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A new class of generalized convex functions and mathematical programming

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Abstract

In this paper, a new class of nonconvex optimization problem is considered, namely (h,φ)-(b,F,ρ)-convexity is defined for (h,φ)-differentiable mathematical programming problem. The sufficiency of the so-called Karush-Kuhn-Tucker optimality conditions are established for the considered (h,φ)-differentiable mathematical programming problem under (generalized) (h,φ)-(b,F,ρ)-convexity hypotheses. Further, the so-called Mond-Weir (h,φ)-dual problem is defined for the considered (h,φ)-differentiable mathematical programming problem and several duality theorems in the sense of Mond-Weir are derived under appropriate (generalized) (h,φ)-(b,F,ρ)-convex assumptions.

Keywords: (h,φ)-differentiable mathematical programming, (h,φ)-(b,F,ρ)-convex function, generalized algebraic operations, optimality conditions, duality
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1 Introduction

Convexity is unequivocally a pivotal concept influencing virtually every facet of mathematical programming. Many nonlinear programming problems feature nonconvex objective and constraint functions, prompting numerous authors to undertake defining diverse nonconvex function classes. The exploration of optimality criteria for addressing such problems has become a focal point in recent years. The literature has witnessed various generalizations of convexity, with extensive studies dedicated to deriving necessary optimality conditions for differentiable and nondifferentiable programming problems across diverse classes. This signifies an ongoing and dynamic effort to address the challenges posed by nonconvexity in mathematical programming, reflecting a comprehensive exploration of optimality concepts in the realm of nonlinear programming (see, for example, [7, 11, 13, 14, 15, 17, 18, 21, 23, 24, 25, 26, 27, 28, 29, 33, 35], and others). Generalized convex functions have received significant attention in the last few decades. Various generalizations of convex functions have appeared in the literature (see, for example, [1, 2, 3, 4, 5, 6, 9, 10, 11, 16, 19, 22, 34], and others).

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Ben-Tal [10] introduced generalized operations of addition and multiplication. Under Ben-Tal's generalized algebraic operations, (h,φ)-convex functions are introduced as generalization of convex functions. Further, some basic properties of (h,φ)-convex functions are discussed by Ben-tal [10]. Xu and Liu [32], [31] established necessary optimality conditions for (h,φ)-optimization programming problem. The concept of (F,ρ)-convexity was introduced by Preda [27] as extension of F-convexity [12] and ρ-convexity [30]. Aghezzaf and Hachimi [7] derived some sufficient optimality conditions and mixed type duality results involving generalized (F,ρ)-convexity. Antczak et al. [8] proved the optimality and duality results for (h,φ)-nondifferentiable multiobjective programming problem under (generalized) (h,φ)-(b,F,ρ)-convex assumptions.

In this paper, a new class of generalized convexity is considered. The functions constituting it are not necessarily differentiable, but they are (h,φ)-differentiable. The concept of (h,φ)-(b,F,ρ)-convexity is defined for (h,φ)-differentiable mathematical programming problem. The sufficient optimality conditions are derived for (h,φ)-differentiable optimization problem under (generalized) (h,φ)-(b,F,ρ)-convexity. Further, for (h,φ)-differentiable optimization problem, its dual problem in the sense of Mond-Weir (h,φ)-dual problem is defined. Then various duality theorems between (h,φ)-differentiable optimization problem and its Mond-Weir (h,φ)-dual problem are established also under (generalized) (h,φ)-(b,F,ρ)-convexity hypotheses.

2 Preliminaries and (generalized) (h,φ)-(b,F,ρ)-convexity

Now, let us recall generalized operations of addition and multiplication introduced by Ben-Tal [10].

a) Let h be an n-dimensional vector-valued continuous function defined on R^n possessing an inverse function h^{-1}. Then, the h-vector addition of $x,y\in R^n$ is defined as follows:

$$x\oplus y=h^{-1}(h(x)+h(y)) \tag{2.1}$$

and the h-scalar multiplication of $x\in R^n$ and $\delta\in R$ is defined as follows:

$$\delta\otimes x=h^{-1}(\delta h(x)). \tag{2.2}$$

b) Let φ be a real-valued continuous function defined on R possessing the inverse function φ^{-1}. Then the φ-scalar addition of two numbers δ and ϑ is defined as follows:

$$\delta[+]\vartheta=\varphi^{-1}(\varphi(\delta)+\varphi(\vartheta)) \tag{2.3}$$

and the φ-scalar multiplication is defined as follows:

$$\lambda[\cdot]\delta=\varphi^{-1}(\lambda\varphi(\delta)). \tag{2.4}$$

c) The (h,φ)-inner product of $x\in R^n$ and $y\in R^n$ is defined by

$$(x^Ty)_{(h,\varphi)}=\varphi^{-1}(h(x)^Th(y)). \tag{2.5}$$

Denote

$$\bigoplus_{i=1}^{m}x_i=x_1\oplus x_2\oplus ...\oplus x_m,\ x_i\in R^n. \tag{2.6}$$

$$\left[\sum_{i=1}^{m}\right]\delta_i=\delta_1[+]\delta_2[+]...[+]\delta_m,\quad \delta_i\in R. \tag{2.7}$$

$$\delta[-]\vartheta=\delta[+]((-1)[\cdot]\vartheta). \tag{2.8}$$

Definition 2.1. Let X be a nonempty subset of R^n. A functional $F:X\times X\times R^n\to R$ is called sublinear if, for any $x,z\in X$,

$$F(x,z;a_1\bigoplus a_2)\leq F(x,z;a_1)[+]F(x,z;a_2),\forall a_1,a_2\in R^n, \tag{2.9}$$

$$F(x,z;\delta\bigotimes a)=\delta[\cdot]F(x,z;a),\forall a\in R^n,\delta\geq 0. \tag{2.10}$$

By (2.10), it is clear that

$$F(x,z;0)=0,\forall x,z\in X. \tag{2.11}$$

Now, we give the definition of a (h, φ)-differentiable function.

