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Pairwise compactness in bi-isotonic spaces

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Abstract

In this article, we have introduced the notion of pairwise compactness in bi-isotonic spaces via finite intersection property and pairwise open cover. Moreover, we have given pairwise compactness of bi-isotonic subspaces with both reduced closure and interior functions. We have characterized pairwise compactness by the neighborhood concept; however, the axioms of bi-isotonic spaces are insufficient to prove the theorem, and we have studied this in bi-closure spaces. For similar reasons, occasionally, bi-closure spaces have been considered with additional explanations, even if some concepts related to compactness have been naturally extended to bi-isotonic spaces. Additionally, interesting results have been obtained considering the pairwise Hausdorffness and compactness relationship. It has also been observed that resembling cross-relationships exist for closed subsets of pairwise compact spaces. Finally, it has been observed that the pairwise compactness of bi-closure spaces is preserved under bi-continuity.

Keywords: Pairwise compact, pairwise Hausdorff space, closure operator, bi-isotonic space
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1 Introduction

Bi-isotonic space is an extension of bitopological space defined by removing the necessity of expansive, sub-additive, and idempotent properties of the Kuratowski closure axioms for two closure functions [4, 5]. The bi-isotonic spaces are a relatively new subject and emerged after the definition of isotonic spaces. Isotonic spaces are generalizations of the closure spaces that Kuratowski defined as an extension of the topological spaces by removing sub-additive property [10]. Stadler et al. and Habil et al. gave the basic principles of isotonic spaces [7, 15, 16]. Subsequently, Erol and Ersoy defined bi-isotonic spaces with two closure operators. Thus, introducing the definitions of open and closed sets allowed the examination of the separation axioms in these spaces and also bicontinuous transformations between these spaces [5]. In addition to this study, Ersoy and Acet examined pairwise separated sets in bi-isotonic spaces and some of their properties based on the previous ideas. They introduced the concepts of pairwise connectedness (disconnectedness) and total disconnection in bi-isotonic spaces [4].

In order to define pairwise compactness in bi-isotonic spaces, it is necessary to wise up the emergence and development process of the concept of pairwise compactness in bitopological spaces. Actually, pairwise compactness in bitopological spaces was introduced in six different ways, including the definitions of Kim [9], Fletcher, Hoyle, and Patty [6], Birsan [1], Swart [17], Phak and Choi [13] and Saegrove [14], chronologically. The first definition, called $p-$compactness, was given by Kim in 1968, but a preliminary concept called adjoint topology was also needed in this

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definition [9]. To be compatible with the conventional structure, Fletcher et al. presented the notion of the pairwise open cover of a set in a bitopological space, hence the definition of pairwise compactness based on this notion and related characterizations in 1969 [6]. Although it is simultaneous with the date of the appearance of this definition of pairwise compactness, another independent explanation was given by Birsan [1]. In order to avoid confusion in the literature, the terms pairwise B-compactness and FHP-compactness have been used, respectively, instead of the notions of pairwise compactness in the meaning of Birsan and Fletcher et al. Both definitions of pairwise B-compactness and FHP-compactness necessitate that if the topological spaces $(X, \mathcal{T})$ and $(X, \mathcal{Q})$ are compact and the bitopological space $(X, \mathcal{T}, \mathcal{Q})$ is pairwise Hausdorff, then $\mathcal{T} = \mathcal{Q}$. However, Swart provided a counterexample to this proposition. In this regard, pairwise compactness in the sense of Swart was defined [17]. Then, the term pairwise S-compact has been used. On the other hand, by taking pairwise closed families instead of pairwise open covers as the starting, the concept of pairwise compactness was introduced by Phak and Choi in 1971 [13]. Meanwhile, a quite different concept called pseudo-compactness was introduced in 1973 by Saegrove, such as a bitopological space $(X, \mathcal{T}, \mathcal{Q})$ is pseudo-compact if and only if every pair continuous function from $(X, \mathcal{T}, \mathcal{Q})$ into $(\mathbb{R}, \mathcal{R}, \mathcal{L})$ is bounded where $\mathbb{R}$ is the real line and $\mathcal{R}$ is the family of open right rays and $\mathcal{L}$ is the family of open left rays [14]. A relatively new concept called near pairwise compactness was introduced in [12], and its characterizations were presented based on the definitions of pairwise almost continuity and pairwise almost openness of a function

In this article, inspired by the above-mentioned studies on bi-isotonic spaces and pairwise compactness in bitopological spaces, pairwise compactness in bi-isotonic spaces is defined for the first time, relevant definitions are given, and theorems are proven.

2 Preliminaries

In this section, we recall the basic concepts and related results of bi-isotonic spaces that are needed in the current paper.

Definition 2.1. Let X be a non-empty set and $P(X)$ denote its power set, i.e., the set of all subsets of X. A triple $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called a generalized bi-closure space such that $\mathrm{cl}_1 : P(X) \to P(X)$ and $\mathrm{cl}_2 : P(X) \to P(X)$ are arbitrary set-valued operators [2, 3].

