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Fixed point theorems for θ-Ω-contraction on (α, η)-b-rectangular metric spaces

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Abstract

In this paper, we consider a new extension of the Banach contraction principle, θ-Ω-contraction inspired by the concept of θ-contraction in (α, η)-b-rectangular metric spaces to study the existence and uniqueness of fixed point theorems for the mappings in metric spaces. Moreover, we discuss some illustrative examples to highlight the realized improvements.

Keywords: Fixed point, b-rectangular metric space, generalized θ-Ω-contraction
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1 Introduction

In recent times, the Banach contraction principle [1] was attracted by many authors (see [2, 7, 8, 9, 11, 14, 20, 21, 22]). In 2014, Jleli et al. [7, 8] introduced the notion of θ-contraction. By using θ-contractions, Jleli et al. [7, 8] proved a fixed point theorem which generalizes Banach contraction principle in a different way than in the known results from the literature. Later, Kari et al. [10, 12, 13, 19] proved new type fixed point theorems in rectangular metric space and generalized asymmetric metric space by using a modified generalized θ-contraction.

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, b-rectangular metric spaces were introduced by George et al. [3], in such a way that triangle inequality is replaced by the b-triangle inequality: $d(x, y) \leq s(d(x, u)+d(u, v)+d(v, y))$ for all pairwise distinct points x, y, u, v. Any metric space is a b-rectangular metric space but in general, b-rectangular metric space might not be a metric space. Various fixed point results were established on such spaces, the readers can refer to [12, 15, 17, 18].

In 2014, Hussain and Salimi [6] introduced the notion of an α-GF-contraction and stated fixed point theorems for α-GF-contractions. On the other hand, Hussain et al. [4] established some new fixed point theorems for generalized α-η-GF-contractions mappings in complete b-metric spaces.

In this paper, we introduce the notion of a generalized α-η-θ-Ω-contraction in b-rectangular metric space. Also, examples are given to illustrate the obtained results we derive some useful corollaries of these results.

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2 Preliminaries

Definition 2.1. [3]. Let X be a nonempty set, $s \geq 1$ be a given real number and $d : X \times X \to [0, +\infty[$ be a function such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y,

(1) $d(x,y) = 0$ if only if $x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ (b-rectangular inequality).

Then (X, d) is called a b-rectangular metric space.

Example 2.2. [12]. Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \{2,3,4,5,6,7\}\}$ and $B = [1,2]$. Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x) \ for \ all \ x, y \in X; \\ d(x,y) = 0 \Leftrightarrow y = x \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = d\left(\frac{1}{6}, \frac{1}{7}\right) = 0,05 \\ d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{3}, \frac{1}{7}\right) = d\left(\frac{1}{5}, \frac{1}{6}\right) = 0,08 \\ d\left(\frac{1}{2}, \frac{1}{6}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{7}\right) = 0,4 \\ d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{6}\right) = d\left(\frac{1}{4}, \frac{1}{7}\right) = 0,24 \\ d\left(\frac{1}{2}, \frac{1}{7}\right) = d\left(\frac{1}{3}, \frac{1}{5}\right) = d\left(\frac{1}{4}, \frac{1}{6}\right) = 0,15 \\ \quad d(x,y) = (|x-y|)^2 \ \ otherwise. \end{cases}$$

Then (X, d) is a b-rectangular metric space with coefficient $s = 3$.

Definition 2.3. [3] Let (X, d) is a b-rectangular metric space and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X and $x \in X$.

(i) We say that $\{x_n\}_{n \in \mathbb{N}}$ converges to x if
$$\lim_{n \to +\infty} d(x, x_n) = 0.$$

(ii) We say that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy if
$$\lim_{n,m \to +\infty} d(x_n, x_m) = 0.$$

Definition 2.4. [3]. Let (X, d) be a b-rectangular metric space. Then X is said to be complete if every Cauchy sequence $\{x_n\}_n$ in X converges to $x \in X$.

Lemma 2.5. [15] Let (X, d) be a b-rectangular metric space.

(a) Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have
$$\frac{1}{s} d(x,y) \leq \lim_{n \to \infty} \inf d(x_n, y_n) \leq \lim_{n \to \infty} \sup d(x_n, y_n) \leq s d(x,y).$$

(b) If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for any $m, n \in \mathbb{N}$, $m \neq n$, converging to $x \neq y$, then
$$\frac{1}{s} d(x,y) \leq \lim_{n \to \infty} \inf d(x_n, y) \leq \lim_{n \to \infty} \sup d(x_n, y) \leq s d(x,y),$$
for all $x \in X$.

Lemma 2.6. [12] Let (X,d) be a b-rectangular metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \lim_{n\to\infty} d(x_n,x_{n+2}) = 0. \tag{2.1}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon>0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$\varepsilon \le \lim_{k\to\infty} \inf d\left(x_{m_{(k)}},x_{n_{(k)}}\right) \le \lim_{k\to\infty} \sup d\left(x_{m_{(k)}},x_{n_{(k)}}\right) \le s\varepsilon,$$

$$\varepsilon \le \lim_{k\to\infty} \inf d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \le \lim_{k\to\infty} \sup d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \le s\varepsilon,$$

$$\varepsilon \le \lim_{k\to\infty} \inf d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right) \le \lim_{k\to\infty} \sup d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right) \le s\varepsilon,$$

$$\frac{\varepsilon}{s} \le \lim_{k\to\infty} \inf d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right) \le \lim_{k\to\infty} \sup d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right) \le s^2\varepsilon.$$

In this section, we give basic definitions of concepts concerning a θ-ϕ-contraction in the setting of generalized metric spaces. The following definition was given by Jleli et al. in [7].

Definition 2.7. [7] Let Θ_c be the family of all functions $\theta :]0,+\infty[\to]1,+\infty[$ such that

(θ_1) θ is increasing;

(θ_2) for each sequence $(x_n) \subset]0,+\infty[$,

$$\lim_{n\to 0} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

(θ_3) θ is continuous.

