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Chebyshev-type fractional inequalities via (k,ψ)-Hilfer Operator

Bouharket Benaissa[a,*], Noureddine Azzouz[b,c]

[a] Laboratory of Informatics and Mathematics, Faculty of Material Sciences, University of Tiaret, Algeria
[b] Faculty of Sciences, University Center Nour Bachir, El Bayadh, Algeria
[c] University Belhadj Bouchaib Ain Temouchent, Ain Temouchent, Algeria

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Abstract

In this paper, we use the (k,ψ)-Hilfer fractional integral of functions with respect to another function to generalize Chebyshev-type fractional integral inequalities. Some inequalities involving (k,ψ)-Hilfer fractional integrals are also to be proved.

Keywords: Chebyshev inequality, Hilfer operator, Fractional operator
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1 Introduction

In applied sciences, integral inequalities are incredibly important. Furthermore, the study of integral inequalities using fractional integration theory has become extremely important; for specific applications, see ([6], [11]). In this paper, we will examine the Chebyshev inequality.

$$T(f,\, g)(x) \geq 0, \tag{1.1}$$

introduced in [3] for the following so-called Chebyshev functional

$$T(f,\, g)(x) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\,\frac{1}{b-a}\int_a^b g(x)\,dx, \tag{1.2}$$

where f and g are two integrable functions and synchronous on $[a,b]$, that is, for all $x,y\in[a,b]$

$$(f(x)-f(y))(g(x)-g(y)) \geq 0. \tag{1.3}$$

Over the previous decade, several authors established different new integral inequalities of type 1.1 using various fractional integral operators, See ([10, 5, 8, 4, 1, 12, 13]). In particular, Belarbi and Dahmani [2] developed the following results about Chebyshev inequality using the Riemann-Liouville fractional integral operator defined by

$$\mathrm{I}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt.$$

*Corresponding author
Email addresses: bouharket.benaissa@univ-tiaret.dz (Bouharket Benaissa), n.azzouz@cu-elbayadh.dz (Noureddine Azzouz)

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Theorem 1.1. Let f and g be two synchronous functions on $[0,+\infty[$, then for all $x>0$, $\alpha>0$.

$$\mathrm{I}^{\alpha}(f\,g)(x) \geq \frac{\Gamma(\alpha+1)}{x^{\alpha}}\mathrm{I}^{\alpha}f(x)\,\mathrm{I}^{\alpha}g(x). \tag{1.4}$$

Theorem 1.2. Let f and g be two synchronous functions on $[0,+\infty[$, then for all $x>0$, $\alpha>0$ and $\alpha>0$, we have

$$\frac{x^{\beta}}{\Gamma(\beta+1)}\,\mathrm{I}^{\alpha}(f\,g)(x)+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\,\mathrm{I}^{\beta}(f\,g)(x) \geq \mathrm{I}^{\alpha}f(x)\,{}_{a^{+}}\mathrm{I}^{\beta}g(x)+\mathrm{I}^{\beta}f(x)\,\mathrm{I}^{\alpha}g(x). \tag{1.5}$$

Theorem 1.3. Let $i=1,2,...,m$ and f_i be n positive and increasing on $[a,b]$, then for all integer $m\geq 1$ we have

$$\mathrm{I}^{\alpha}\left(\prod_{i=1}^{m} f_i\right)(x) \geq \left(\frac{\Gamma(\alpha+1)}{x^{\alpha}}\right)^{m-1}\prod_{i=1}^{m}\mathrm{I}^{\alpha}\left(f_i\right)(x). \tag{1.6}$$

On the other hand, the (k,ψ)-Hilfer integral fractional operators are defined as follows [7]:

Definition 1.4. Let $k>0$ and ψ be an increasing positive monotone function on $[a;b]$ such that ψ' is continuous on (a,b). The left and right-sided (k,ψ)-Hilfer fractional integral operators of a function f with respect to the function ψ on $[a,b]$ are defined respectively as:

$$ {}_{a^{+}}\mathrm{J}_{k}^{\alpha,\psi}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}f(t)dt, \quad a<x\leq b. \tag{1.7}$$

$$ {}_{b^{-}}\mathrm{J}_{k}^{\alpha,\psi}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\frac{\alpha}{k}-1}f(t)dt, \quad a\leq x<b.$$

Our aim in this study is to establish Chebyshev fractional inequalities involving the (k,ψ)-Hilfer integral fractional operator defined in (1.7) with two parameters. Chebyshev fractional inequalities will be derived according to specific choices of the function ψ. This paper is organized as follows: in Section 2, we present some preliminary results; in Section 3, the main results are stated and proved; and in Section 4, some derived Chebyshev fractional inequalities are given.