For $i \in \{1, 2\}$, the arbitrary set-valued operators $\mathrm{cl}_i : P(X) \to P(X)$ are called closure operators, $\mathrm{int}_i : P(X) \to P(X)$ defined by $\mathrm{int}_i A = X \backslash \mathrm{cl}_i(X \backslash A)$ are called interior operators, and $\mathscr{N}_i : X \to P(P(X))$ defined by $\mathscr{N}_i(x) = \{N \in P(X) : x \in \mathrm{int}_i N\}$ are called neighborhood operators on generalized bi-closure space X.
For any closure operator $\mathrm{cl}_i : P(X) \to P(X)$, Kuratowski closure axioms are well-known as follows:

K0) $\mathrm{cl}_i(\varnothing) = \varnothing$ (grounded),

K1) $A \subseteq B \Rightarrow \mathrm{cl}_i(A) \subseteq \mathrm{cl}_i(B)$ (isotone),

K2) $A \subseteq \mathrm{cl}_i(A)$ (expansive),

K3) $\mathrm{cl}_i(A \cup B) \subseteq \mathrm{cl}_i(A) \cup \mathrm{cl}_i(B)$ (sub-additive),

K4) $\mathrm{cl}_i(\mathrm{cl}_i(A)) = \mathrm{cl}_i(A)$ (idempotent).

If the closure operators cl_1 and cl_2 satisfy the axioms (**K0**) and (**K1**), the generalized bi-closure space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called a bi-isotonic space [4, 5]. Additionally, a bi-isotonic space satisfying the axiom (**K2**) is called a bi-neighborhood space, a bi-neighborhood space satisfying the axiom (**K4**) is called a bi-closure space [2, 3], and a bi-closure space satisfying the axiom (**K3**) is called a bitopological space [8, 11].

Definition 2.2. A subset F of a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called closed if $\mathrm{cl}_1\mathrm{cl}_2(F) = F$. The complement of a closed set is called open [5]. Under the light of this definition, the following proposition is obvious.

Proposition 2.3. A subset A of a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is closed if and only if $\mathrm{cl}_1(A) = A$ and $\mathrm{cl}_2(A) = A$ [5].

It is obvious that a subset A of a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is an open set if and only if $A = X \backslash \mathrm{cl}_1\mathrm{cl}_2(X \backslash A)$, which means that $A = X \backslash \mathrm{cl}_i(X \backslash A)$ or $\mathrm{int_i}(A) = A$ for all $i \in \{1, 2\}$.

Proposition 2.4. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $Y \subseteq X$. The induced operators $\mathrm{cl}_i^Y : P(Y) \to P(Y)$ are isotonic provided that $\mathrm{cl}_i^Y(F) = \mathrm{cl}_i(F) \cap Y$ for all $F \subseteq Y$ and $i \in \{1,2\}$ [5].

Definition 2.5. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $Y \subseteq X$. $\left(Y, \mathrm{cl}_1^Y, \mathrm{cl}_2^Y\right)$ is called a subspace of $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ such that cl_1^Y and cl_2^Y are induced isotonic operators [5].

Definition 2.6. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $\left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ be bi-isotonic spaces. If $f : (X, \mathrm{cl}_i) \to \left(Y, \mathrm{cl}'_i\right)$ is continuous (open, closed or homeomorphism) then $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to \left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ is called $i-$continuous ($i-$open, $i-$closed or $i-$homeomorphism) [5].

If $f : X \to Y$ is a bijective and bi-continuous map and $f^{-1} : Y \to X$ is also bi-continuous, then it is called bi-homeomorphism [5].

Proposition 2.7. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $\left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ be bi-isotonic spaces. A map $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to \left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ is bi-continuous if and only if $\mathrm{cl}_i\left(f^{-1}(B)\right) \subseteq f^{-1}\left(\mathrm{cl}'_i(B)\right)$ for all $B \in P(Y)$ and $i \in \{1,2\}$ [5].

Proposition 2.8. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $\left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ be bi-isotonic spaces. A map $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to \left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ is bi-continuous if and only if $f(\mathrm{cl}_i(A)) \subseteq \mathrm{cl}'_i(f(A))$ for all $A \in P(X)$ and $i \in \{1,2\}$ [5].

Definition 2.9. A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called pairwise Hausdorff space if there are $U \in \mathscr{N}_1(x)$ and $V \in \mathscr{N}_2(y)$ such that $U \cap V = \varnothing$ for all distinct points $x, y \in X$ [5].

Proposition 2.10. A bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a pairwise Hausdorff space if and only if there is $U \in \mathscr{N}_1(x)$ such that $y \notin \mathrm{cl}_2(U)$ and there is $V \in \mathscr{N}_2(y)$ such that $x \notin \mathrm{cl}_1(V)$ for all distinct points $x, y \in X$ [5].

3 Pairwise compactness in bi-isotonic spaces

Definition 3.1. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $K, L \subset X$ such that $\mathrm{cl}_1(K) = K$ and $\mathrm{cl}_2(L) = L$. The family of subsets K is called cl_1- closed family and denoted by $\mathscr{F}_{\mathrm{cl}_1}$. The family of subsets L is called cl_2- closed family and denoted by $\mathscr{F}_{\mathrm{cl}_1}$. The family of subsets K or L is called a pairwise closed family and is denoted by $\mathscr{F}$.

Definition 3.2. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space, and the families $\mathscr{F}_{\mathrm{cl}_1} \cap \mathscr{F}$ and $\mathscr{F}_{\mathrm{cl}_2} \cap \mathscr{F}$ involve at least one non-empty set. If every pairwise closed family $\mathscr{F}$ having the finite intersection property has a non-empty intersection, then the bi-isotonic space X is called pairwise FHP-compact.

Definition 3.3. In a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, a non-empty family $\mathscr{F}_{\mathrm{cl}_1}$ is said to have cl_1- closed finite intersection property relative to cl_2 if the intersection over any finite subfamily $\mathscr{F}_{\mathrm{cl}_2}$ is non-empty.