Definition 2.8. [7] Let Θ_G be the family of all functions $\theta :]0,+\infty[\to]1,+\infty[$ such that

(θ_1) θ is increasing;

(θ_2) for each sequence $(x_n) \subset]0,+\infty[$,

$$\lim_{n\to 0} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

(θ_3) there exist $r \in]0,1[$ and $l \in]0,+\infty[$ such hat $\lim_{t\to 0} \frac{\theta(t)-1}{t^r} = l$.

Definition 2.9. [7] Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a θ-contraction if there exist $\theta \in \Theta$ and $k \in]0,1[$ such that for any $x,y \in X$,

$$d(Tx,Ty) > 0 \Rightarrow \theta[d(Tx,Ty)] \le [\theta(d(x,y))]^k.$$

Theorem 2.10. [7] Let (X,d) be a complete generalized metric space and $T : X \to X$ be a θ-ϕ-contraction. Then T has a unique fixed point.

In 2014, Hussain et al. [4] proposed a weaker definition that completeness, which is called α-completeness for generalized metric spaces.

Definition 2.11. [4] Let $T : X \to X$ and $\alpha,\eta : X\times X \to [0,+\infty[$. We say that T is a triangular (α,η)-admissible mapping if
(T_1) $\alpha(x,y) \ge 1 \Rightarrow \alpha(Tx,Ty) \ge 1$, $x,y \in X$;
(T_2) $\eta(x,y) \le 1 \Rightarrow \eta(Tx,Ty) \le 1$, $x,y \in X$;
(T_3) $\left\{\begin{array}{l} \alpha(x,y) \ge 1 \\ \alpha(y,z) \ge 1 \end{array}\right. \Rightarrow \alpha(x,z) \ge 1$ for all $x,y,z \in X$;

(T_4) $\left\{\begin{array}{l} \eta(x,y) \le 1 \\ \eta(y,z) \le 1 \end{array}\right. \Rightarrow \eta(x,z) \le 1$ for all $x,y,z \in X$.

Definition 2.12. [4] Let (X, d) be a b-rectangular metric space and $\alpha, \eta : X \times X \to [0, +\infty[$ be two mappings.

(a) T is an α-continuous mapping on (X, d) if for a given point $x \in X$ and a sequence (x_n) in X, $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, imply that $Tx_n \to Tx$.

(b) T is an η sub-continuous mapping on (X, d) if for a given point $x \in X$ and a sequence (x_n) in X, $x_n \to x$ and $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, imply that $\mathrm{T}x_n \to Tx$.

(c) T is an (α, η)-continuous mapping on (X, d) if for a given point $x \in X$ and a sequence (x_n) in X, $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, imply that $Tx_n \to Tx$.

Hussain et al. [5] gave the following definition.

Definition 2.13. [5] Let (X, d) be a generalized metric space and $\alpha, \eta : X \times X \to [0, +\infty[$ be two mappings. The space X is said to be

(a) α-complete, if every Cauchy sequence (x_n) in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in X;

(b) η-sup-complete if every Cauchy sequence (x_n) in X with $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, converges in X;

(c) (α, η)-complete if every Cauchy sequence (x_n) in X with $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, converges in X.

Definition 2.14. [5] Let (X, d) be a generalized metric space and $\alpha, \eta : X \times X \to [0, +\infty[$ be two mappings.

(a) (X, d) is α-regular if $x_n \to x$, where $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, implies $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

(b) (X, d) is η-sub-regular, if $x_n \to x$, where $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, implies $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.

(c) (X, d) is (α, η)-regular if $x_n \to x$, where $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$, imply that $\alpha(x_n, x) \geq 1$ or $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.

3 Main results

Definition 3.1. Let Δ denote the set of all functions $\Omega : \mathbf{R}_+^5 \to \mathbb{R}_+$ satisfying: for all $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}_+$ with $t_1 t_2 t_3 t_4 t_5 = 0$ there exists $\pi \in]0, 1[$ such that $\Omega(t_1, t_2, t_3, t_4, t_5) = \pi$.

Example 3.2. If $\Omega(t_1, t_2, t_3, t_4, t_5) = \min\{t_1, t_2, t_3, t_4, t_5\} + \pi$ where $\pi \in]0, 1[$ then $\Delta \in \Omega$.

Example 3.3. If $\Omega(t_1, t_2, t_3, t_4, t_5) = \frac{\min\{t_1, t_2, t_3, t_4, t_5\}}{\max\{t_1, t_2, t_3, t_4, t_5 + 1\}} + \pi$ where $\pi \in]0, 1[$ then $\Delta \in \Omega$.

In this paper, we present the concept θ-Ω-contraction in generalize metric spaces and we prove some fixed point results for such spaces.

Definition 3.4. Let $d(X, d)$ be a (α, η)-b-rectangular metric space and T be a self mapping on X. Suppose that $\alpha, \eta : X \times X \to [0, +\infty[$ are two functions. We say that T is an (α, η)-Ω-θ_C-contraction if for all $x, y \in X$ with $(\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1)$ and $d(Tx, Ty) > 0$ we have

$$\theta\left(s^2 d(Tx, Ty)\right) \leq [\theta(M(x, y))]^{\Omega(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(T^2x, y))} \tag{3.1}$$

where $\theta \in \Theta_C, \Omega \in \Delta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), d(Tx, y), d\left(T^2x, y\right), d\left(T^2x, Ty\right), d\left(T^2x, Tx\right)\right\}.$$

Theorem 3.5. Let (X, d) be an (α, η)-complete b-rectangular metric and let $\alpha, \eta : X \times X \to [0, +\infty[$ be two functions. Let $T : X \times X \to X$ be a self mapping satisfying the following conditions:

(i) T is a triangular (α, η)-admissible mapping;

(ii) T is an (α,η)-θ-Ω-contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$;

(iv) $T$is (α,η)-continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ for all $x, y \in X$.

Proof . Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$. Since T is a triangular (α,η)-admissible mapping, $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) \geq 1 = \alpha(x_1, x_2)$ or $\eta(x_0, x_1) = \eta(x_0, Tx_0) \leq 1 \Rightarrow \alpha(Tx_0, Tx_1) \leq 1 = \alpha(x_1, x_2)$.