Definition 2.2. [10] Let f be a real-valued function defined on R^n, denote $\hat{f}(t) = \varphi(f(h^{-1}(t)))$, $t \in R^n$. For simplicity, write $\hat{f}(t) = \varphi f h^{-1}(t)$. The function f is said to be (h, φ)-differentiable at $x \in R^n$, if $\hat{f}(t)$ is differentiable at $t = h(x)$, and denoted by $\nabla^* f(x) = h^{-1}(\nabla \hat{f}(t) \mid_{t=h(x)})$. In addition, it is said that f is (h, φ)-differentiable on $X \subset R^n$ if it is (h, φ)-differentiable at each $x \in X$. A vector-valued function is called (h, φ)-differentiable on $X \subset R^n$ if each of its components is (h, φ)-differentiable at each $x \in X$.

It is clear that every differentiable function f is (h, φ)-differentiable function (if h and φ are the identity functions, respectively), but the converse is not true.

Example 2.3. Let $f(x) = |x|$ be a nondifferentiable function at $x = 0$ and let $h(t) = t,\ t \in R$ and $\varphi(\delta) = \delta^2,\ \delta \in R$, then the function f is (h, φ)-differentiable at $x = 0$.

Let h be a n-dimensional vector-valued continuous function defined on R^n and φ be such a real-valued continuous function defined on R that it has the inverse function φ^{-1}. The following results are proposed by Ben-Tal [10].

Lemma 2.4. [10] Assume that f is a real-valued function defined on R^n and (h, φ)-differentiable at $\bar{x} \in R^n$. Then, the following statements hold:

a) Let $x^i \in R^n, \lambda_i \in R, i = 1, ..., m$. Then

$$\bigotimes_{i=1}^{m}(\lambda_i \otimes x^i) = h^{-1}\left(\sum_{i=1}^{m} \lambda_i h(x^i)\right),$$
$$\bigotimes_{i=1}^{m} x^i = h^{-1}\left(\sum_{i=1}^{m} h(x^i)\right).$$

b) Let $\mu_i, \delta_i \in R$, i=1,...,m. Then

$$\left[\sum_{i=1}^{m}\right](\mu_i[\cdot]\delta_i) = \varphi^{-1}\left(\sum_{i=1}^{m} \mu_i \varphi(\delta_i)\right),$$
$$\left[\sum_{i=1}^{m}\right]\delta_i = \varphi^{-1}\left(\sum_{i=1}^{m} \varphi(\delta_i)\right).$$

Lemma 2.5. [10] The following statements hold:

a) $\delta[\cdot](\vartheta[\cdot]\sigma) = \vartheta[\cdot](\delta[\cdot]\sigma) = (\delta[\cdot]\vartheta)[\cdot]\sigma$ for $\delta, \vartheta, \sigma \in R$,
b) $\vartheta[\cdot]\left[\sum_{i=1}^{m}\right](\delta_i) = \left[\sum_{i=1}^{m}\right](\vartheta[\cdot]\delta_i)$, $\delta_i, \vartheta, \in R$,
c) $\sigma[\cdot](\delta[-]\vartheta) = (\sigma[\cdot]\delta)[-](\sigma[\cdot]\vartheta)$ for $\delta, \vartheta, \sigma \in R$,
d) $\left[\sum_{i=1}^{m}\right](\delta[+]\vartheta) = \left[\sum_{i=1}^{m}\right](\delta_i)[+]\left[\sum_{i=1}^{m}\right](\vartheta_i)$, $\delta_i, \vartheta_i \in R$,
d) $\left[\sum_{i=1}^{m}\right](\delta[-]\vartheta) = \left[\sum_{i=1}^{m}\right](\delta_i)[-]\left[\sum_{i=1}^{m}\right](\vartheta_i)$, $\delta_i, \vartheta_i \in R$.

Now we introduce the definitions of (h, φ)-(b, F, ρ)-convex functions and generalized $(h, \varphi) - (b, F, \rho)$-convex functions for (h, φ)-differentiable mathematical programming.

Definition 2.6. Let $f : X \longrightarrow R$ be a (h, φ)-differentiable function at $\bar{x}$ on X. It is said that f is a (h, φ)-(b, F, ρ)-convex function at $\bar{x}$ on X if, there exist a sublinear functional $F : X \times X \times R \to R$, $b : X \times X \to R$, $d : X \times X \to R$, and a real number ρ such that, the following inequality

$$b(x, \bar{x})[\cdot](f(x)[-]f(\bar{x})) \geq F(x, \bar{x}; \nabla^* f(\bar{x}))[+](\rho_f[\cdot]d^2(x, \bar{x})) \tag{2.12}$$

holds for all $x \in X$. If (2.12) is satisfied for each $\bar{x} \in X$, then f is a (h, φ)-(b, F, ρ)-convex function on X.

Remark 2.7. Note that the Definition 2.6 generalizes and extends several generalized convexity notions. Indeed, there are the following special cases:

a) In the case when $b(x,\bar{x}) = 1$ and $F(x,\bar{x};\nabla^* f(\bar{x}))[+](\rho_f[\cdot]d^2(x,\bar{x})) = \langle x[-]\bar{x}, \nabla^* f(\bar{x})\rangle$, then the definition of a (h,φ)-(b,F,ρ)-convex function reduces to the definition of a $h-\varphi$-convex function introduced by Ben-Tal [10].
b) If $b(x,\bar{x}) = 1$, then the definition of a (h,φ)-(b,F,ρ)-convex function reduces to the definition of a (h,φ)-(F,ρ)-convex function.
c) If $b(x,\bar{x}) = 1$ and $F(x,\bar{x};\nabla^* f(\bar{x}))[+](\rho_f[\cdot]d^2(x,\bar{x})) = \langle \eta(x,\bar{x}), \nabla^* f(\bar{x})\rangle$, then the definition of a (h,φ)-(b,F,ρ)-convex function reduces to the definition of a (h,φ)-invex function introduced by Yu and Liu [31].
d) If f is a Lipschitz function and f is (h,φ)-nondifferentiable, then we get the definition of a (h,φ)-(b,F,ρ)-convex function introduced by Antczak et al. [8].
e) If $b(x,\bar{x}) = 1$ and $F(x,\bar{x};\nabla^* f(\bar{x}))[+](\rho_f[\cdot]d^2(x,\bar{x})) = F(x,\bar{x};\nabla^* f(\bar{x}))$, then the definition of a (h,φ)-(b,F,ρ)-convex function reduces to the definition of a (h,φ)-F-convex function.