2 Preliminaries

The space $L_p^W[a,b]$ of all real-valued Lebesgue measurable functions $f\neq 0$ on $[a,b]$ with norm condition :

$$\| f \|_p^W=\left(\int_a^b \mid f(x)\mid^p W(x)dx\right)^{\frac{1}{p}}<\infty,\ p\geq 1,$$

is known as weighted Lebesgue space, where W is a weighted function (positive and measurable).

1. Put $p=1$ and $W\equiv 1$, the space $L_p^W[a,b]$ reduces to the classical Lebesgue space $L([a,b])$.
2. Choose $p=1$ and $W(x)=\psi'(x)$, we get

$$L_{X_{\psi}'}([a,b])=\left\{f:\| f \|_{X_{\psi}'}=\int_a^b \mid f(x)\mid \psi'(x)dx\ <\infty\right\}. \tag{2.1}$$

In the next theorem, we show that the (k,ψ)-Hilfer integral fractional operators are well defined on $L_{X_{\psi}'}([a,b])$.

Theorem 2.1. For all functions $f\in L_{X_{\psi}'}([a,b])$ we have ${}_{a^{+}}\mathrm{J}_{k}^{\alpha,\psi}f \in L_{X_{\psi}'}([a,b])$ and ${}_{b^{-}}\mathrm{J}_{k}^{\alpha,\psi}f \in L_{X_{\psi}'}([a,b])$. Moreover the operators ${}_{a^{+}}\mathrm{J}_{k}^{\alpha,\psi}$ and ${}_{b^{-}}\mathrm{J}_{k}^{\alpha,\psi}$ are bounded on $L_{X_{\psi}'}([a,b])$. Explicitly

$$\left\|{}_{a^{+}}\mathrm{J}_{k}^{\alpha,\psi}f\right\|_{X_{\psi}'}\leq C\,\|f\|_{X_{\psi}'}, \qquad \left\|{}_{b^{-}}\mathrm{J}_{k}^{\alpha,\psi}f\right\|_{X_{\psi}'}\leq C\,\|f\|_{X_{\psi}'},$$

where $C=\dfrac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}$.

Proof . Let $f \in L_{X_{\psi'}}([a,b])$ then, using Fubini's Theorem we get

$$\begin{aligned}
\left\|{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f\right\|_{X_{\psi'}} &= \int_a^b |{}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(x)\,|\,\psi'(x)\,dx \\
&\leq \frac{1}{k\,\Gamma_k(\alpha)}\int_a^b\int_a^x |f(s)|\,\psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1}\psi'(x)\,ds\,dx \\
&= \frac{1}{k\,\Gamma_k(\alpha)}\int_a^b |f(s)|\left(\int_s^b (\psi(x)-\psi(s))^{\frac{\alpha}{k}-1}\psi'(x)\,dx\right)\psi'(s)\,ds \\
&= \frac{1}{\alpha\,\Gamma_k(\alpha)}\int_a^b |f(s)|\,(\psi(b)-\psi(s))^{\frac{\alpha}{k}}\,\psi'(s)\,ds \\
&\leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\int_a^b |f(s)|\,\psi'(s)\,ds \\
&= C\,\|f\|_{X_{\psi'}}.
\end{aligned}$$

Similarly, we establish that

$$\left\|{}_{b^-}\mathrm{J}_k^{\alpha,\psi} f\right\|_{X_{\psi'}} \leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\ \|f\|_{X_{\psi'}}.$$

□

Remark 2.2. If the function f is continuous, then

$$\begin{aligned}
\|\,f\,\|_{X_{\psi'}} &= \int_a^b |\,f(x)\,|\,\psi'(x)dx \\
&\leq \max_{a\leq x\leq b}|\,f(x)\,|\int_a^b \psi'(x)dx \\
&\leq (\psi(b)-\psi(a))\ \max_{a\leq x\leq b}|\,f(x)\,| \quad < \infty.
\end{aligned}$$

Thus, continued functions belong to the space $L_{X_{\psi'}}([a,b])$. In all that follows, we will assume that the considered functions are in $L_{X_{\psi'}}([a,b])$.