Continuing this process, we have $\alpha(x_{n-1}, x_n) \geq 1$ or $\eta(x_{n-1}, x_n) \leq 1$, for all $n \in \mathbb{N}$. By (T_3) and (T_4), one has

$$\alpha(x_m, x_n) \geq 1 \text{ or } \eta(x_m, x_n) \leq 1, \qquad \forall m, n \in \mathbb{N},\ m \neq n. \tag{3.2}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = Tx_{n_0}$. Then x_{n_0} is a fixed point of T and the proof is finished. Hence we assume that $x_n \neq Tx_n$, i.e., $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. We have

$$x_n \neq x_m,\ \forall m, n \in \mathbb{N}, m \neq n. \tag{3.3}$$

Indeed, suppose that $x_n = x_m$ for some $m = n + k > n$ Then we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$. Denote $d_m = d(x_m, x_{m+1})$. Then (3.1) implies that

$$\begin{aligned}\theta(d_n) = \theta(d_m) &= \theta(d(Tx_{m-1}, Tx_m)) \\ &\leq \left(s^2 Tx_{m-1}, Tx_m\right) \\ &\leq (\theta M(x_{m-1}, x_m))^{\Omega(d_{m-1}, d_{m-1}, d_m, 0, d_{m+1})},\end{aligned}$$

where

$$\begin{aligned}M(x_{m-1}, x_m) \quad = \quad & \{d(x_{m-1}, x_m), d(x_{m-1}, x_m), d(x_m, x_{m+1}), d(x_m, x_m), d(x_{m+1}, x_m), \\ & d(x_{m+1}, x_{m+1}), d(x_{m+1}, x_m), d(x_{m+1}, x_m), d(x_{m+1}, x_{m+1})\}.\end{aligned}$$

Then $M(x_{m-1}, x_m) = \max\{, d(x_{m-1}, x_m), , d(x_m, x_{m+1})\}$ and there exists $\pi \in]0,1[$ such that $\Omega(d_{m-1}, d_m, 0, d_{m+1}) = \pi$. If $M(x_{m-1}, x_m) = d(x_m, x_{m+1})$, then we have

$$\theta(d_m) \leq [\theta(d_m)]^{\pi} < \theta(d_m).$$

This is a contradiction. So $M(x_{m-1}, x_m) = d(x_{m-1}, x_m)$ and $d_n = d_m < d_{m-1}$. Continuing this process, we can prove that $d_n = d_m < d_m < d_{m-1} < d_{m-2} < .. < d_n$, which is a contradiction. Thus we can assume that (3.2) and (3.3) hold. Letting $x = x_{n-1}$ and $y = x_n$ in (3.1) for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &\leq \theta\left(s^2 d(x_n, x_{n+1})\right) \\ &\leq (\theta(d(x_{n-1}, x_n)))^{\lambda}.\end{aligned}$$

Repeating this step, we conclude that

$$\theta(d(x_n, x_{n+1})) \leq (\theta(d(x_{n-1}, x_n)))^{\pi} \leq (\theta(d(x_{n-2}, x_{n-1})))^{\lambda^2} \leq ... \leq \theta(d(x_0, x_1))^{\lambda^n}.$$

By using (θ_1) and (θ_3), we get

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{3.4}$$

Therefore, $d(x_n, x_{n+1})_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = \alpha$. Now, we claim that $\alpha = 0$. Arguing by contraction, we assume that $\alpha > 0$. Since $d(x_n, x_{n+1})_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, we have $d(x_n, x_{n+1}) \geq \alpha$, for all $n \in \mathbb{N}$. By the property of θ, we get

$$1 < \theta(\alpha) \leq \theta(d(x_0, x_1))^{\pi^n}. \tag{3.5}$$

By letting $n \to \infty$ in (3.5), we obtain $1 < \theta(\alpha) \leq 1$. This is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

Letting $x = x_{n-1}$ and $y = x_{n+1}$ in (3.1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\theta(d(x_n, x_{n+2})) &\leq \theta\left(d\left(s^2 x_n, x_{n+2}\right)\right)\\ &\leq (\theta M(x_{n-1}, x_{n+1}))^{\Omega(d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+1}))}\\ &= (\theta M(x_{n-1}, x_{n+1}))^{\Omega(d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1}),0)},\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1}, x_{n+1}) = \max\{&d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_n), d(x_{n+1}, x_{n+1}),\\ &d(x_{n+1}, x_n), d(x_{n+1}, x_n)\}.\end{aligned}$$

Since $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$,

$$M(x_{n-1}, x_{n+1}) = \max\{, d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}$$

and there exists $\pi \in]0,1[$ such that

$$\Omega(d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1}),0) = \pi.$$

So we have

$$\theta(d(x_n, x_{n+2})) \leq [\theta(\max\{, d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\})]^{\pi}. \tag{3.7}$$

Take $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Then one can write $\theta(a_n) \leq [\theta(\max\{, b_{n-1}))]^{\pi}$. By (θ_1) and (θ_3), we get $a_n < \max\{a_{n-1}, b_{n-1}\}$. By (3.4), we have $b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}$, which implies that

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}, \ \forall n \in \mathbb{N}.$$

Therefore, the sequence $\max\{a_{n-1}, b_{n-1}\}_{n\in\mathbb{N}}$ is monotone non-increasing. Thus there exists $\lambda \geq 0$ such that $\lim_{n\to\infty} \max\{a_n, b_n\} = \lambda$. By (3.6), we assume that $\lambda > 0$ and then we get

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\}.$$

Taking the $\limsup_n \to \infty$ in (3.7), and using the properties of θ_3, we obtain

$$\theta\left(\lim_{n\to\infty} \sup a_n\right) < \theta\left(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}\right).$$

Therefore, $\theta(\lambda) < \theta(\lambda)$. By (θ_1) and (θ_3), we get $\lambda < \lambda$. This is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{3.8}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n\to\infty} d(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary. By Lemma 2.6, there is an $\varepsilon > 0$ such that for an integer k there exist two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ $m_{(k)} >$, $n_{(k)} > k$, such that

$$I)\ \varepsilon \leq \lim_{k\to\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq \lim_{k\to\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon,$$

$$II)\ \varepsilon \leq \lim_{k\to\infty} \inf d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\varepsilon,$$

$$III)\ \varepsilon \leq \lim_{k\to\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq s\varepsilon,$$

$$IV)\ \frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq \lim_{k\to\infty} \sup d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq s^2\varepsilon.$$