Definition 2.8. Let $f : X \longrightarrow R$ be a (h,φ)-differentiable function at $\bar{x}$ on X. It is said that f is a (h,φ)-(b,F,ρ)-quasi-convex function at $\bar{x}$ on X if, there exist a sublinear functional $F : X \times X \times R \to R$, $b : X \times X \to R$, $d : X \times X \to R$, and a real number ρ such that, the following relation

$$b(x,\bar{x})[\cdot]f(x) \leq b(x,\bar{x})[\cdot]f(\bar{x}) \Longrightarrow F\left(x,\bar{x};\nabla^* f(\bar{x})\right)[+]\left(\rho_f[\cdot]d^2(x,\bar{x})\right) \leq 0 \tag{2.13}$$

holds for all $x \in X$. If (2.13) is satisfied for each $\bar{x} \in X$, then f is a (h,φ)-(b,F,ρ)-quasi-convex function on X.

Definition 2.9. Let $f : X \longrightarrow R$ be a (h,φ)-differentiable function at $\bar{x}$ on X. It is said that f is a (h,φ)-(b,F,ρ)-pseudo-convex function at $\bar{x}$ on X if, there exist a sublinear functional $F : X \times X \times R \to R$, $b : X \times X \to R$, $d : X \times X \to R$, and a real number ρ such that, the following relation

$$b(x,\bar{x})[\cdot]f(x) < b(x,\bar{x})[\cdot]f(\bar{x}) \Longrightarrow F\left(x,\bar{x};\nabla^* f(\bar{x})\right)[+]\left(\rho_f[\cdot]d^2(x,\bar{x})\right) < 0 \tag{2.14}$$

holds for all $x \in X$. If (2.14) is satisfied for each $\bar{x} \in X$, then f is a (h,φ)-(b,F,ρ)-pseudo-convex function on X.

Definition 2.10. Let $f : X \longrightarrow R$ be a (h,φ)-differentiable function at $\bar{x}$ on X. It is said that f is a strictly (h,φ)-(b,F,ρ)-pseudo-convex function at $\bar{x}$ on X if, there exist a sublinear functional $F : X \times X \times R \to R$, $b : X \times X \to R$, $d : X \times X \to R$, and a real number ρ such that, the following relation

$$F\left(x,\bar{x};\nabla^* f(\bar{x})\right)[+]\left(\rho_f[\cdot]d^2(x,\bar{x})\right) \geq 0 \Longrightarrow b(x,\bar{x})[\cdot]f(x) > b(x,\bar{x})[\cdot]f(\bar{x}) \tag{2.15}$$

holds for all $x \in X$. If (2.15) is satisfied for each $\bar{x} \in X$, then f is a strictly (h,φ)-(b,F,ρ)-pseudo-convex function on X.

Definition 2.11. $\bar{x} \in X$ is called a global minimizer of f if the following inequality$f(\bar{x}) \leq f(x)$ holds for all $x \in X$.

Proposition 2.12. Let function f be (h,φ)-differentiable on X. If $\bar{x} \in X$ is a global minimizer of the function f, then $\nabla^* f(\bar{x}) = 0$.

Proposition 2.13. Let φ be a continuous function with $\varphi(0) = 0$, F be a sublinear function and f be a (h,φ)-differentiable (h,φ)-(b,F,ρ)-convex function on X. If $\nabla^* f(\bar{x}) = 0$, then $\bar{x}$ is a minimizer of f.

Proof . Assume that f is a (h,φ)-differentiable (h,φ)-(b,F,ρ)-convex function on X. Hence, by Definition 2.6, the inequality

$$b(x,\bar{x})[\cdot](f(x)[-]f(\bar{x})) \geq F(x,\bar{x};\nabla^* f(\bar{x}))[+](\rho_f[\cdot]d^2(x,\bar{x})) \tag{2.16}$$

holds for all $x \in X$. Since φ is a continuous function with $\varphi(0) = 0$, F is a sublinear function, $\nabla^* f(\bar{x}) = 0$ and (2.16), we obtain

$$b(x,\bar{x})[\cdot](f(x)[-]f(\bar{x})) \geq 0 \tag{2.17}$$

where $b(x,\bar{x}) > 0$, the following inequality

$$f(\bar{x}) \leq f(x)$$

holds for all $x \in X$. Thus, by Definition 2.11, $\bar{x}$ is a minimizer of f.□

Proposition 2.14. Let φ be a continuous function with $\varphi(0) = 0$, F be a sublinear function and f be a (h,φ)-differentiable (h,φ)-(b,F,ρ)-pseudo-convex function on X. If $\nabla^* f(\bar{x}) = 0$, then $\bar{x}$ is a minimizer of f.

Proof . The proof of this proposition follows from Definitions 2.9 and 2.11.□

3 (h, φ)-differentiable mathematical programming problem

In this paper, we consider the following (h, φ)-differentiable programming problem:

$$P_{(h,\varphi)} \quad \begin{cases} \min f(x) \\ \text{subject to } g_j(x) \leq 0, j \in J = \{1, ..., m\} \end{cases}$$

where $f : X \to R$, $g_j : X \to R$, $j \in J$ are (h, φ)-differentiable functions defined on X. Let D denote the set of all feasible solutions of $\mathrm{P}_{(h,\varphi)}$, that is,

$$D = \{x \in X : g_j(x) \leq 0, j \in J\}.$$

Further, by $J(\bar{x})$, we denote the set of inequality constraint indices that are active at a feasible solution $\bar{x}$, that is, $J(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\}$.