Definition 3.4. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space. If every family of subsets of X having cl_1- closed finite intersection property relative to cl_2 has a non-empty intersection, then X is called cl_1- compact relative to cl_2 .

Definition 3.5. If a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is both cl_1- compact relative to cl_2 and cl_2- compact relative to cl_1, then it is called pairwise B-compact.

Definition 3.6. In a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ if any pairwise closed family $\mathscr{F}$ having a finite intersection property has a non-empty intersection, then the bi-isotonic space X is called pairwise S-compact.

According to this definition, if (X, cl_1) compact or (X, cl_2) compact, then $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ pairwise S-compact. From all these definitions, it is clear that bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is

pairwise S-compact $\Rightarrow$pairwise B-compact$\Rightarrow$pairwise FHP-compact.

Example 3.7. Let the operators $\mathrm{cl}_1 : P(\mathbb{R}) \to P(\mathbb{R})$ and $\mathrm{cl}_2 : P(\mathbb{R}) \to P(\mathbb{R})$ be, respectively, defined

$$\mathrm{cl}_1(C) = \begin{cases} C, & C \text{ is finite} \\ \mathbb{R}, & C \text{ is infinite} \end{cases}$$

$$\mathrm{cl}_1(C) = \begin{cases} \varnothing\ , & C = \varnothing \\ (-\infty, \lambda]\ , & \sup C = \lambda \\ \mathbb{R}\ , & \sup C = \infty \end{cases}$$

over the real numbers set. Obviously, although $(\mathbb{R}, \mathrm{cl}_1)$ is compact and $(\mathbb{R}, \mathrm{cl}_2)$ is not, $(\mathbb{R}, \mathrm{cl}_1, \mathrm{cl}_2)$ is pairwise compact.

Definition 3.8. Let (X, cl_i) be an isotonic space for each $i \in \{1, 2\}$ and $\mathscr{B}_i = \{A_j \subset X \,|\mathrm{int}_i(A_j) = A_j,\ j \in J\}$ be any open cover of in isotonic space (X, cl_i). If a cover $\mathscr{B}$ is given as $\mathscr{B} \subset \mathscr{B}_1 \cup \mathscr{B}_2$, then it is called a pairwise open cover of X in bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$.

Theorem 3.9. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space. In this case, the following properties are equivalent:

i. X is a pairwise compact space.

ii. Every family $\mathscr{F} = \{F_j \,|j \in J\}$, where J is an arbitrary indexing set, of pairwise closed subsets of X having empty intersection has a finite subfamily $\{F_{j_1}, F_{j_2}, \ldots, F_{j_n}\}$ with empty intersection.

iii. Every pairwise open cover $\mathscr{B} = \{A_j \,|j \in J\}$ of X has a finite subcover $\{A_{j_1}, A_{j_2}, \ldots, A_{j_n}\}$.

Proof . (i)⇔(ii) As is due to Definition 3.6, the necessary and sufficient condition of a bi-isotonic space to be pairwise compact is that if every finite subfamily $\{F_{j_1}, F_{j_2}, \ldots, F_{j_n}\}$ of any pairwise closed family $\mathscr{F} = \{F_j \,|j \in J\}$ satisfies $\bigcap_{k=1}^{n} F_{j_k} \neq \varnothing$, then the elements of $\mathscr{F}$ satisfies $\bigcap_{j\in I} F_j \neq \varnothing$. Obviously, the contrapositive equivalent of this condition is that every pairwise closed family $\mathscr{F} = \{F_j \,|j \in J\}$ satisfying $\bigcap_{j\in I} F_j = \varnothing$ has a finite subfamily $\{F_{j_1}, F_{j_2}, \ldots, F_{j_n}\}$ satisfying $\bigcap_{k=1}^{n} F_{j_k} = \varnothing$.

(ii)⇒(iii) Assume that every family $\mathscr{F} = \{F_j \,|j \in J\}$ of pairwise closed subsets of X satisfying $\bigcap_{j\in I} F_j = \varnothing$ has a finite subfamily $\{F_{j_1}, F_{j_2}, \ldots, F_{j_n}\}$ satisfying $\bigcap_{k=1}^{n} F_{j_k} = \varnothing$. Consider arbitrary pairwise open cover $\mathscr{B}$ of X such as pairwise open cover is defined by $\mathscr{B} = \{A_j \,|j \in J\}$ where $\mathrm{int}_1(A_j) = A_j$ or $\mathrm{int}_2(A_j) = A_j$. If we call $F_j = X - A_j$ for all $j \in J$, then

$$F_j = X - A_j = X - \mathrm{int}_i(A_j) = \mathrm{cl}_i(X - A_j) = \mathrm{cl}_i(F_j)$$

for $i = 1$ or $i = 2$. In the other words, $F_j \in \mathscr{F}$, that is, F_j are cl_1-closed or cl_2-closed. The complement of $X = \bigcup_{j\in I} A_j$ gives us $\varnothing = \bigcap_{j\in I} (X - A_j) = \bigcap_{j\in I} F_j$. By the hypothesis, there is a finite pairwise closed subfamily $\{F_{j_1}, F_{j_2}, \ldots, F_{j_n}\}$ such that $\bigcap_{k=1}^{n} F_{j_k} = \varnothing$. If the last equality is complemented again, there is a pairwise open subfamily $\{A_{j_1}, A_{j_2}, \ldots, A_{j_n}\}$ satisfying $\bigcup_{k=1}^{n} (X - F_{j_k}) = \bigcup_{k=1}^{n} A_{j_k} = X$.

