delta_{n-1}$. Therefore, $\max\{\gamma_n, \delta_n\} \leq \max\{\gamma_{n-1}, \delta_{n-1}\}$ for all $n \in \mathbb{N}$. Then the sequence $\{\max\{\gamma_n, \delta_n\}\}_{n\in\mathbb{N}}$ is monotone nonincreasing, and so it converges to some $l \geq 0$. By (3.46), for $l > 0$, we have

$$\lim_{n\to\infty} \sup\ \gamma_n = \lim \sup_{n\to\infty} \max\{\gamma_n, \delta_n\} = \lim \sup_{n\to\infty} \max\{\gamma_{n-1}, \delta_{n-1}\} = l. \tag{3.53}$$

Taking $\limsup_{n\to\infty}$ in (3.49) and the properties of ϕ, we obtain

$$l = \lim_{n\to\infty} \sup\ \gamma_n \leq \lim_{n\to\infty} \sup \max\{\gamma_{n-1}, \delta_{n-1}\} \leq \phi(\lim_{n\to\infty} \max\{\gamma_{n-1}, \delta_{n-1}\}) = \phi(l) < l,$$

which is a contradiction. Thus

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{3.54}$$

Letting $x = x_n$ and $y = x_{n+2}$ (3.43), we obtain

$$d(x_{n+2}, x_n) \leq \phi(M(x_{n+1}, x_{n-1})), \tag{3.55}$$

where

$$\begin{aligned} M(x_{n+1}, x_{n-1}) &= \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\} \\ &= \max\{d(x_{n+1}, x_{n-1}), d(x_n, x_{n-1})\}. \end{aligned}$$

Thus

$$d(x_{n+2}, x_n) \leq \phi(\max\{d(x_{n+1}, x_{n-1}), d(x_n, x_{n-1})\}). \tag{3.56}$$

Take $\kappa = d(x_{n+2}, x_n)$ and $\pi_n = d(x_{n+1}, x_n)$. Then, from (3.55), we have

$$\kappa_n \leq \phi(\max\{\kappa_{n-1}, \pi_{n-1}\}), \tag{3.57}$$

which implies that

$$\kappa_n \leq max\{\kappa_{n-1}, \pi_{n-1}\}. \tag{3.58}$$

So we have $\pi_n \leq \pi_{n-1}$. Therefore, $\max\{\kappa_n, \pi_n\} \leq \max\{\kappa_{n-1}, \pi_{n-1}\}$ for all $n \in \mathbb{N}$. Then the sequence $\{\max\{\kappa_n, \pi_n\}\}_{n\in\mathbb{N}}$ is monotone nonincreasing, and so it converges to some $l \geq 0$. By (3.46), for $h > 0$, we have

$$\lim_{n\to\infty} \sup\ \kappa_n = \lim \sup_{n\to\infty} \max\{\kappa_n, \pi_n\} = \lim \sup_{n\to\infty} \max\{\kappa_{n-1}, \pi_{n-1}\} = l.$$

Taking the $\limsup_{n\to\infty}$ in (3.58) and the properties of ϕ, we obtain

$$\begin{aligned} l &= \lim_{n\to\infty} \sup\ \kappa_n \\ &\leq \lim_{n\to\infty} \sup \max\{\kappa_{n-1}, \delta_{n-1}\} \\ &\leq \phi(\lim_{n\to\infty} \max\{\kappa_{n-1}, \pi_{n-1}\}) \\ &= \phi(h) < h, \end{aligned}$$

which is a contradiction. Thus

$$\lim_{n\to\infty} d(x_{n+2}, x_n) = 0. \tag{3.59}$$

Step 3. We prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Firstly, we show $\{x_n\}_{n\in\mathbb{N}}$ is a right-Cauchy sequence, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \qquad \forall k \in \mathbb{N}.$$

The cases $k = 1$ and $k = 2$, are proved, respectively by (3.47) and (3.54). Now, we take $k \geq 3$ arbitrary. It is sufficient to show two cases.

Case I. Suppose that $k = 2m + 1$, where $m \geq 1$. Then by the quadrilateral inequality, we obtain

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+2m}, x_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+2m} \phi^p(d(x_0, x_1)) \\ &\leq \sum_{p=n}^{\infty} \phi^p(d(x_0, x_1)) \to 0 \ \ as\ n \to \infty. \end{aligned}$$

Case II. Suppose that $k = 2m$, where $m \geq 2$. Then by using (3.52) and the quadrilateral inequality, we obtain

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+2m}) \\ &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{n+2m-1}, x_{n+2m}) \\ &\leq \sum_{p=n+2}^{n+2m-1} \phi^p(d(x_0, x_1)) \\ &\leq \sum_{p=n+2}^{\infty} \phi^p(d(x_0, x_1)) \to 0 \ as\ n \to \infty \end{aligned}$$

By combining the expressions, we have $\limsup_{n\to\infty} d(x_n, x_{n+k}) = 0$, for all $k \in \mathbb{N}$. We conclude that $\{x_n\}$ is a right-Cauchy sequence in (X, d). Secondly, we show $\{x_n\}_{n\in\mathbb{N}}$ is a left-Cauchy sequence, that is,

$$\lim_{n\to\infty} d(x_{n+k}, x_n) = 0, \qquad \forall k \in \mathbb{N}.$$

The cases $k = 1$ and $k = 2$, are proved, respectively by (3.48) and (3.59). Now, we take $k \geq 3$ arbitrary. It is sufficient to show two cases.