Definition 3.1. A point $\bar{x} \in D$ is said to be an optimal solution of $\mathrm{P}_{(h,\varphi)}$ if and only if there exists no other feasible point $x \in D$ such that $f(x) < f(\bar{x})$.

Definition 3.2. $(\bar{x}, \bar{\mu}) \in D \times R^m$ is said to be a Karush-Kuhn-Tucker point for $\mathrm{P}_{(h,\varphi)}$ if the Karush-Kuhn-Tucker necessary optimality conditions

$$\nabla^* f(\bar{x}) \oplus \left(\bigoplus_{j=1}^{m} (\mu_j \otimes \nabla^* g_j(\bar{x})) \right) = 0, \tag{3.1}$$

$$\mu_j[\cdot]g_j(\bar{x}) = 0, j \in J(\bar{x}), \tag{3.2}$$

$$\overline{\mu} \geq 0. \tag{3.3}$$

are satisfied at $\bar{x}$ with Lagrange multiplier $\overline{\mu}$.

Now we prove the sufficient optimality conditions for the considered (h, φ)-differentiable programming problem $\mathrm{P}_{(h,\varphi)}$.

Theorem 3.3. Let $(\bar{x}, \bar{\mu}) \in D \times R^m$ be a Karush-Kuhn-Tucker point of $\mathrm{P}_{(h,\varphi)}$. Further, assume the following hypotheses are fulfilled:

a) the objective function f is (h, φ)-(b_f, F, ρ_f)-convex at $\bar{x}$ on D,
b) each inequality constraint function g_j, $j \in J(\bar{x})$, is (h, φ)-(b_{g_j}, F, ρ_{g_j})-convex at $\bar{x}$ on D,
c) $\rho_f[\cdot]d^2(x, \bar{x})[+] \left(\sum_{j=1}^{m} \bar{\mu_j}[\cdot](\rho_{g_j}[\cdot]d^2(x, \bar{x})) \right) \geq 0$.

Then $\overline{x}$ is an optimal solution of $\mathrm{P}_{(h,\varphi)}$.

Proof . By assumption, $(\bar{x}, \bar{\mu}) \in D \times R^m$ is a Karush-Kuhn-Tucker point of $\mathrm{P}_{(h,\varphi)}$. Then, by Definition 3.2, the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\bar{x}$ with Lagrange multiplier $\bar{\mu} \in R^m$. We proceed by contradiction. Suppose, contrary to the result, that $\bar{x}$ is not an optimal solution of the problem $\mathrm{P}_{(h,\varphi)}$. Hence, by Definition 3.1, there exists another $\widehat{x} \in D$, such that

$$f(\widehat{x}) < f(\bar{x}). \tag{3.4}$$

that is

$$f(\widehat{x})[-]f(\bar{x}) < 0. \tag{3.5}$$

Since $b_f(\widehat{x}, \bar{x}) > 0$, we have

$$b_f(\widehat{x}, \bar{x})[\cdot](f(\widehat{x})[-]f(\bar{x})) < 0. \tag{3.6}$$

Using hypotheses a)-c), by Definition 2.6, the following inequalities hold

$$b_f(\widehat{x}, \bar{x})[\cdot](f(\widehat{x})[-]f(\bar{x})) \geq F(\widehat{x}, \bar{x}; \nabla^* f(\bar{x}))[+](\rho_f[\cdot]d^2(\widehat{x}, \bar{x})), \tag{3.7}$$

$$b_{g_j}(\widehat{x}, \bar{x})[\cdot](g_j(\widehat{x})[-]g_j(\bar{x})) \geq F(\widehat{x}, \bar{x}; \nabla^* g_j(\bar{x}))[+](\rho_{g_j}[\cdot]d^2(\widehat{x}, \bar{x})), \quad j \in J(\bar{x}). \tag{3.8}$$

Combining (3.6) and (3.7), we get

$$F(\widehat{x}, \bar{x}; \nabla^* f(\bar{x}))[+](\rho_f[\cdot]d^2(\widehat{x}, \bar{x})) < 0. \tag{3.9}$$

Multiplying inequalities (3.8) by the corresponding Lagrange multiplier, and then adding both sides, we get, for all $x \in X$,

$$\sum_{j=1}^{m} b_{g_j}(\widehat{x}, \bar{x})[\cdot](\mu_j[\cdot]g_j(\widehat{x})[-]\mu_j[\cdot]g_j(\bar{x})) \geq \sum_{j=1}^{m} \mu_j[\cdot]F(\widehat{x}, \bar{x}; \nabla^* g_j(\bar{x}))[+]\sum_{j=1}^{m} \mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x}, \bar{x})). \tag{3.10}$$

Since $g_j(\bar{x}) = 0$, $\widehat{x}, \bar{x} \in D$, with above inequalities, we obtain

$$\sum_{j=1}^{m} \mu_j[\cdot]F(\widehat{x}, \bar{x}; \nabla^* g_j(\bar{x}))[+]\sum_{j=1}^{m} \mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x}, \bar{x})) \leq 0. \tag{3.11}$$

Thus,

$$F(\widehat{x}, \bar{x}; \bigoplus_{j=1}^{m} \mu_j \otimes \nabla^* g_j(\bar{x}))[+]\sum_{j=1}^{m} \mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x}, \bar{x})) \leq 0. \tag{3.12}$$

Combining (3.9) and (3.12), by the sublinearity of the functional F, we get

$$F\left(\widehat{x}, \bar{x}; \nabla^* f(\bar{x}) \oplus \left(\bigoplus_{j=1}^{m} (\mu_j \otimes \nabla^* g_j(\bar{x}))\right)\right)[+](\rho_f[\cdot]d^2(\widehat{x}, \bar{x}))[+]\sum_{j=1}^{m} \mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x}, \bar{x})) < 0. \tag{3.13}$$