(iii)⇒(ii) Assume that every pairwise cover of X has a finite subcover. Let us consider an arbitrary family of pairwise closed (i.e., cl_1-closed or cl_2-closed) subsets $F_j \subset X$ satisfying $\bigcap_{j\in I} F_j = \varnothing$. If we call $A_j = X - F_j$ for all $j \in J$, then

$$A_j = X - F_j = X - \mathrm{cl}_i(F_j) = \mathrm{cl}_i(X - F_j) = \mathrm{int}_i(A_j)$$

for $i = 1$ or $i = 2$, which means that $\mathrm{int}_1(A_j) = A_j$ or $\mathrm{int}_2(A_j) = A_j$. Besides, the complement of $\bigcap_{j\in I} F_j = \varnothing$ requires $\bigcup_{j\in J} X - F_j = \bigcup_{j\in J} A_j = X$. Thus, the family

$$\mathscr{B} = \{A_j \subset X \,|\mathrm{int}_1(A_j) = A_j \text{ or } \mathrm{int}_2(A_j) = A_j,\ j \in J\}$$

becomes a pairwise open cover of X, and the assumption requires that there is a pairwise open cover $\{A_{j_1}, A_{j_2}, \dots, A_{j_n}\}$ such that $\bigcup\limits_{k=1}^{n} A_{j_k} = X$. Once again, by complementing, we get

$$\bigcap_{k=1}^{n} X - A_{j_k} = \bigcap_{k=1}^{n} X - \mathrm{int}_i\left(A_{j_k}\right) = \bigcap_{k=1}^{n} F_{j_k} = \varnothing.$$

Consequently, every pairwise closed family $\{F_j \,|j \in J\}$ satisfying $\bigcap\limits_{j\in I} F_j = \varnothing$ has a finite subfamily $\{F_{j_1}, F_{j_2}, \dots, F_{j_n}\}$ satisfying $\bigcap\limits_{k=1}^{n} F_{j_k} = \varnothing$. This completes the proof. □

Definition 3.10. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $\left(Y, {\mathrm{cl}_1}^Y, {\mathrm{cl}_2}^Y\right)$ be its subspace with induced isotonic operators ${\mathrm{cl}_1}^Y$ and ${\mathrm{cl}_2}^Y$ on $Y \subset X$. If the families $\mathscr{B}_i^Y = \left\{A_j \,\middle|A_j = \mathrm{int}_i^Y\left(A_j\right), j \in J\right\}$ of open subsets in isotonic spaces $\left(Y, {\mathrm{cl}_i}^Y\right)$ for all $i \in \{1, 2\}$ are open covers of Y, then the cover $\mathscr{B}^Y$ associated with $\mathscr{B}^Y \subset \mathscr{B}_1^Y \cup \mathscr{B}_2^Y$ is called the pairwise open cover of the bi-isotonic subspace $\left(Y, {\mathrm{cl}_1}^Y, {\mathrm{cl}_2}^Y\right)$.

Definition 3.11. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $Y \subset X$. The bi-isotonic subspace $\left(Y, {\mathrm{cl}_1}^Y, {\mathrm{cl}_2}^Y\right)$ is called pairwise compact, provided every pairwise open cover $\mathscr{B}^Y$ has a finite subcover.

Theorem 3.12. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $Y \subset X$. The bi-isotonic subspace $\left(Y, {\mathrm{cl}_1}^Y, {\mathrm{cl}_2}^Y\right)$ is pairwise compact if and only if every family $\{F_j |\, j \in J\}$ of ${\mathrm{cl}_1}^Y-$ closed or ${\mathrm{cl}_2}^Y-$ closed subsets having empty intersection has a finite subfamily $\{F_{j_1}, F_{j_2}, \dots, F_{j_n}\}$ with empty intersection.

Proof . $\Rightarrow$ Suppose that the subspace $Y \subset X$ is pairwise compact. Then, Definition 3.11 requires that every pairwise open cover of Y has a finite subcover in the bi-isotonic subspace. Let us consider a family

$$\left\{F_j \,\middle|F_j = \mathrm{cl}_1^Y\left(F_j\right) \text{ or } F_j = \mathrm{cl}_2^Y\left(F_j\right), j \in J\right\}$$

where $\bigcap\limits_{j\in I} F_j = \varnothing$. If we call $A_j = Y - F_j$ for all $j \in J$, then we get

$$\mathrm{int}_i^Y\left(A_j\right) = Y - \mathrm{cl}_i^Y\left(Y - A_j\right) = Y - \mathrm{cl}_i^Y\left(F_j\right) = Y - F_j = A_j$$

for $i = 1$ or $i = 2$. The complement of $\bigcap\limits_{j\in I} F_j = \varnothing$ gives us $\bigcup\limits_{j\in J}\left(Y - F_j\right) = \bigcup\limits_{j\in J} A_j = Y$. The family $\mathscr{B}^Y = \left\{A_j \,\middle|A_j = \mathrm{int}_1^Y\left(A_j\right) \text{ or } A_j = \mathrm{int}_2^Y\left(A_j\right), j \in J\right\}$ becomes a pairwise open cover of Y in subspace. By the hypothesis, there is a pairwise open subcover $\{A_{j_1}, A_{j_2}, \dots, A_{j_n}\}$ such that $\bigcup\limits_{k=1}^{n} A_{j_k} = Y$. With the aid of subsequent complement, we get $\bigcap\limits_{k=1}^{n}\left(Y - A_{j_k}\right) = \bigcap\limits_{k=1}^{n} F_{j_k} = \varnothing$. Consequently, every family of pairwise closed subsets satisfying $\bigcap\limits_{j\in I} F_j = \varnothing$ has a subfamily $\{F_{j_1}, F_{j_2}, \dots, F_{j_n}\}$ satisfying $\bigcap\limits_{k=1}^{n} F_{j_k} = \varnothing$.