From (3.1) and by setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ we have

$$M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \max\{d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), d\left(x_{m_{(k)+2}}, x_{m_{(k)+1}}\right), d\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)+2}}, x_{n_{(k)+1}}\right), d\left(x_{m_{(k)+2}}, x_{m_{(k)+1}}\right)\}.$$

Taking the limit as $k \to \infty$ and using Lemma 2.5, we have

$$\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \le \max\{s\varepsilon, 0, 0, s\varepsilon, s\varepsilon, s\varepsilon, s\varepsilon)\} = s\varepsilon.$$

Now, letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (3.1), we obtain

$$\begin{aligned}
&\theta\left[d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] \le \theta\left[s^2 d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] \\
&\le \left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^{\Omega\left[d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, Tx_{m_{(k)}}\right), d\left(x_{n_{(k)}}, Tx_{n_{(k)}}\right), d\left(Tx_{m_{(k)}}, x_{n_{(k)}}\right), d\left(T^2 x_{m_{(k)}}, x_{n_{(k)}}\right)\right]} \\
&= \left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^{\Omega\left[d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), d\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right)\right]}.
\end{aligned}$$

Since Ω is a continuous function,

$$\begin{aligned}
&\lim_{k\to\infty} \Omega\left[M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right] \\
&= \lim_{k\to\infty} \Omega\left[d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), d\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right)\right] \\
&= \Omega\left[\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), d\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right)\right] \\
&\le \Omega\left[s\varepsilon, 0, 0, s\varepsilon, s\varepsilon\right].
\end{aligned}$$

So there exists $\pi \in]0,1[$ such that $\Omega\left[s\varepsilon, 0, 0, s\varepsilon, s\varepsilon\right] = \pi$. Then

$$\theta\left(d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right)\left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^{\pi}.$$

Letting $k \to \infty$ in the above inequality and applying the continuity of θ, we have

$$\theta\left(d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \le \left[\theta\left(\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^{\pi}.$$

Therefore, $\theta(\varepsilon) \le [\theta(s\varepsilon)]^{\pi} < \theta(\varepsilon)$, which is a contradiction. Thus $\lim_{n,m\to\infty} d\left(x_m, x_n\right) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X,d), there exists $z \in X$ such that

$$\lim_{n\to\infty} d\left(x_n, z\right) = 0.$$

Now, we show that $d(Tz,z) = 0$. Arguing by contradiction, we assume that $d(Tz,z) > 0$. Since $x_n \to z$ as $n \to \infty$ for all $n \in \mathbf{N}$, from Lemma 2.5, we conclude that

$$\frac{1}{s} d\left(z, Tz\right) \le \lim_{n\to\infty} \sup d\left(Tx_n, Tz\right) \le s d\left(z, Tz\right). \tag{3.9}$$

Hence

$$s d\left(z, Tz\right) = s^2 \frac{1}{s} d\left(z, Tz\right) \le \lim_{n\to\infty} \sup s^2 d\left(Tx_n, Tz\right). \tag{3.10}$$

Since T is (α,η)-continuous, we conclude that $\lim_{n\to\infty} Tx_n = Tz$. Then

$$\lim_{n\to\infty} d\left(Tx_n, Tz\right) = d\left(z, Tz\right). \tag{3.11}$$

Letting $x = x_n$ and $y = z$ in (3.1), we obtain

$$\begin{aligned}
\theta\left(s d\left(z, Tz\right)\right) &\le \theta\left(s^2 d\left(Tx_n, Tz\right)\right) \qquad (3.12)\\
&\le \left[\theta\left(M\left(x_n, z\right)\right)\right]^{\Omega\left(d\left(x_n, Tx_n\right), d\left(z, Tz\right), d\left(x_n, Tz\right), d\left(z, Tx_n\right), d\left(T^2 x_n, z\right)\right)},
\end{aligned}$$

$$\begin{aligned} M\left(x_n, z\right) &= \max\left\{d\left(x_n, z\right), d\left(x_n, Tx_n\right), d\left(z, Tz\right), d\left(Tx_n, z\right), d\left(T^2x_n, z\right), d\left(T^2x_n, Tz\right), d\left(T^2x_n, Tx_n\right)\right\} \\ &= d\left(z, Tz\right) \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \Omega(d\left(x_n, Tx_n\right), d\left(z, Tz\right), d\left(x_n, Tz\right), d\left(z, Tx_n\right), d\left(T^2x_n, z\right)) \\ &= \Omega(\lim_{n\to\infty} d\left(x_n, Tx_n\right), d\left(z, Tz\right), d\left(x_n, Tz\right), d\left(z, Tx_n\right), d\left(T^2x_n, z\right)) \\ &\leq \Omega(0, d\left(z, Tz\right), d\left(z, Tz\right), d\left(z, Tz\right), 0). \end{aligned}$$

Then there exists $\pi \in]0,1[$ such that $\theta\left(sd\left(z, Tz\right)\right) \leq \theta\left(sd\left(z, Tz\right)\right)^{\pi}$. This is a contradiction. So $z = Tz$. Now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Then we have

$$d\left(z, u\right) = d\left(Tz, Tu\right) > 0.$$

Letting $x = z$ and $y = u$ (3.1), we have

$$\begin{aligned} \theta\left(d\left(z, u\right)\right) &= \theta\left(d\left(Tu, Tz\right)\right) \\ &\leq \theta\left(s^2 d\left(Tz, Tu\right)\right) \\ &\leq \left[\theta\left(M\left(z, u\right)\right)\right]^{\Omega(d(z,u), d(z,Tz), d(u,Tu), d(u,Tz), d\left(T^2z,u\right))} \\ &= \left[\theta\left(M\left(z, u\right)\right)\right]^{\Omega(d(z,u), d(z,z), d(u,u), d(u,z), d(z,u))} \\ &= \left[\theta\left(M\left(z, u\right)\right)\right]^{\Omega(d(z,u), 0, 0, d(u,z), d(z,u))} \\ &= \left[\theta\left(M\left(z, u\right)\right)\right]^{\pi}, \end{aligned}$$

where

$$\begin{aligned} M\left(z, u\right) &= \max\{d\left(z, u\right), d\left(z, Tz\right), d\left(u, Tu\right), d\left(Tz, u\right), d\left(T^2z, Tz\right), d\left(T^2z, u\right), d\left(T^2z, Tu\right)\} \\ &= \max\{d\left(z, u\right), d\left(z, z\right), d\left(u, u\right), d\left(z, u\right), d\left(z, z\right), d\left(z, u\right), d\left(z, u\right)\} \\ &= d(z, u). \end{aligned}$$