The (k,ψ)-Hilfer integral fractional operators are notable for their ability to generate specific types of k-fractional integrals depending on the choice of the function ψ.

1. Taking $\psi(\tau)=\tau$, the (k,ψ)-Hilfer yields to the k- Riemann-Liouville fractional integral operator of order $\alpha>0$

$$ {}_{a^+}\mathcal{RL}_k^{\alpha} f(x) = \frac{1}{k\,\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt,, \quad x>a,$$

$$ {}_{b^-}\mathcal{RL}_k^{\alpha} f(x) = \frac{1}{k\,\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt,, \quad x<b.$$

2. Using $\psi(\tau)=\ln\tau$, the (k,ψ)-Hilfer reduces to the k-Hadamard fractional integral operator of order $\alpha>0$

$$ {}_{a^+}\mathcal{H}_k^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1} f(t)\frac{dt}{t}, \quad x>a>1,$$

$$ {}_{b^-}\mathcal{H}_k^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\frac{\alpha}{k}-1} f(t)\frac{dt}{t}, \quad 1<x<b.$$

3. Putting $\psi(\tau)=\frac{\tau^{\rho+1}}{\rho+1}$ where $\rho>0$, the (k,ψ)-Hilfer makes it similar to the k-Katugompola fractional integral operator of order $\alpha>0$

$$ {}_{a^+}\mathcal{K}_k^{\alpha} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^x \left(x^{\rho+1}-t^{\rho+1}\right)^{\frac{\alpha}{k}-1} f(t)t^{\rho}dt,\ x>a, \tag{2.2}$$

$$ {}_{b^-}\mathcal{K}_k^{\alpha} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_x^b \left(t^{\rho+1}-x^{\rho+1}\right)^{\frac{\alpha}{k}-1} f(t)t^{\rho}dt,\ x<b.$$

4. Setting $\psi(\tau)=\frac{(\tau-a)^{\theta}}{\theta}$ ($\psi(\tau)=-\frac{(b-\tau)^{\theta}}{\theta}$) respectively where $\theta>0$, the left sided (right sided) (k,ψ)-Hilfer respectively is reduced to the k-fractional conformable integral operator of order $\alpha>0$ [9].

$$
{}_{a^+}\mathcal{C}_k^{\alpha} f(x)=\frac{\theta^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^x \left((x-a)^{\theta}-(t-a)^{\theta}\right)^{\frac{\alpha}{k}-1}\frac{f(t)}{(t-a)^{1-\theta}}dt,\ x>a, \tag{2.3}
$$

$$
{}_{b^-}\mathcal{C}_k^{\alpha} f(x)=\frac{\theta^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_x^b \left((b-x)^{\theta}-(b-t)^{\theta}\right)^{\frac{\alpha}{k}-1}\frac{f(t)}{(b-t)^{1-\theta}}dt,\ x<b.
$$

3 Main results

The Chebyshev-type inequalities are presented below with the (k,ψ)-Hilfer operator.

Theorem 3.1. Let $\alpha,\beta,\ k>0$, f and g be two synchronous functions on $[a,b]$ and ψ be a positive increasing function on $[a,b]$ having a continuous derivative ψ' on (a,b), then for $x>a$ the following inequalities hold:

$$
\begin{aligned}
&\frac{(\psi(x)-\psi(a))^{\frac{\beta}{k}}}{\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,g)(x)+\frac{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_{a^+}\mathrm{J}_k^{\beta,\psi}(f\,g)(x)\\
&\geq {}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\beta,\psi}g(x)+{}_{a^+}\mathrm{J}_k^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x).
\end{aligned} \tag{3.1}
$$

and

$$
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}(f\,g)(x)\geq\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x). \tag{3.2}
$$

Proof . Let f and g be two synchronous functions on $[a,b]$, then according to (1.3) we have for all $t,s\in[a,b]$

$$
f(t)g(t)+f(s)g(s)\geq f(t)g(s)+f(s)g(t). \tag{3.3}
$$

Multiplying by $\frac{\psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{k}-1}}{k\,\Gamma_k(\alpha)}$ and integrating with respect to t over (a,x), we get

$$
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)g(x)+f(s)g(s)\,\frac{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\geq g(s)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)+f(s)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi}g(x). \tag{3.4}
$$

Multiplying inequality (3.4) by $\frac{\psi'(s)(\psi(x)-\psi(s))^{\frac{\beta}{k}-1}}{k\,\Gamma_k(\beta)}$ and integrating with respect to s over (a,x), we get the desired inequality (3.1).