Case I. Suppose that $k = 2m + 1$, where $m \geq 1$. Then by the quadrilateral inequality, we obtain

$$\begin{aligned} d(x_{n+k}, x_n) &\leq d(x_{n+2m+1}, x_n) \\ &\leq d(x_{n+2m+1}, x_{n+2m}) + d(x_{n+2m}, x_{n+2m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \sum_{p=n}^{n+2m} \phi^p(d(x_1, x_0)) \\ &\leq \sum_{p=n+2}^{\infty} \phi^p(d(x_1, x_0)) \to 0 \ \ as\ n \to \infty \end{aligned}$$

Case II. Suppose that $k = 2m$, where $m \geq 2$. Then by the quadrilateral inequality, we obtain

$$\begin{aligned} d(x_{n+k}, x_n) &\leq d(x_{n+2m}, x_n) \\ &\leq d(x_{n+2m}, x_{n+2m-1}) + d(x_{n+2m-1}, x_{n+2m-2}) + \cdots + d(x_{n+2}, x_n) \\ &\leq \sum_{p=n+2}^{n+2m-1} \phi^p(d(x_1, x_0)) \\ &\leq \sum_{p=n+2}^{\infty} \phi^p(d(x_1, x_0)) \to 0 \ \ as\ n \to \infty. \end{aligned}$$

By combining the expressions, we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \quad \forall k \in \mathbb{N}.$$

We conclude that $\{x_n\}$ is a left-Cauchy sequence in (X, d). Hence, by completeness of (X, d), there exist $z, u \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(u, x_n) = 0.$$

So, from Lemma 2.5, we get $z = u$ and hence

$$\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(z, x_n) = 0.$$

Step 4. We prove that $z = Tz$, i.e., $d(Tz, z) = 0$ and $d(z, Tz) = 0$. Arguing by contradiction, we assume that $d(Tz, z) > 0$ or $d(z, Tz) > 0$. First, assume that $d(Tz, z) > 0$. As in the proof of Theorem 3.1, we conclude that

$$\lim_{n\to+\infty} d(Tx_n, Tz) = d(z, Tz) \tag{3.60}$$

and so there exists $n_2 \in \mathbb{N}$ such that $d(Tx_n, Tz) \geq d(z, Tz) > 0$, for all $n \geq n_2$. Letting $x = Tx_n$ and $y = Tz$ in (3.43), we obtain

$$d(Tx_n, Tz)) \leq \phi(M(x_n, z)),$$

where $\lim_{n\to\infty} M(x_n, z) = d(z, Tz)$. Taking $\limsup$ as $n \to \infty$ in (3.60) and using the properties of ϕ, we obtain

$$\begin{aligned} d(z, Tz) &= \lim_{n\to\infty} d(Tx_n, Tz)) \\ &\leq \lim_{n\to+\infty} \sup\ \phi(M(x_n, z)) \\ &\leq \phi(\lim_{n\to\infty} M(x_n, z)), \end{aligned}$$

which is a contradiction. If $d(Tz, z) > 0$, then by similar method, we get a contradiction. Therefore $d(z, Tz) = 0$ and $d(Tz, z) = 0$,and hence $z = Tz$.

Step 5. (Uniqueness)

Suppose that there are two distinct points $z, u \in X$ such that $Tz = z$ and $Tu = u$. Then $d(z, u) = d(Tz, Tu) = d(Tz, Tu) > 0$. Letting $x = z$ and $y = u$ in (3.1), we obtain

$$d(z, u) \leq \phi(M(z, u)) = \phi(d(z, u)) < d(z, u),$$

where $M(z, u) = \max\{d(z, u), d(z, Tz), d(u, Tu)\} = d(z, u)$. This is a contradiction. So $z = u$. $\square$

Example 3.3. Let $X = \mathbb{R}_+$. Define $d : X \times X \to [0, +\infty[$ by $d(x, y) = \max\{y - x, 0\}$. Then (X, d) is a complete generalized metric space. Define a mapping $T : X \to X$ by

$$T(x) = \ln(\frac{x}{3} + 1), \forall x \in X.$$

Consider the functions $\phi, \psi : [0, +\infty[\to [0, +\infty[$ defined by $\psi(t) = 2t$, for all $t \in [0, +\infty[$, $\phi(t) = \frac{t}{3}$, for all $t \in [0, +\infty[$. For all $(x, y) \in X^2$, we have $d(T^2y, x) = \max\{x - T^2y, 0\}$ and

$$T^2y = \ln(\frac{1}{3}\ln(\frac{y}{3} + 1) + 1),$$

and hence $\max\{x - y, 0\} \leq \max\left\{x - \ln(\frac{1}{3}\ln(\frac{y}{3} + 1) + 1), 0\right\}$. Thus $d(y, x) \leq d(T^2y, x)$. On the other hand,

$$d(Tx, Ty) = \max\{\ln(\frac{y}{3} + 1) - \ln(\frac{x}{3} + 1), 0\},$$

$$M(x, y) = \max\{\max\{y - x, 0\}, \max\{\ln(\frac{x}{3} + 1) - x, 0\}, \max\{\ln(\frac{y}{3} + 1) - y, 0\}\}.$$

1. If $x \geq y$, then we have $d(Tx, Ty) = 0, M(x, y) = 0$. So

$$\psi(d(Tx, Ty)) = \psi(M(x, y)) - \phi(M(x, y)).$$

2. If $y > x$, then we have

$$d(Tx,Ty) = \ln(\frac{y}{3}+1) - \ln(\frac{x}{3}+1), \quad M(x,y) = \max\{y-x,0,0\} = y-x.$$

So

$$\psi(d(Tx,Ty)) = 2\ln(\frac{y}{3}+1) - 2\ln(\frac{x}{3}+1), \psi(M(x,y)) = 2(y-x), \phi(M(x,y)) = \frac{y-x}{3}.$$

Thus

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max(d(x,y),d(y,Ty))).$$

So 0 is a unique fixed point of T.

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