Therefore, we have $\theta\left(d\left(z, u\right)\right) \leq \left[\theta\left(d\left(z, u\right)\right)\right]^{\pi} < \theta\left(d\left(z, u\right)\right)$, which implies that $d\left(z, u\right) < d\left(z, u\right)$. This is a contradiction. Therefore, $u = z$. □

Corollary 3.6. Let (X, d) be an (α, η)-complete b-rectangular metric space and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. Let $T : X \times X \to X$ be a self mapping satisfying the following:

(i) $\theta[s^2 d(Tx, Ty)] \leq [\theta(M(x, y))]^k, k \in (0, 1), \quad \theta \in \Theta_C$.

(ii) T is a triangular (α, η)-admissible mapping;

(iii) T is an (α, η)-θ-Ω-contraction;

(iv) there exists $x_0 \in X$ such that $\alpha\left(x_0, Tx_0\right) \geq 1$ or $\eta\left(x_0, Tx_0\right) \leq 1$;

(v) T is (α, η)-continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha\left(x, y\right) \geq 1$ or $\eta\left(x, y\right) \leq 1$ for all $x, y \in X$.

Theorem 3.7. Let $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions and (X, d) be an (α, η)-b-rectangular complete metric space. Let $T : X \to X$ be a mapping satisfying the following conditions:

(i) T is a triangular (α, η)-admissible mapping;

(ii) T is an (α, η)-Ω-θ-contraction;

(iii) $\alpha\left(z, Tz\right) \geq 1$ or $\eta\left(z, Tz\right) \leq 1$, for all $z \in$ Fix (T).

Then T has the property P, $(T^n x = Tx)$.

Proof . Let $z \in$ Fix (T^n) for some fixed $n > 1$. Since $\alpha(z, Tz) \geq 1$ or $\eta(z, Tz) \leq 1$ and T is a triangular (α, η)-admissible mapping,

$$\alpha\left(Tz, T^2z\right) \geq 1 \text{ or } \eta\left(T^2z, Tz\right) \leq 1.$$

Continuing this process, we have

$$\alpha\left(T^nz, T^{n+1}z\right) \geq 1 \text{ or } \eta\left(T^nz, T^{n+1}z\right) \leq 1$$

for all $n \in \mathbb{N}$. By (T_3) and (T_4), we get

$$\alpha\left(T^mz, T^nz\right) \geq 1 \text{ or } \eta\left(T^mz, T^nz\right) \leq 1, \quad \forall\, m, n \in \mathbb{N}, \; n \neq m.$$

Assume that $z \notin$ Fix (T), i.e., $d(z, Tz) > 0$. Letting $x = T^{n-1}z$ and $y = z$ in (3.1), we get

$$d(z, Tz) = d(T^nz, Tz) = d\left(TT^{n-1}z, Tz\right),$$

which implies that

$$\begin{aligned}
\theta(d\left(TT^{n-1}z, Tz\right) &\leq \left[\theta\left(M(T^{n-1}z, z)\right)\right]^{\Omega(d\left(T^{n-1}z,z\right),d\left(T^{n-1}z,TT^{n-1}z\right),d(z,Tz),d\left(TT^{n-1}z,z\right),d\left(z,T^2T^{n-1}z\right))} \\
&= \left[\theta\left(M(T^{n-1}z, z)\right)\right]^{\Omega(d\left(T^{n-1}z,z\right),d\left(T^{n-1}z,T^nz\right),d(z,Tz),d(T^nz,z),d\left(z,T^{n+1}z\right))} \\
&= \left[\theta\left(M(T^{n-1}z, z)\right)\right]^{\Omega(d\left(T^{n-1}z,z\right),d\left(T^{n-1}z,T^nz\right),d(z,Tz),d(T^nz,z),0)}.
\end{aligned}$$

Thus there exists $\pi \in]0, 1[$ such that

$$\Omega(d\left(T^{n-1}z, z\right), d\left(T^{n-1}z, T^nz\right), d(z, Tz), d(T^nz, z), 0) = \pi.$$

Then

$$d(z, Tz) = d(T^nz, Tz) = d\left(TT^{n-1}z, Tz\right) \leq \left[\theta\left(M(T^{n-1}z, z)\right)\right]^{\pi},$$

where

$$\begin{aligned}
M\left(z, T^{n-1}z\right) &= \max\{d\left(T^{n-1}z, z\right), d\left(T^{n-1}z, TT^{n-1}z\right), d(z, Tz), d\left(T^{n-1}z, z\right), d\left(T^2T^{n-1}z, z\right), \\
&\quad d\left(T^2T^{n-1}z, Tz\right), d\left(T^2T^{n-1}z, T^{n-1}z\right)\} \\
&= \max\{d\left(T^{n-1}z, z\right), d\left(T^{n-1}z, T^nz\right), d(z, Tz), d\left(T^{n-1}z, z\right), d(TT^nz, z), d(TT^nz, Tz), \\
&\quad d\left(TT^nz, T^{n-1}z\right)\} \\
&= \max\{d\left(T^{n-1}z, z\right), d\left(T^{n-1}z, z\right), d(z, Tz), d\left(T^{n-1}z, z\right), d(Tz, z), d(Tz, Tz), d\left(Tz, T^{n-1}z\right)\}.
\end{aligned}$$

Since $d\left(T^{n-1}z, T^nz\right) \to 0$, taking the limit as $n \to \infty$, we obtain $\lim\limits_{n \to +\infty} M\left(z, T^{n-1}z\right) = d(z, Tz)$. Since θ is an increasing and contentious function,

$$\theta(d(z, Tz)) \leq [\theta(d(z, Tz))]^{\pi} < \theta(d(z, Tz)),$$

which is a contradiction. So $d(z, Tz) > 0$. Thus $Fix(T^n) = Fix(T)$. Therefore, T has the property (P). □

Assuming the following conditions, we prove that Theorem 3.5 still holds for T not necessarily continuous.