Putting $\beta=\alpha$ in the inequality (3.1), we'll get the required inequality (3.2). □

Remark 3.2. We present some special cases of the above Theorem 3.1.

1. By putting $k=1$, $b=+\infty$ and $a=0$, we obtain Theorem 7 and Theorem 6 in [10].
2. If we choose $\psi(x)=\ln x$, we obtain Theorem 3 in [5].
3. Putting $k=1$, $a=0$ and $\psi(\tau)=\frac{(\tau)^{\xi+\eta}}{\xi+\eta}$ gives Theorems 2.2 and 2.1 in [8].
4. Taking $k=1$, $a=0$ and $\psi(\tau)=\frac{(\tau)^{\theta}}{\theta}$ yields Theorems 6 and 5 in [13].

Corollary 3.3. Let $i=1,2,...,m$ and $f,\ f_i\in L_{X^1_{\psi}},[a,b]$. If the functions f and f_i are positive and increasing on $[a,b]$, then for all integer $m\geq1$ we have

$$
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}\prod_{i=1}^{m}f_i(x)\geq\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m}{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f_i(x)\right), \tag{3.5}
$$

and

$$
{}_{a^+}\mathrm{J}_k^{\alpha,\psi}f^m(x)\geq\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left({}_{a^+}\mathrm{J}_k^{\alpha,\psi}f(x)\right)^m. \tag{3.6}
$$

Proof . For $m=1$ the equality holds. Let $m\neq 1$, by using the inequality (3.2) with $g=\prod_{i=2}^{m} f_i$, we obtain

$$\begin{aligned}
{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}\prod_{i=1}^{m} f_i(x) \geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_1(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}\prod_{i=2}^{m} f_i(x)\\
\geq& \frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_1(x)\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_2(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}\prod_{i=3}^{m} f_i(x)\right)\\
=&\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^2\left(\prod_{i=1}^{2}{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_i(x)\right){}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}\prod_{i=3}^{m} f_i(x)\\
\geq&\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^3\left(\prod_{i=1}^{3}{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_i(x)\right){}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}\prod_{i=4}^{m} f_i(x)\\
&\vdots\\
\geq&\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m-1}{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_i(x)\right){}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_m(x)\\
=&\left(\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\right)^{m-1}\left(\prod_{i=1}^{m}{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi} f_i(x)\right),
\end{aligned}$$

which yields to the desired inequality (3.5). Putting $f_1=f_2=\cdots=f_m=f$ in the inequality (3.5), we get the inequality (3.6). □

Remark 3.4. We present some particular cases of Corollary 3.3.

1. By putting $k=1$, $b=+\infty$, and $a=0$ in the above Corollary, we obtain Theorem 8 and Corollary 2 in [10].
2. Setting $k=1$, $a=0$ and $\psi(\tau)=\frac{(\tau)^{\xi+\eta}}{\xi+\eta}$ gives Theorem 2.3 in [8].
3. Taking $k=1$, $a=0$ and $\psi(\tau)=\frac{(\tau)^{\theta}}{\theta}$ yields Theorem 7 in [13].

Corollary 3.5. Let f and g be two functions defined on $L_{X_{\psi}}{}'\,([a,b])$, such that f is monotone, g is differentiable on (a,b) and there exists a real numbers $m:=\inf_{x\in[a,b]} g'(x)$ and $M:=\sup_{x\in[a,b]} g'(x)$.