Conversely, assume that every family $\{F_j |\, j \in J\}$ of $\mathrm{cl}_1^Y -$ closed or $\mathrm{cl}_2^Y -$ closed subsets of X satisfying $\bigcap\limits_{j\in I} F_j = \varnothing$ has a subfamily $\{F_{j_1}, F_{j_2}, \dots, F_{j_n}\}$ satisfying $\bigcap\limits_{k=1}^{n} F_{j_k} = \varnothing$. Let $\mathscr{B}^Y$ be any pairwise induced open cover of Y, which means that the cover $\mathscr{B}^Y = \{A_j \,|j \in J\}$ is a family of subsets of $\mathrm{int}_1^Y\left(A_j\right) = A_j$ or $\mathrm{int}_2^Y\left(A_j\right) = A_j$. If we denote $F_j = Y - A_j$ for all $j \in J$, then we have

$$\mathrm{cl}_i^Y\left(F_j\right) = \mathrm{cl}_i^Y\left(Y - A_j\right) = Y - \mathrm{int}_i^Y\left(A_j\right) = Y - A_j = F_j$$

for $i = 1$ or $i = 2$. Obviously, the complement of $Y = \bigcup\limits_{j\in I} A_j$ is obtained as $\varnothing = \bigcap\limits_{j\in I}\left(Y - A_j\right) = \bigcap\limits_{j\in I} F_j$. By assumption, there is a subfamily $\{F_{j_1}, F_{j_2}, \dots, F_{j_n}\}$ satisfying $\bigcap\limits_{k=1}^{n} F_{j_k} = \varnothing$. Finally, there is a finite subcover $\{A_{j_1}, A_{j_2}, \dots, A_{j_n}\}$ with once more complement $\bigcup\limits_{k=1}^{n}\left(Y - F_{j_k}\right) = \bigcup\limits_{k=1}^{n} A_{j_k} = Y$. □

Corollary 3.13. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $Y \subset X$. The bi-isotonic subspace $\left(Y, \mathrm{cl}_1{}^Y, \mathrm{cl}_2{}^Y\right)$ is pairwise compact if and only if a family $\mathscr{F}^Y$ of pairwise induced closed subsets in Y having finite intersection property provides $\bigcap\limits_{F \in \mathscr{F}^Y} F \neq \varnothing$.

Now, let us define the pairwise neighborhood family of a point in a bi-isotonic space as follows.

Definition 3.14. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $x \in X$. The function $\mathscr{N} : X \to P(P(X))$ given by $\mathscr{N}(x) \subset \mathscr{N}_1(x) \cup \mathscr{N}_2(x)$ is called a neighborhood function on bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ if family $\mathscr{N}_i(x)$ is a neighborhood family of x in the isotonic space (X, cl_i) for each $i \in \{1, 2\}$. Also, the family $\mathscr{N}(x)$ is called a pairwise neighborhood family of the point x in the bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$.

Theorem 3.15. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space, and $N_x \in \mathscr{N}(x)$ denote a neighborhood of any point $x \in X$. If there are a finite number of points $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup\limits_{k=1}^{n} N_{x_k}$, then the bi-isotonic space X is pairwise compact.

Proof . Let $N_x \in \mathscr{N}(x)$ denote a neighborhood of any point $x \in X$. Assume that there are a finite number of points $x_1, x_2, ..., x_n \in X$, where $X = \bigcup\limits_{k=1}^{n} N_{x_k}$ is satisfied and consider any pairwise open cover

$$\mathscr{B} = \{A_j \subset X \,|\mathrm{int}_1(A_j) = A_j \text{ or } \mathrm{int}_2(A_j) = A_j,\ j \in J\}$$

of X means that $\bigcup\limits_{j \in J} A_j = X$. Absolutely, there is at least a $j = j(x)$ such that $x \in \mathrm{int}_i(A_j)$ for all $x \in X$, that is, $A_{j(x)} \in \mathscr{N}(x)$. By the assumption, the finite family of subsets $A_{j(x_k)}$, $1 \leq k \leq n$ covers X such that $\bigcup\limits_{k=1}^{n} A_{j(x_k)} = X$. This is sufficient to prove the bi-isotonic space X is pairwise compact. □

Remark 3.16. The definition and characterizations of pairwise compactness have naturally been extended to bi-isotonic spaces, just like many known concepts of bitopological spaces. However, working with bi-isotonic spaces has not been sufficient for some extensions. For example, since the existences of the axioms (**K2**) and (**K4**) in addition to the axioms (**K0**) and (**K1**) are needed to prove the inverse of the above theorem, the inverse of this theorem below has been given in bi-closure space, which is a particular case of bi-isotonic space.

Theorem 3.17. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-closure space, and $N_x \in \mathscr{N}(x)$ denote a neighborhood of any $x \in X$. If the bi-closure space X is pairwise compact. Then there are finite points $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup\limits_{k=1}^{n} N_{x_k}$.