By Karush-Kuhn-Tucker condition (3.1) together with (2.11) and hypotheses c), we get that the following inequalities

$$F\left(\widehat{x}, \bar{x}; \nabla^* f(\bar{x}) \oplus \left(\bigoplus_{j=1}^{m} (\mu_j \otimes \nabla^* g_j(\bar{x}))\right)\right)[+](\rho_f[\cdot]d^2(\widehat{x}, \bar{x}))[+]\sum_{j=1}^{m} \mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x}, \bar{x})) \geq 0 \tag{3.14}$$

which is a contradiction to (3.13). Thus, the proof of this theorem is completed. □

Theorem 3.4. Let $(\bar{x}, \bar{\mu}) \in D \times R^m$ be a Karush-Kuhn-Tucker point of $\mathrm{P}_{(h,\varphi)}$. Further, assume the following hypotheses are fulfilled:

a) the objective function f is strictly (h, φ)-(b_f, F, ρ_f)-convex at $\bar{x}$ on D,
b) each inequality constraint function g_j, $j \in J(\bar{x})$, is (h, φ)-(b_{g_j}, F, ρ_{g_j})-convex at $\bar{x}$ on D,
c) $\rho_f[\cdot]d^2(x, \bar{x})[+]\left(\sum_{j=1}^{m} \bar{\mu_j}[\cdot](\rho_{g_j}[\cdot]d^2(x, \bar{x}))\right) \geq 0.$

Then $\bar{x}$ is an optimal solution of $\mathrm{P}_{(h,\varphi)}$.

Theorem 3.5. Let $(\bar{x}, \bar{\mu}) \in D \times R^m$ be a Karush-Kuhn-Tucker point of $\mathrm{P}_{(h,\varphi)}$. Further, assume the following hypotheses are fulfilled:

a) the objective function f is (h, φ)-(b_f, F, ρ_f)-pseudo-convex at $\bar{x}$ on D,
b) $\mu_j[\cdot]g_j$, $j \in J(\bar{x})$, is (h, φ)-(b_{g_j}, F, ρ_{g_j})-quasi-convex at $\bar{x}$ on D,
c) $\rho_f[\cdot]d^2(x, \bar{x})[+]\left(\sum_{j=1}^{m} \bar{\mu_j}[\cdot](\rho_{g_j}[\cdot]d^2(x, \bar{x}))\right) \geq 0.$

Then $\bar{x}$ is an optimal solution of $\mathrm{P}_{(h,\varphi)}$.

Proof . By assumption, $(\bar{x}, \bar{\mu}) \in D \times R^m$ is a Karush-Kuhn-Tucker point of $\mathrm{P}_{(h,\varphi)}$. Then, by Definition 3.2, the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\bar{x}$ with Lagrange multiplier $\bar{\mu} \in R^m$. We proceed by contradiction. Suppose, contrary to the result, that $\bar{x}$ is not an optimal solution of the problem $\mathrm{P}_{(h,\varphi)}$. Hence, by Definition 3.1, there exists another $\widehat{x} \in D$, such that

$$f(\widehat{x}) < f(\bar{x}). \tag{3.15}$$

Since $b_f(\widehat{x},\bar{x})>0$, we have

$$b_f(\widehat{x},\bar{x})[\cdot]f(\widehat{x})<b_f(\widehat{x},\bar{x})[\cdot]f(\bar{x}). \tag{3.16}$$

Since f is (h,φ)-(b_f,F,ρ_f)-pseudo-convex at $\bar{x}$ on D, by Definition 2.9, the inequality

$$F\left(\widehat{x},\bar{x};\nabla^* f(\bar{x})\right)[+]\left(\rho_f[\cdot]d^2(\widehat{x},\bar{x})\right)<0 \tag{3.17}$$

holds. Since $\widehat{x}\in D$, the Karush-Kuhn-Tucker necessary optimality conditions (3.2)-(3.3) imply

$$g_j(\widehat{x})[-]g_j(\bar{x})\le 0,\quad j\in J(\bar{x}). \tag{3.18}$$

Using $b_{g_j}(\widehat{x},\bar{x})>0,\ j\in J(\bar{x})$, we get

$$\sum_{j=1}^{m}b_{g_j}(\widehat{x},\bar{x})[\cdot]g_j(\widehat{x})\le\sum_{j=1}^{m}b_{g_j}(\widehat{x},\bar{x})[\cdot]g_j(\bar{x}). \tag{3.19}$$

Since $\mu_j[\cdot]g_j,\ j\in J(\bar{x})$, is (h,φ)-(b_{g_j},F,ρ_{g_j})-quasi-convex at $\bar{x}$ on D, by Definition 2.8, we get

$$\sum_{j=1}^{m}F\left(\widehat{x},\bar{x};\mu_j[\cdot]\nabla^* g_j(\bar{x})\right)[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x},\bar{x}))\le 0. \tag{3.20}$$

Thus,

$$F(\widehat{x},\bar{x};\bigoplus_{j=1}^{m}(\mu_j\otimes\nabla^* g_j(\bar{x})))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x},\bar{x}))\le 0. \tag{3.21}$$

Combining (3.17) and (3.21), by the sublinear of the functional F, we get that the following inequalities

$$F\left(\widehat{x},\bar{x};\nabla^* f(\bar{x})\oplus\left(\bigoplus_{j=1}^{m}(\mu_j\otimes\nabla^* g_j(\bar{x}))\right)\right)[+](\rho_f[\cdot]d^2(\widehat{x},\bar{x}))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x},\bar{x}))<0. \tag{3.22}$$