- If f is an increasing function, then

$$\begin{aligned}
{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(f\,g)(x)\geq&\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}g(x)-m\,\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}I_d(x)\\
&+m\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(I_d f)(x).
\end{aligned}\tag{3.7}$$

- If f is a decreasing function, then

$$\begin{aligned}
{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(f\,g)(x)\geq&\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}g(x)-M\,\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}I_d(x)\\
&+M\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(I_d f)(x),
\end{aligned}\tag{3.8}$$

where $I_d(x)$ is the identity function.

Proof . Taking $G(x)=g(x)-m\,x$, thus G is differentiable and increasing on (a,b). By using the inequality (3.2), we get

$$\begin{aligned}
{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(f\,G)(x)\geq&\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}(g-m\,x)(x)\\
=&\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}g(x)-m\,\frac{\Gamma_k(\alpha+k)}{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k}^{\alpha,\psi}I_d(x),
\end{aligned}\tag{3.9}$$

we also have

$$ {}_{a^+}\mathrm{J}_k^{\alpha,\psi} f(g - mI_d)(x) = {}_{a^+}\mathrm{J}_k^{\alpha,\psi}(fg)(x) - m\, {}_{a^+}\mathrm{J}_k^{\alpha,\psi}(I_d f)(x). \tag{3.10} $$

Combining inequalities (3.9) and (3.10), we obtain the required inequality (3.7). Considering the situation of a decreasing function f and taking $G(x) = g(x) - M\,x$, we will obtain the inequality (3.8) by sketching the proof of the inequality (3.7). □

Remark 3.6. We give some specific results of Corollary 3.5.

1. Using $k = 1$, $b = +\infty$ and $a = 0$, we obtain Theorem 10 in [10].
2. Taking $k = 1$, $a = 0$ and $\psi(\tau) = \frac{(\tau)^{\xi+\eta}}{\xi+\eta}$ yields Theorem 2.4 in [8].
3. Setting $k = 1$, $a = 0$ and $\psi(\tau) = \frac{(\tau)^{\theta}}{\theta}$ gives Theorem 8 in [13].

4 Applications

The previously mentioned result is now applied to Chebyshev inequalities involving two specific operators: the k-Katugompola operator and the k-fractional conformable integral operator.

4.1 Chebyshev-type inequalities via k-Katugompola operator

Putting $\psi(\tau) = \frac{\tau^{\rho+1}}{\rho+1}$ where $\rho > 0$, the left side (k,ψ)-Hilfer ${}_{a^+}\mathrm{J}_k^{\alpha,\psi}$ reduces to the left side k-Katugompola fractional integral operator (2.2) of order $\alpha > 0$ and the following results hold.

Corollary 4.1. Let f and g be two synchronous functions on $[a, b]$ and let $\alpha, \beta,\, k > 0$, then for $x > a$ we have

$$ \frac{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\, {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) + \frac{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\, {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) \geq {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathrm{J}_k^{\beta,\psi} g(x) + {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathrm{J}_k^{\alpha,\psi} g(x), $$

and

$$ {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) \geq \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x), {}_{a^+}\mathcal{K}_k^{\alpha} g(x). $$

Corollary 4.2. Let $i = 1, 2, ..., m$ and $f,\, f_i \in L_{X^1_{\psi}},[a, b]$. If the functions f and f_i are positive increasing functions on $[a, b]$, then for all integer $m \geq 1$ we have

$$ {}_{a^+}\mathcal{K}_k^{\alpha} \prod_{i=1}^{m} f_i(x) \geq \left(\frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}} \right)^{m-1} \left(\prod_{i=1}^{m} {}_{a^+}\mathcal{K}_k^{\alpha} f_i(x) \right), $$

and

$$ {}_{a^+}\mathcal{K}_k^{\alpha} f^m(x) \geq \left(\frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}} \right)^{m-1} \left({}_{a^+}\mathcal{K}_k^{\alpha} f(x) \right)^m. $$

Corollary 4.3. Let f and g be two functions defined on L_{X_ψ}, $([a, b])$, such that f is monotone and g is differentiable on $]a, b[$ and there exists a real numbers $m := \inf\limits_{x\in[a,b]} g'(x)$ and $M := \sup\limits_{x\in[a,b]} g'(x)$.