Theorem 3.8. Let $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions and (X, d) be an (α, η)-complete generalized metric space. Let $T : X \to X$ be a mapping satisfying the following assertions:

(i) T is triangular (α, η)-admissible;

(ii) T is (α, η)-θ-Ω-contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$;

(iv) (X, d) is (α, η)-regular.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(z,u) \geq 1$ or $\eta(z,u) \leq 1$ for all $z,u \in Fix(T)$.

Proof . Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$. Similar to the proof of Theorem 3.5, we can conclude that

$$(\alpha(x_n, x_{n+1}) \geq 1 \text{ or } \eta(x_n, x_{n+1}) \leq 1)$$

and

$$x_n \to z \text{ as } n \to \infty, \tag{3.13}$$

where $x_{n+1} = Tx_n$. From (iv), $\alpha(x_{n+1}, z) \geq 1$ or $\eta(x_{n+1}, z) \leq 1$, holds for $n \in \mathbb{N}$. Suppose that $Tz = x_{n_{0+1}} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$. From Theorem 3.5, we know that the members of the sequence $\{x_n\}$ are distinct. Hence, we have $Tz \neq Tx_n$, i.e., $d(Tz, Tx_n) > 0$ for all $n > n_0$. Thus we can apply (3.1) to x_n and z for all $n > n_0$ to get

$$\begin{aligned}
\theta(d(Tx_n, Tz)) &\leq \theta\left(s^2 d(Tx_n, Tz)\right) \\
&\leq [\theta(M(x_n, z))]^{\Omega(d(x_n,z), d(x_n,Tx_n), d(z,Tz), d(x_n,Tz), d(T^2x_n, z))} \\
&= \leq [\theta(M(x_n, z))]^{\Omega(d(x_n,z), d(x_n,x_{n+1}), d(z,Tz), d(x_n,Tz), d(x_{n+2}, z))}.
\end{aligned}$$

Thus

$$\theta(d(Tx_n, Tz)) \leq [\theta(M(x_n, z))]^{\Omega(d(x_n,z), d(x_n,x_{n+1}), d(z,Tz), d(x_n,Tz), d(x_{n+2}, z))}, \tag{3.14}$$

where

$$\begin{aligned}
M(x_n, z) &= \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), d(Tx_n, z), d\left(T^2x_n, Tz\right), d\left(T^2x_n, z\right), d\left(T^2x_n, Tx_n\right) \\
&= \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), d(x_{n+1}, z), d(x_{n+2}, Tz), d(x_{n+2}, z), d(x_{n+2}, x_{n+1}).
\end{aligned}$$

So

$$\begin{aligned}
\lim_{n\to\infty} M(x_n, z) &= \max\{\lim_{n\to\infty} d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), d(x_{n+1}, z), d(x_{n+2}, Tz), d(x_{n+2}, z), d(x_{n+2}, x_{n+1})\} \\
&= \max\{0, 0, d(z, Tz), 0, \lim_{n\to\infty} d(x_{n+2}, Tz), 0, 0, \}.
\end{aligned}$$

Since $0 \leq d(x_{n+2}, Tz) \leq s(d(x_{n+2}, x_n) + d(x_n, z) + d(z, Tz))$,

$$\lim_{n\to\infty} d(x_{n+2}, Tz) \leq d(z, Tz). \tag{3.15}$$

If $d(z,Tz) > 0$, then by (3.15) and the fact that θ and Ω are continuous and by taking the limit as $n \to \infty$ in (3.14) we get a contradiction. Therefore, $d(z,Tz) = 0$, that is, z is a fixed point of T and so $z = Tz$. Thus, z is a fixed point of T. The proof of the uniqueness is similar to that of Theorem 3.5. □

Definition 3.9. Let (X,d) be an (α,η)-generalized metric space and T be a self mapping on X. Suppose that $\alpha, \eta : X \times X \to [0, +\infty[$ are two functions. We say that T is an (α,η)-Ω-θ_G-contraction if for all $x, y \in X$ with $(\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1)$ and $d(Tx, Ty) > 0$ we have

$$\theta\left(s^2 d(Tx, Ty)\right) \leq [\theta(M(x,y))]^{\Omega(d(x,y), d(x,Tx), d(y,Ty), d(Tx,y), d(T^2x, y))}, \tag{3.16}$$

where $\theta \in \Theta_G, \Omega \in \Delta$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(Tx,y), d\left(T^2x, y\right), d\left(T^2x, Ty\right), d\left(T^2x, Tx\right)\right\}.$$

Theorem 3.10. Let (X,d) be an (α,η)-complete b-rectangular metric space and let $\alpha, \eta : X \times X \to [0, +\infty[$ be two functions. Let $T : X \times X \to X$ be a self mapping satisfying the following conditions:

(i) T is a triangular (α,η)-admissible mapping;

(ii) T is an (α,η)-θ-Ω-contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$;

(iv) T is (α, η)-continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in X$.