By Karush-Kuhn-Tucker condition (3.1) together with (2.11) and hypotheses c), we get that the following inequalities

$$F\left(\widehat{x},\bar{x};\nabla^* f(\bar{x})\oplus\left(\bigoplus_{j=1}^{m}(\mu_j\otimes\nabla^* g_j(\bar{x}))\right)\right)[+](\rho_f[\cdot]d^2(\widehat{x},\bar{x}))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(\widehat{x},\bar{x}))\ge 0 \tag{3.23}$$

which is a contradiction to (4.15). Thus, the proof of this theorem is completed. □

Theorem 3.6. Let $(\bar{x},\bar{\mu})\in D\times R^m$ be a Karush-Kuhn-Tucker point of $\mathrm{P}_{(h,\varphi)}$. Further, assume the following hypotheses are fulfilled:

a) the objective function f is strictly (h,φ)-(b_f,F,ρ_f)-pseudo-convex at $\bar{x}$ on D,
b) $\mu_j[\cdot]g_j,\ j\in J(\bar{x})$, is (h,φ)-(b_{g_j},F,ρ_{g_j})-quasi-convex at $\bar{x}$ on D,
c) $\rho_f[\cdot]d^2(x,\bar{x})[+]\left(\sum_{j=1}^{m}\bar{\mu_j}[\cdot](\rho_{g_j}[\cdot]d^2(x,\bar{x}))\right)\ge 0.$

Then $\overline{x}$ is an optimal solution of $\mathrm{P}_{(h,\varphi)}$.

Example 3.7. Consider the following nonconvex optimization problem

$$\begin{aligned}&\text{minimize } f(x)=|x|\\ &\text{s.t. } g(x)=-x\le 0.\end{aligned}\qquad \text{(P1)}$$

Note that $D=\{x\in R: x\ge 0\}$ is the set of all feasible solutions of (P1). Let $F: R\times R\times R\to R$ defined by $F(x,\overline{x};\zeta)=\zeta|x|$, $h(t)=t$, $\varphi(u)=u^2$, $b_f=b_g=1$, $\rho_f=0$, $\rho_g=-1$, and $d(x,\overline{x})=|x-\overline{x}|$. Note that $\overline{x}=0$ is a feasible solution of the problem (P1). Since $\hat{f}(t)=\varphi(f(h^{-1}(t)))=t^2$ and $\hat{g}(t)=\varphi(g(h^{-1}(t)))=t^2$ are differentiable, we conclude that f,g are (h,φ)-differentiable at $\overline{x}=0$, and we obtain $\nabla^* f(\overline{x})=h^{-1}(\nabla\hat{f}(t)\mid_{t=h(\overline{x})})=h^{-1}(\nabla(t^2)\mid_{t=0})=0$ and $\nabla^* g(\overline{x})=h^{-1}(\nabla\hat{g}(t)\mid_{t=h(\overline{x})})=h^{-1}(\nabla(t^2)\mid_{t=0})=0$. Further, by the definition of a (h,φ)-(b,F,ρ)-convex function 2.6, f is (h,φ)-(b_f,F,ρ_f)-convex at $\overline{x}$ on D, and g is (h,φ)-(b_g,F,ρ_g)-convex at $\overline{x}$ on D. Then, it can also be shown that the Karush-Kuhn-Tucker optimality conditions (3.1)-(3.3) are fulfilled at $\overline{x}=0$ with Lagrange multiplier $\overline{\mu}=0$. Hence, by Theorem 3.3, $\overline{x}$ is an optimal solution of (P1) (see Figure 1).

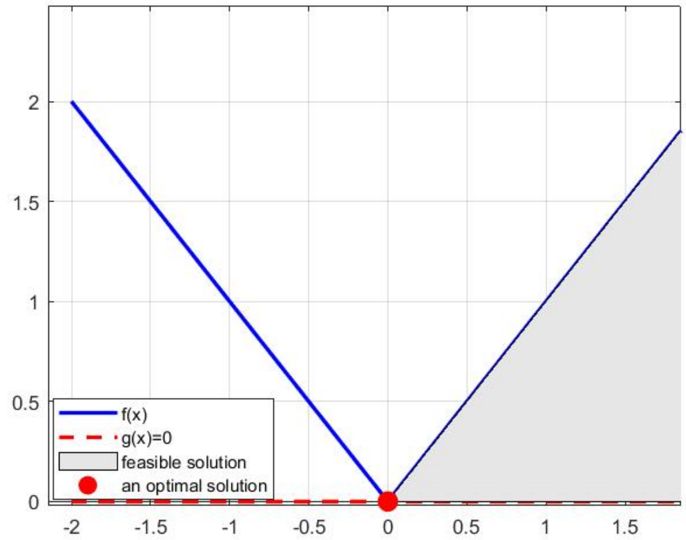

Figure 1: Graphical view of problem (P1) in Example 3.7.

4 Mond-Weir duality

In this section, for the considered (h,φ)-differentiable optimization problem $(\mathrm{P}_{(h,\varphi)})$, we give the definition of its Mond-Weir (h,φ)-dual problem $(\mathrm{D}_{(h,\varphi)})$. Then, we prove several duality results between $(\mathrm{P}_{(h,\varphi)})$ and $(\mathrm{D}_{(h,\varphi)})$ under appropriate (h,φ)-(b,F,ρ)-convexity and/or generalized (h,φ)-(b,F,ρ)-convexity hypotheses. We define the following (h,φ)-dual problem in the sense of Mond-Weir related for the (h,φ)-differentiable optimization problem $(\mathrm{P}_{(h,\varphi)})$ as follows:

$$\begin{aligned} & f(y) \to \max \\ \text{s.t.} \quad & \nabla^* f(y) \oplus \left(\bigoplus_{j=1}^{m} (\mu_j \otimes \nabla^* g_j(y)) \right) = 0 \\ & [\sum_{j=1}^{m}] \, (\mu_j [\cdot] g_j(y)) \geq 0, \; j \in J \qquad \mathrm{D}_{(h,\varphi)} \\ & y \in R^n, \mu_j \geq 0, j \in J. \end{aligned}$$

Further, let $\Psi_{(h,\varphi)} = \{(y,\mu) \in R^n \times R^m : \; \nabla^* f(y) \oplus \left(\bigoplus_{j=1}^{m} (\mu_j \otimes \nabla^* g_j(y)) \right) = 0, \; [\sum_{j=1}^{m}] \, (\mu_j[\cdot]g_j(y)) \geq 0, \; \mu_j \geq 0\}$ be the feasible solution set of the problem $(\mathrm{D}_{(h,\varphi)})$. Let us denote, $Y_{(h,\varphi)} = \{y \in R^n : \; (y,\mu) \in \Psi_{(h,\varphi)}\}$.