- If f is an increasing function, then

$$ \begin{aligned} {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) \geq & \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathrm{J}_k^{\alpha,\psi} g(x) - m \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x) {}_{a^+}\mathcal{K}_k^{\alpha} I_d(x) \\ & + m\, {}_{a^+}\mathcal{K}_k^{\alpha}(I_d f)(x). \end{aligned} $$

- If f is a decreasing function, then

$$ \begin{aligned} {}_{a^+}\mathcal{K}_k^{\alpha}(f\,g)(x) \geq & \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathcal{K}_k^{\alpha} g(x) - M \frac{(\rho+1)^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x^{\rho+1} - a^{\rho+1})^{\frac{\beta}{k}}}\, {}_{a^+}\mathcal{K}_k^{\alpha} f(x)\, {}_{a^+}\mathcal{K}_k^{\alpha} I_d(x) \\ & + M\, {}_{a^+}\mathcal{K}_k^{\alpha}(I_d f)(x), \end{aligned} $$

where $I_d(x)$ is the identity function.

4.2 Chebyshev-type inequalities via k-fractional conformable integral operator

Set $\psi(\tau) = \frac{(\tau-a)^\theta}{\theta}$ where $\theta > 0$, the left side (k,ψ)-Hilfer ${}_{a^+}\mathrm{J}_k^{\alpha,\psi}$ reduces to the left side k-fractional conformable integral operator (2.3) of order $\alpha > 0$ and we derive what follows.

Corollary 4.4. Let f and g be two synchronous functions on $[a,b]$ and let $\alpha, \beta,\ k > 0$, then for $x > a$ we have

$$\frac{(x-a)^{\frac{\theta\beta}{k}}}{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x) + \frac{(x-a)^{\frac{\theta\beta}{k}}}{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}\,{}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x) \geq {}_{a^+}\mathcal{C}_k^{\alpha} f(x)\,{}_{a^+}\mathrm{J}_k^{\beta,\psi} g(x) + {}_{a^+}\mathcal{C}_k^{\alpha} f(x)\,{}_{a^+}\mathrm{J}_k^{\alpha,\psi} g(x),$$

and

$${}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x) \geq \frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha} f(x),{}_{a^+}\,\mathcal{C}_k^{\alpha} g(x).$$

Corollary 4.5. Let $i = 1,2,...,m$ and $f,\ f_i \in L_{X^1_\psi},[a,b]$. If the functions f and f_i are positive increasing functions on $[a,b]$, then for all integer $m \geq 1$ we have

$${}_{a^+}\mathcal{C}_k^{\alpha} \prod_{i=1}^{m} f_i(x) \geq \left(\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\right)^{m-1} \left(\prod_{i=1}^{m} {}_{a^+}\mathcal{C}_k^{\alpha} f_i(x)\right),$$

and

$${}_{a^+}\mathcal{C}_k^{\alpha} f^m(x) \geq \left(\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\right)^{m-1} \left({}_{a^+}\mathcal{C}_k^{\alpha} f(x)\right)^m.$$

Corollary 4.6. Let f and g be two functions defined on $L_{X_\psi},([a,b])$, such that f is monotone and g is differentiable on $]a,b[$ and there exists a real numbers $m := \inf_{x\in[a,b]} g'(x)$ and $M := \sup_{x\in[a,b]} g'(x)$.

- If f is an increasing function, then

$${}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x) \geq \frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha} f(x)\,{}_{a^+}\mathrm{C}_k^{\alpha,\psi} g(x) - m\,\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha} f(x)\,{}_{a^+}\mathcal{C}_k^{\alpha} I_d(x) + m\,{}_{a^+}\mathcal{C}_k^{\alpha}(I_d f)(x).$$

- If f is a decreasing function, then

$${}_{a^+}\mathcal{C}_k^{\alpha}(f\,g)(x) \geq \frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha} f(x)\,{}_{a^+}\mathrm{C}_k^{\alpha,\psi} g(x) - M\,\frac{\theta^{\frac{\beta}{k}}\,\Gamma_k(\alpha+k)}{(x-a)^{\frac{\theta\beta}{k}}}\,{}_{a^+}\mathcal{C}_k^{\alpha} f(x)\,{}_{a^+}\mathcal{C}_k^{\alpha} I_d(x) + M\,{}_{a^+}\mathcal{C}_k^{\alpha}(I_d f)(x),$$

where $I_d(x)$ is the identity function.

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