Proof . Let X be a pairwise compact bi-closure space. As is due to Theorem 3.9 (iii), every pairwise open cover $\mathscr{B}$ of X has a finite subcover. If any neighborhood of any $x \in X$ is given, there is $\mathscr{N}(x) \subset \mathscr{N}_1(x) \cup \mathscr{N}_2(x)$, which means that $N_x \in \mathscr{N}_1(x)$ or $N_x \in \mathscr{N}_2(x)$. Then $x \in \mathrm{int}_1(N_x)$ or $x \in \mathrm{int}_2(N_x)$ since $\mathscr{N}_i(x) = \{N_x \in P(X) : x \in \mathrm{int}_i(N_x)\}$. As the points x are sweeping the set X, $\bigcup\limits_{x \in X} \mathrm{int}_i(N_x) = X$ is satisfied for $i = 1$ or $i = 2$. Moreover, the axiom (**K4**) gives us $\mathrm{int}_i(\mathrm{int}_i(N_x))\,\mathrm{int}_i(N_x)$, and the family of subsets $\mathrm{int}_i(N_x)$ becomes a pairwise open family covering X. By the assumption of the bi-closure space X being pairwise compact, there is a finite pairwise open subcover $\{\mathrm{int}_i(N_{x_1}), \mathrm{int}_i(N_{x_2}), \ldots, \mathrm{int}_i(N_{x_n})\}$ such that $X = \bigcup\limits_{k=1}^{n} \mathrm{int}_i(N_{x_k})$. Moreover, by the aid of axiom (**K2**), $x_k \in \mathrm{int}_i(N_{x_k}) \subset N_{x_k}$, i.e., $x_k \in N_{x_k}$ is satisfied, and this proves there are finite numbers of points $x_1, x_2, ..., x_n$ satisfying $X = \bigcup\limits_{k=1}^{n} N_{x_k}$. □

Remark 3.18. The following definition is not up to the mark for bi-isotonic spaces. The space has to be at least a bi-neighborhood or more. Otherwise, $N \in \mathscr{N}(x)$ does not require $x \in N$ at the deficiency of axiom (**K2**).

Definition 3.19. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-neighborhood space and $A \subset X$. A point $x \in X$ is called a limit point of A in X if every neighborhood $N \in \mathscr{N}(x)$ of the point x satisfies $(N - \{x\}) \cap A = \emptyset$.

Theorem 3.20. Every infinite subset of a pairwise compact bi-closure space has at least one limit point.

Proof . Let A be an arbitrary infinite subset of a pairwise compact bi-closure space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$. Assume that A has no limit point. In this case, every point x has some neighborhood $N_x \in \mathscr{N}(x)$ satisfying $(N_x - \{x\}) \cap A = \varnothing$. Theorem 3.17 requires the existence of finite numbers of points $x_1, x_2, ..., x_n$ satisfying $X = \bigcup_{k=1}^{n} N_{x_k}$ since the bi-closure space X is pairwise compact. By the fact that $A \subset X$, there is $A \subset \bigcup_{k=1}^{n} N_{x_k}$. However, each neighborhood $N_{x_k} \in \mathscr{N}(x_k)$ for $1 \leq k \leq n$ provides

$$A \cap (N_{x_k} - \{x_k\}) = \emptyset,$$

since the points $x_1, x_2, ..., x_n \in X$ are not limit points of A. Therefore, A is $\varnothing$ or a finite set. This contradicts the assumption. Consequently, A has at least one limit point. □
Before investigating pairwise compactness in pairwise Hausdorff bi-closure spaces, let us express the implication of being compact in Hausdorff closure spaces by the following supporting lemma.

Lemma 3.21. Compact subsets of Hausdorff closure space are closed.

Proof . Consider a compact subset C of a Hausdorff closure space (X, cl). It is known from [7] that for any two distinct points $x, y \in X$, there is $U \in \mathscr{N}(x)$ such that $y \notin \mathrm{cl}(U)$ if and only if the isotonic space (X, cl) is Hausdorff. Then, it is true for closure space, which is the special case of isotonic space. If we call $\mathrm{cl}(U) = F$, then $\mathrm{cl}(F) = F$ (from (**K4**)). In this regard, F is a closed set. Moreover, $U \subset \mathrm{cl}(U) = F$ (from (**K2**)). $U \in \mathscr{N}(x)$ and $U \subset F$ imply that $F \in \mathscr{N}(x)$ (from (**K1**)). As a result, $F \in \mathscr{N}(x)$ requires $x \in F$ (from (**K2**)). Let $\mathscr{F}(x)$ be a closed neighborhood family of the point x. Since there are some closed sets $F \in \mathscr{F}(x)$, do not include all y points different from the point x. This means that

$$\bigcap_{F \in F(x)} F = \{x\}.$$

Now, let us consider any $x \in \mathrm{cl}(C)$. It is also known from [4] that $\mathrm{cl}(C) = \{x \in X | \forall N \in \mathscr{N}(x) \ : \ C \cap N \neq \varnothing\}$ is satisfied in isotonic spaces, thus in closure spaces, too. By the compactness of C, the induced closed neighborhood family $\mathscr{F}^C(x) = \{C \cap F | F \in \mathscr{F} \ : \ x \in F\}$ has finite intersection property and a non-empty intersection such that

$$\varnothing \neq \bigcap_{C \cap F \in \mathscr{F}^C(x)} (C \cap F) = C \cap \bigcap_{F \in \mathscr{F}(x)} F = C \cap \{x\}.$$

Consequently, $C \cap \{x\} \neq \varnothing$ gives us $x \in C$, that is, $\mathrm{cl}(C) \subset C$. From (**K2**), it is known that $C \subset \mathrm{cl}(C)$, and this completes the proof. □

Theorem 3.22. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise Hausdorff bi-closure space. Then cl_1-compact subsets of X are cl_2-closed.