Theorem 4.1. (Mond-Weir weak duality). Let x and (y,λ,μ) be any feasible solutions of the problems $(\mathrm{P}_{(h,\varphi)})$ and $(\mathrm{D}_{(h,\varphi)})$, respectively. Further, assume that at least one of the following hypotheses is fulfilled:

a) objective function f is (h,φ)-(b_f,F,ρ_f)-convex at y on $D \cup Y_{(h,\varphi)}$, each constraint function $g_j, j \in J$ is (h,φ)-(b_{g_j},F,ρ_{g_j})-convex at y on $D \cup Y_{(h,\varphi)}$, with $\rho_f[\cdot]d^2(x,\bar{x})[+] \left(\sum_{j=1}^{m} \bar{\mu_j}[\cdot](\rho_{g_j}[\cdot]d^2(x,\bar{x})) \right) \geq 0$.

b) f is a (h,φ)-(b_f,F,ρ_f)-pseudo-convex function at y on $D \cup Y_{(h,\varphi)}$ and $\mu_j[\cdot]g_j, j \in J$ is a (h,φ)-(b_g,F,ρ_g)-quasi-convex function at y on $D \cup Y_{(h,\varphi)}$, with $\rho_f[\cdot]d^2(x,\bar{x})[+] \left(\sum_{j=1}^{m} \bar{\mu_j}[\cdot](\rho_{g_j}[\cdot]d^2(x,\bar{x})) \right) \geq 0$.

Then $f(x) \not< f(y)$.

Proof . Let x and (y,μ) be any feasible solutions of the problems $(\mathrm{P}_{(h,\varphi)})$ and $(\mathrm{D}_{(h,\varphi)})$, respectively. The proof of this theorem under hypothesis a). If $x = y$, then the weak duality trivially holds. Now, we prove the weak duality theorem when $x \neq y$. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$f(x) < f(y) \tag{4.1}$$

holds. That is

$$f(x)[-]f(y) < 0. \tag{4.2}$$

Since $b_f(x,y) > 0$, therefore, we have

$$b_f(x,y)[\cdot](f(x)[-]f(y)) < 0. \tag{4.3}$$

Since f is (h,φ)-(b_f, F, ρ_f)-convex at y on $D \cup Y_{(h,\varphi)}$, each constraint function $g_j, j \in J$ is (h,φ)-(b_{g_j}, F, ρ_{g_j})-convex at y on $D \cup Y_{(h,\varphi)}$, by Definition 2.6, the following inequalities

$$b_f(x,y)[\cdot](f(x)[-]f(y)) \geq F(x,y;\nabla^* f(y))[+](\rho_f[\cdot]d^2(x,y)), \tag{4.4}$$

$$b_{g_j}(x,y)[\cdot](g_j(x)[-]g_j(y)) \geq F(x,y;\nabla^* g_j(x))[+](\rho_{g_j}[\cdot]d^2(x,y)), \quad j \in J(y) \tag{4.5}$$

hold. Combining (4.3) and (4.4), we get

$$F(x,y;\nabla^* f(y))[+](\rho_f[\cdot]d^2(x,y)) < 0. \tag{4.6}$$

Multiplying inequalities (4.5) by the corresponding Lagrange multiplier

$$b_{g_j}(x,y)[\cdot](\mu_j[\cdot]g_j(x)[-]\mu_j[\cdot]g_j(y)) \geq \mu_j[\cdot]F(x,y;\nabla^* g_j(y))[+]\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y)) \; j \in J. \tag{4.7}$$

Using second condition in $\mathrm{D}_{(h,\varphi)}$ together with $x \in D$ and $y \in D$, we get

$$\mu_j[\cdot]F(x,y;\nabla^* g_j(y))[+]\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y)) \leq 0, \; j \in J. \tag{4.8}$$

Thus,

$$F(x,y;\bigoplus_{j=1}^{m}\mu_j \otimes \nabla^* g_j(y))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y)) \leq 0. \tag{4.9}$$

Adding both sides of (4.6) and (4.9), we obtain that the following inequality

$$F\left(x,y;\nabla^* f(y) \oplus \left(\bigoplus_{j=1}^{m}(\mu_j \otimes \nabla^* g_j(y))\right)\right)[+](\rho_f[\cdot]d^2(x,y))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y)) < 0. \tag{4.10}$$

Using first condition in $\mathrm{D}_{(h,\varphi)}$ together with (2.11) and $(\rho_f[\cdot]d^2(x,y))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y)) \geq 0$, we get that the following inequalities

$$F\left(x,y;\nabla^* f(y) \oplus \left(\bigoplus_{j=1}^{m}(\mu_j \otimes \nabla^* g_j(y))\right)\right)[+](\rho_f[\cdot]d^2(x,y))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y)) \geq 0 \tag{4.11}$$

which is a contradiction to (4.10). Thus, the proof of the Mond-Weir weak duality theorem between the optimization problems $(\mathrm{P}_{(h,\varphi)})$ and $(\mathrm{D}_{(h,\varphi)})$ is completed under hypothesis a).