Proof . Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$. Similar to the proof of Theorem 3.5, we can conclude that $(\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1)$ and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. By (θ_3), there exist $r \in]0,1[$ and $l \in]0,+\infty[$ such that $\lim_{n\to\infty} \frac{\theta(d(x_n,x_{n+1}))-1}{d(x_n,x_{n+1})^r} = l$. Suppose that $l < \infty$. Then there exists $n_1 \in \mathbb{N}$ such that

$$\left|\frac{\theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r}\right| < \frac{l}{2}, \forall n \geq n_1.$$

By taking $M = \frac{2}{l}$, we have $n\left[d(x_n, x_{n+1})^r\right] < Mn\left[\theta(d(x_n, x_{n+1})) - 1\right]$, for all $n \geq n_1$. Suppose now that $l = \infty$. Let $N > 0$ be an arbitrary positive number. Then there exists $n_2 \in \mathbb{N}$ such that

$$\left|\frac{\theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r}\right| > N, \quad \forall n \geq n_2.$$

By taking $M = \frac{1}{N}$, we have $n\left[d(x_n, x_{n+1})^r\right] < Mn\left[\theta(d(x_n, x_{n+1})) - 1\right]$, for all $n \geq n_2$. Thus, in all cases, there exist $M > 0$ and $q \in \mathbb{N}$ $\quad (q = \max(n_1, n_2)$ such that

$$n\left[d(x_n, x_{n+1})^r\right] < Mn\left[\theta(d(x_n, x_{n+1})) - 1\right], \quad \forall n \geq n_q.$$

By induction, we obtain

$$n\left[d(x_n, x_{n+1})^r\right] < Mn\left[\theta(d(x_n, x_{n+1})) - 1\right] < \cdots < Mn\left[(\theta(d(x_0, x_1)))^{k^n} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} n\left[d(x_n, x_{n+1})^r\right] = 0$. So there exists $n_3 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}}$, for all $n \geq n_3$. By (θ_3), there exist $r \in]0,1[$ and $h \in]0,+\infty[$ such that $\lim_{n\to\infty} \frac{\theta(d(x_n,x_{n+2}))-1}{d(x_n,x_{n+2})^r} = h$. Suppose that $h < \infty$. Then there exists $n_4 \in \mathbb{N}$ such that

$$\left|\frac{\theta(d(x_n, x_{n+2})) - 1}{d(x_n, x_{n+2})^r}\right| < \frac{h}{2}, \quad \forall n \geq n_1.$$

By taking $p = \frac{2}{h}$, we have $n\left[d(x_n, x_{n+2})^r\right] < Pn\left[\theta(d(x_n, x_{n+2})) - 1\right]$, for all $n \geq n_4$. Suppose now that $h = \infty$. Let $Q > 0$ be an arbitrary positive number. Then there exists $n_5 \in \mathbb{N}$ such that

$$\left|\frac{\theta(d(x_n, x_{n+2})) - 1}{d(x_n, x_{n+2})^r}\right| > Q, \quad \forall n \geq n_5.$$

By taking $P = \frac{1}{Q}$, we have $n\left[d(x_n, x_{n+2})^r\right] < Pn\left[\theta(d(x_n, x_{n+2})) - 1\right]$, for all $n \geq n_5$. Thus, in all cases, there exist $P > 0$ and $w \in \mathbb{N}$ $\quad (w = \max(n_5, n_4))$ such that $n\left[d(x_n, x_{n+2})^r\right] < Pn\left[\theta(d(x_n, x_{n+2})) - 1\right]$, for all $n \geq w$. By induction, we obtain

$$n\left[d(x_n, x_{n+2})^r\right] < Pn\left[\theta(d(x_n, x_{n+2})) - 1\right] < \cdots < Pn\left[(\theta(d(x_0, x_2)))^{k^n} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} n\left[d(x_n, x_{n+2})^r\right] = 0$. So there exists $n_6 \in \mathbb{N}$ such that $d(x_n, x_{n+2}) \leq \frac{1}{n^{\frac{1}{r}}}$, for all $n \geq n_6$. If $m > n$ and $m = n + 2k + 1$ with $k \in \mathbb{N}$, then we have

$$\begin{aligned}
d(x_n, x_m) \leq& sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + sd(x_{n+2}, x_{n+2k+1}) \\
\leq& sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) + s^2 d(x_{n+4}, x_{n+5}) + s^2 d(x_{n+5}, x_{n+2k+1}) \\
\leq& sd(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) + s^2 d(x_{n+4}, x_{n+5}) \\
&+ s^3 d(x_{n+5}, x_{n+6}) + \ldots + s^{2k} d(x_{n+2k}, x_{n+2k+1}) \\
=& \sum_{i=n}^{n+2k} s^{2k-n} \frac{1}{i^{\frac{1}{r}}}.
\end{aligned}$$

If $m > n$ and $m = n + 2k$ with $k \in \mathbb{N}$, then we have

$$\begin{aligned}
d(x_n, x_m) \leq & sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + sd(x_{n+2}, x_{n+2k}) \\
\leq & sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + sd(x_{n+3}, x_{n+4}) + s^2 d(x_{n+4}, x_{n+5}) + s^2 d(x_{n+5}, x_{n+5}) + s^2 d(x_{n+6}, x_{n+2k1}) \\
\leq & sd(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + s^2 d(x_{n+4}, x_{n+5}) + s^2 d(x_{n+5}, x_{n+6}) \\
& + s^3 d(x_{n+6}, x_{n+7}) + \ldots + s^{2k-1} d(x_{n+2k-1}, x_{n+2k}) \\
= & \sum_{i=n}^{n+2k-1} s^{2k-n} \frac{1}{i^{\frac{1}{r}}}.
\end{aligned}$$

So

$$d(x_n, x_m) \leq \sum_{i=n}^{\infty} s^{m-1} \frac{1}{i^{\frac{1}{r}}}. \tag{3.17}$$

Since $0 < r < 1$, the series $\sum_{i=n}^{\infty} s^{m-1} \frac{1}{i^{\frac{1}{r}}}$ converges. Therefore, by taking the limit as $n, m \to \infty$ in (3.17), we get $\lim_{n\to\infty} d(x_n, x_m) = 0$. Hence x_n is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$. and since T is (α, η)-continuous, $\lim_{n\to\infty} d(Tx_n, Tz) = 0$. Then

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz.$$

This proves that z is a fixed point of T. □

Corollary 3.11. Let (X, d) be an (α, η)-complete b-rectangular metric space. Let $\alpha, \eta : X \times X \to [0, +\infty[$ be two functions. Let $T : X \times X \to X$ be a self mapping satisfying the following:

(i) $\theta[s^2 d(Tx, Ty)] \leq [\theta(M(x, y))]^k, k \in (0, 1), \quad \theta \in \Theta_G$;

(ii) T is a triangular (α, η)-admissible mapping;

(iii) T is an (α, η)-θ-Ω-contraction;

(iv) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$;

(v) $T$is a(α, η)-continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in X$.