Proof . Let C be a cl_1-compact subset of pairwise Hausdorff bi-closure space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$. According to Proposition 2.10, the pairwise Hausdorffness requires the existence of $N_y \in \mathscr{N}_2(y)$ such that $x \notin \mathrm{cl}_1(N_y)$ for all distinct points $x, y \in X$. $\mathrm{cl}_1(N_y)$ is a cl_1-closed set, since $\mathrm{cl}_1(\mathrm{cl}_1(N_y)) = \mathrm{cl}_1(N_y)$ from the axiom (**K0**). Assume that $\mathscr{F}_{\mathrm{cl}_1}(y)$ is a family of cl_1-closed subsets, including the point y. In this case,

$$\bigcap_{\mathrm{cl}_1(N_y) \in \mathscr{F}_{\mathrm{cl}_1}(y)} \mathrm{cl}_1(N_y) = \{y\}$$

since the points x different from y satisfy $x \notin \mathrm{cl}_1(N_y)$. Let's take any $y \in \mathrm{cl}_2(C)$. Here

$$\mathrm{cl}_2(C) = \{y \in X | \forall N_y \in \mathscr{N}_2(y) : \ C \cap N_y \neq \emptyset\}$$

is known, and this means that $C \cap N_y \neq \emptyset$ for each $N_y \in \mathscr{N}_2(y)$. Additionally, $C \cap \mathrm{cl}_1(N_y) \neq \emptyset$ since $C \cap N_y \subset C \cap \mathrm{cl}_1(N_y)$ by virtue of $y \in N_y \subseteq \mathrm{cl}_1(N_y)$ (from (**K2**)). Under the assumption cl_1-compactness of C, there are finite numbers of points $y_1, y_2, ..., y_n \in X$ such that $C \subset \bigcup_{k=1}^{n} N_{y_k}$ for cl_1-closed subsets $\mathrm{cl}_1(N_y) \in \mathscr{F}_{\mathrm{cl}_1}(y)$. Thus, we find

$$\bigcap_{k=1}^{n} (C \cap \mathrm{cl}_1(N_{y_k})) \neq \emptyset,$$

which means that the family of the induced cl_1-closed sets

$$\mathscr{F}^C_{\mathrm{cl}_1}(y) = \{\, C \cap \mathrm{cl}_1(N_y) |\, \mathrm{cl}_1(N_y) \in \mathscr{F}_{\mathrm{cl}_1}(y)\,\}$$

has finite intersection property and

$$\bigcap_{C\cap \mathrm{cl}_1(N_y)\in F^C_{\mathrm{cl}_1}(y)} (C \cap \mathrm{cl}_1(N_y)) \neq \emptyset.$$

This implies that

$$\emptyset \neq C \cap \bigcap_{\mathrm{cl}_1(N_y)\in \mathscr{F}_{\mathrm{cl}_1}(y)} \mathrm{cl}_1(N_y) = C \cap \{y\}\,.$$

Consequently, $y \in C$ and this proves $\mathrm{cl}_2(C) \subseteq C$. Further $C \subseteq \mathrm{cl}_2(C)$ from the axiom (**K2**), and the proof is completed. □

The following corollary is easily proved via a similar vein under consideration the existence of $N_x \in \mathscr{N}_1(x)$ such that $y \notin \mathrm{cl}_1(N_x)$ for all distinct points $x, y \in X$ arising from the pairwise Hausdorffness.

Corollary 3.23. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise Hausdorff bi-closure space. Then cl_i-compact subsets of X are cl_j-closed for $i, j \in \{1, 2\}$ such that $i \neq j$.

Theorem 3.24. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise compact bi-isotonic space. Then cl_1-closed subsets of X are cl_2-compact.

Proof . Let C be a cl_1-closed non-empty subset of X. Consider any family $\mathscr{F}_{\mathrm{cl}^Y_2} = \{\, C \cap F_j |\, \mathrm{cl}_2(F_j) = F_j,\, j \in J\}$ of cl^Y_2-closed subsets of C such that $\bigcap_{j\in I}(C \cap F_j) = \emptyset$. Then the family $\mathscr{F}_{\mathrm{cl}_2} = \{\, F_j |\, \mathrm{cl}_2(F_j) = F_j,\, j \in J\}$ consists of cl_2-closed subsets of X. Furthermore, $\mathscr{F}_{\mathrm{cl}_2} \cup \{C\}$ consists of pairwise closed subsets of X and has an empty intersection. There is a finite subfamily $\{F_{j_1}, F_{j_2}, \ldots, F_{j_n}, C\}$ satisfying $C \cap \bigcap_{k=1}^{n} F_{j_k} = \emptyset$ since X is pairwise compact. By the fact that C is a non-empty subset of X, there is a finite subfamily $\{C \cap F_{j_1}, C \cap F_{j_2}, \ldots, C \cap F_{j_n}\}$ of cl^Y_2-closed subsets of C satisfying $\bigcap_{k=1}^{n}(C \cap F_{j_k}) = \emptyset$. This is sufficient to show C is cl_2-compact. □
The following corollary can be easily proven in a similar way to the proof of this theorem.

Corollary 3.25. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise compact bi-isotonic space. cl_2-closed subsets of X are cl_1-compact.

Considering Theorem 3.22, Corollary 3.23, Theorem 3.24, and Corollary 3.25, the following corollary is obvious.