The proof of this theorem under hypothesis b). We proceed by contradiction. Suppose, contrary to the result, that (4.3) holds. Since the function f is a (h,φ)-(b_f, F, ρ_f)-pseudo-convex function at y on $D \cup Y_{(h,\varphi)}$, by Definition 2.9, the inequality

$$F\left(x,y;\nabla^* f(y)\right)[+]\left(\rho_f[\cdot]d^2(x,y)\right) < 0 \tag{4.12}$$

holds. Since $\mu_j[\cdot]g_j, j \in J$ is a (h,φ)-(b_g, F, ρ_g)-quasi-convex function at y on $D \cup Y_{(h,\varphi)}$, by the foregoing above relations, Definition 2.8, we get

$$\sum_{j=1}^{m} F\left(x,y;\mu_j[\cdot]\nabla^* g_j(y)\right)[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y)) \leq 0. \tag{4.13}$$

Thus,

$$F(x,y;\bigoplus_{j=1}^{m}(\mu_j \otimes \nabla^* g_j(y)))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y)) \leq 0. \tag{4.14}$$

Combining (4.12) and (4.14), we get that the following inequalities

$$F\left(x,y;\nabla^{*}f(y)\oplus\left(\bigoplus_{j=1}^{m}(\mu_j\otimes\nabla^{*}g_j(y))\right)\right)[+](\rho_f[\cdot]d^2(x,y))[+]\sum_{j=1}^{m}\mu_j[\cdot](\rho_{g_j}[\cdot]d^2(x,y))<0 \tag{4.15}$$

which is a contradiction to (4.11). This means that the proof of the Mond-Weir weak duality theorem between optimization problems ($\mathrm{P}_{(h,\varphi)}$) and ($\mathrm{D}_{(h,\varphi)}$) is completed under hypothesis b). □

If stronger (generalized) (h,φ)-(F,ρ)-convexity hypotheses are imposed on the functions constituting the considered (h,φ)-differentiable programming problem, then the stronger result is true.

Theorem 4.2. (Mond-Weir weak duality). Let x and (y,λ,μ) be any feasible solutions of the problems ($\mathrm{P}_{(h,\varphi)}$) and ($\mathrm{D}_{(h,\varphi)}$), respectively. Further, assume that at least one of the following hypotheses is fulfilled:

a) objective function f is strictly (h,φ)-(b_f,F,ρ_f)-convex at y on $D\cup Y_{(h,\varphi)}$, each constraint function $g_j, j\in J$ is (h,φ)-(b_{g_j},F,ρ_{g_j})-convex at y on $D\cup Y_{(h,\varphi)}$, with
$\rho_f[\cdot]d^2(x,\bar{x})[+]\left(\sum_{j=1}^{m}\bar{\mu_j}[\cdot](\rho_{g_j}[\cdot]d^2(x,\bar{x}))\right)\geq 0.$

b) f is a (h,φ)-(b_f,F,ρ_f)-pseudo-convex function at y on $D\cup Y_{(h,\varphi)}$, and $\mu_j[\cdot]g_j, j\in J$ is a (h,φ)-(b_g,F,ρ_g)-quasi-convex function at y on $D\cup Y_{(h,\varphi)}$, with
$\rho_f[\cdot]d^2(x,\bar{x})[+]\left(\sum_{j=1}^{m}\bar{\mu_j}[\cdot](\rho_{g_j}[\cdot]d^2(x,\bar{x}))\right)\geq 0.$

Then $f(x)\not< f(y)$.

Theorem 4.3. (Mond-Weir strong duality). Let $\overline{x}\in D$ be an optimal solution of the optimization problem ($\mathrm{P}_{(h,\varphi)}$). Further, assume that the Kuhn-Tucker constraint qualification [32] be satisfied at $\overline{x}$. Then there exist $\overline{\mu}\in R^m$, $\overline{\mu}\geqq 0$ such that $(\overline{x},\overline{\mu})$ is feasible for the problem ($\mathrm{D}_{(h,\varphi)}$) and the objective functions of ($\mathrm{P}_{(h,\varphi)}$) and ($\mathrm{D}_{(h,\varphi)}$) are equal at these points. If also all hypotheses of the Mond-Weir weak duality (Theorem 4.1) Theorem 4.2 are satisfied, then $(\overline{x},\overline{\mu})$ is an optimal solution of a maximum type in the problem ($\mathrm{D}_{(h,\varphi)}$).

Proof . Since $\overline{x}\in D$ is an optimal solution of the optimization problem ($\mathrm{P}_{(h,\varphi)}$) and the Kuhn-Tucker constraint qualification [32] is satisfied at $\overline{x}$, by Definition3.2, there exist $\overline{\mu}\in R^m$, $\overline{\mu}\geq 0$ such that $(\overline{x},\overline{\mu})$ is an optimal solution of the problem ($\mathrm{D}_{(h,\varphi)}$). This means that the optimal value of ($\mathrm{P}_{(h,\varphi)}$) and ($\mathrm{D}_{(h,\varphi)}$) are equal. If we assume that all hypotheses of the Mond-Weir weak duality (Theorem 4.1) Theorem 4.2 are fulfilled, $(\overline{x},\overline{\mu})$ is an optimal solution of a maximum type in the dual problem ($\mathrm{D}_{(h,\varphi)}$) in the sense of Mond-Weir. □

5 Conclusion

In this paper, a nonlinear programming problem has been considered in which the involved functions are not necessarily differentiable, but they are (h,φ)-differentiable. The concept of (h,φ)-(b,F,ρ)-convexity has been defined for (h,φ)-differentiable mathematical programming problem. Several sufficient optimality conditions have been derived for such nonlinear optimization problems with (h,φ)-differentiable functions under (generalized) (h,φ)-(b,F,ρ)-convexity hypotheses. Further, the so-called Mond-Weir duality theory has been investigated for (h,φ)-differentiable optimization problem. Various duality theorems between the (h,φ)-differentiable optimization problem and its Mond-Weir (h,φ)-dual problem have been proved also under (generalized) (h,φ)-convexity hypotheses.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar optimality results for other classes of (h,φ)-differentiable optimization problems. We shall investigate these questions in subsequent papers.

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