Example 3.12. Let $X = [1, +\infty[$, $a \in]0, 1[$. Define $d : X \times X \to [0, +\infty[$ by $d(x, y) = (|x - y|)^2$. Then (X, d) is a b-rectangular metric space. Define a mapping $T : X \to X$ by $T(t) = a\sqrt{t}$, for all $t \in [1, +\infty[$, and

$$\alpha(x, y) = \frac{\max\{x, y\} + a}{\min\{x, y\} + a}, \quad \eta(x, y) = \frac{\min\{x, y\} + a}{\max\{x, y\} + a}, \quad \Omega(t_1, t_2, t_3, t_4, t_5) = \sqrt{a}, \quad \text{for all } x, y, t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}_+.$$

Then T is an (α, η)-continuous triangular and$(\alpha, \eta)-$ admissible mapping.

Case 1: $0 \leq x \leq y$. $d(Tx, Ty) = \left(a\sqrt{y} - a\sqrt{x}\right)^2$ and

$$M(d(x, y)) = \max\left\{d(x, y), d\left(x, a\sqrt{x}\right), d\left(y, a\sqrt{y}\right), d\left(y, a\sqrt{x}\right), d\left(a^2\sqrt{\sqrt{x}}, y\sqrt{y}\right), d\left(a^2\sqrt{\sqrt{x}}, a\sqrt{y}\right), \ d\left(a^2\sqrt{\sqrt{x}}, a\sqrt{x}\right)\right\}.$$

Since $x \leq y$ and $a \in]0, 1[$,

$$\begin{aligned}
M(d(x, y)) = \max & \left\{(y - x)^2, \left(x - a\sqrt{x}\right)^2, \left(y - a\sqrt{y}\right)^2, \left(y - a\sqrt{x}\right)^2, \left(y\sqrt{y} - a^2\sqrt{\sqrt{x}}\right)^2, \left(a\sqrt{y} - a^2\sqrt{\sqrt{x}}^2\right), \right. \\
& \left. \left(a\sqrt{x} - a^2\sqrt{\sqrt{x}}^2,\right)\right\}.
\end{aligned}$$

Thus $M(d(x,y)) \geq (y-x)^2 \geq a(y-x)^2$. On the other hand, $a(y-x) = \sqrt{a}\sqrt{a}(y-x)^2$, which implies that

$$\theta\left(M(x,y)\right) \geq \theta\left(d(x,y)\right) = e^{(y-x)^2}.$$

Thus

$$\theta\left(d(x,y)\right)^{\Omega\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx),d\left(T^2x,y\right)\right)} = e^{\sqrt{a}(y-x)^2} = e^{\sqrt{a}(\sqrt{y}-\sqrt{x})(\sqrt{y}+\sqrt{x})^2}$$

and $\theta\left(d(Tx,Ty)\right) = e^{a(\sqrt{y}-\sqrt{x})^2}$. Since $x, y \in [1,\infty[$, we have $e^{a(\sqrt{y}-\sqrt{x})} \leq e^{\sqrt{a}(\sqrt{y}-\sqrt{x})^2(\sqrt{y}+\sqrt{x})^2}$. Thus

$$\theta\left(d(Tx,Ty)\right) \leq \theta\left(d(x,y)\right)^{\Omega(d(x,y),d(x,Tx),d(y,Ty),d(y,Tx),d(T^2x,y)}.$$

Case 2: $x > y > 0$. $d(Tx,Ty) = \left(a\sqrt{x} - a\sqrt{y}\right)^2$ and

$$M(d(x,y)) = \max\left\{d\left(x,y\right), d\left(x, a\sqrt{x}\right), d\left(y, a\sqrt{y}\right), d\left(y, a\sqrt{x}\right), d\left(a^2\sqrt{\sqrt{x}}, y\sqrt{y}\right), d\left(a^2\sqrt{\sqrt{x}}, a\sqrt{y}\right), d\left(a^2\sqrt{\sqrt{x}}, a\sqrt{x}\right)\right\}.$$

Since $x > y$ and $a \in]0,1[$,

$$\begin{aligned} M(d(x,y)) = \max\Bigg\{ & (x-y)^2, \left(x - a\sqrt{x}\right)^2, \left(y - a\sqrt{y}\right)^2, \left(|y - a\sqrt{x}|\right)^2, \left(|y\sqrt{y} - a^2\sqrt{\sqrt{x}}|\right)^2, \left(a|\sqrt{y} - a^2\sqrt{\sqrt{x}}|\right)^2, \\ & \left(a\sqrt{x} - a^2\sqrt{\sqrt{x}},\right)^2\Bigg\}. \end{aligned}$$

Thus $M(d(x,y)) \geq (y-x)^2 \geq a(x-y)^2$. On the other hand, $a(y-x) = \sqrt{a}\sqrt{a}(x-y)^2$, which implies that

$$\theta\left(M(x,y)\right) \geq \theta\left(d(x,y)\right) = e^{(x-y)^2}.$$

Thus

$$\theta\left(d(x,y)\right)^{\Omega\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx),d\left(T^2x,y\right)\right)} = e^{\sqrt{a}(x-y)^2} = e^{\sqrt{a}(\sqrt{x}-\sqrt{y})(\sqrt{y}+\sqrt{x})^2}$$

and

$$\theta\left(d(Tx,Ty)\right) = e^{a(\sqrt{x}-\sqrt{y})^2}.$$

Since $x, y \in [1,\infty[$, we have $e^{a(\sqrt{x}-\sqrt{y})^2} \leq e^{\sqrt{a}(\sqrt{x}-\sqrt{y})^2(\sqrt{y}+\sqrt{x})^2}$. Thus

$$\theta\left(d(Tx,Ty)\right) \leq \theta\left(d(x,y)\right)^{\Omega(d(x,y),d(x,Tx),d(y,Ty),d(y,Tx),d(T^2x,y)},$$

where $\theta \in \Theta_C \cap \Theta_G$. Hence (3.12) and (3.16) are satisfied. Therefore, T has a unique fixed point $z = 1$.

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