Corollary 3.26. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise compact and pairwise Hausdorff bi-closure space. A subset $C \subset X$ is cl_i-compact if and only if cl_j-closed for $i, j \in \{1, 2\}$ such that $i \neq j$.

The following theorem proceeds under considerations any compact subsets of Hausdorff closure space are closed from Lemma 3.21 and cl_i-closed subsets of pairwise compact bi-closure space are cl_j-compact from Corollary 3.26.

Theorem 3.27. Let (X, cl_1) and (X, cl_2) be Hausdorff closure spaces and $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise compact bi-closure space. Then $\mathrm{cl}_1 = \mathrm{cl}_2$.

Proof . It should be shown that both each cl_1-closed subset is cl_2-closed and each cl_2-closed subset is cl_1-closed in bi-closure space X. For that purpose, suppose that C is any cl_1-closed subset of X. According to Corollary 3.26, C is a cl_2-compact subset. Moreover, according to Lemma 3.21, C is cl_2-closed because (X, cl_2) is Hausdorff closure space. It is easy to prove that cl_2-closed subsets are cl_1-closed in the same vein. □

Since axiom (**K2**) is needed in the proof of the theorem below, the space must be at least a bi-neighborhood space.

Theorem 3.28. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-neighborhood space. The union of a finite number of pairwise compact subsets of X is pairwise compact.

Proof . Let Y_1 and Y_2 be two pairwise compact subsets of X, and let us show their union $Y = Y_1 \cup Y_2$ is pairwise compact. To that end, consider any family $\{F_j \,|j \in J\}$ of cl_1^Y-closed or cl_2^Y-closed i.e., $F_j = \mathrm{cl}_1^Y(F_j)$ or $F_j = \mathrm{cl}_2^Y(F_j)$ subsets $F_j \subset X$ satisfying $\bigcap\limits_{j\in I} F_j = \emptyset$. Then for $s = 1$ and $s = 2$, subsequent relations exist

$$\begin{gathered} Y_s \subseteq Y \\ Y_s \cap \mathrm{cl}_i(F_j) \subseteq Y \cap \mathrm{cl}_i(F_j) \\ \mathrm{cl}_i^{Y_s}(F_j) \subseteq \mathrm{cl}_i^{Y}(F_j) = F_j. \end{gathered}$$

Moreover, the subsets F_j are $\mathrm{cl}_1^{Y_s}-$closed or $\mathrm{cl}_2^{Y_s}-$closed since $F_j \subseteq \mathrm{cl}_i^{Y_s}(F_j)$ from axiom (**K2**). Therefore, there is a finite subfamily $\{F_{j_k} \,|1 \leqslant k \leqslant m\}$ of $\mathrm{cl}_1^{Y_1}-$closed or $\mathrm{cl}_2^{Y_1}-$closed subsets such that $\bigcap\limits_{k=1}^{m} F_{j_k} = \emptyset$ since Y_1 is pairwise compact and also, there is a finite subfamily $\{F_{j_k} \,|m+1 \leqslant k \leqslant n\}$ of $\mathrm{cl}_1^{Y_2}-$closed or $\mathrm{cl}_2^{Y_2}-$closed subsets such that $\bigcap\limits_{k=m+1}^{n} F_{j_k} = \emptyset$ since Y_2 is pairwise compact. Consequently, the family $\{F_j \,|j \in J\}$ of cl_1^Y-closed or cl_2^Y-closed subsets where $\bigcap\limits_{j\in I} F_j = \emptyset$ has a finite subfamily

$$\{F_{j_k} \,|1 \leqslant k \leqslant m\} \cup \{F_{j_k} \,|m+1 \leqslant k \leqslant n\} = \{F_{j_k} \,|1 \leqslant k \leqslant n\}$$

where $\bigcap\limits_{k=1}^{m} F_{j_k} = \emptyset$. This proves the pairwise compactness of $Y = Y_1 \cup Y_2$. □

Theorem 3.29. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise Hausdorff bi-closure space. The intersection of any number of cl_i-compact subsets of X is cl_j-closed for $i, j \in \{1, 2\}$ such that $i \neq j$.

Proof . Let C_s for any $s \in I$ be a cl_i-compact subset of X where $i \in \{1, 2\}$. Let us prove the subset $C = \bigcap\limits_{s\in I} C_s$ is cl_i-compact. Based on Corollary 3.23, cl_i-compact subsets are cl_j-closed for $i, j \in \{1, 2\}$ such that $i \neq j$ in pairwise Hausdorff bi-closure space. This means C_s is cl_j-closed, that is $\mathrm{cl}_j(C_s) = C_s$. Applying the isotony property to $C = \bigcap\limits_{s\in I} C_s \subseteq C_s$ gives us

$$\mathrm{cl}_j(C) = \mathrm{cl}_j\left(\bigcap_{s\in I} C_s\right) \overset{\text{(from (K1))}}{\subseteq} \bigcap_{s\in I} \mathrm{cl}_j(C_s) = \bigcap_{s\in I} C_s = C.$$

Moreover, from (**K2**), there is the inclusion $C \subseteq \mathrm{cl}_j(C)$, proving the subset C is cl_j-closed. □

Here, the subset of C is cl_i- is compact since $C \subset C_s$ because cl_j-closed subsets are cl_i-compact in pairwise compact bi-closure space from Theorem 3.24 and Corollary 3.25, and the following corollary is obvious.

Corollary 3.30. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a pairwise compact and pairwise Hausdorff bi-closure space. The intersection of any number of cl_i-compact subsets of X is cl_i-compact for $i \in \{1, 2\}$.

Theorem 3.31. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $(Y, \mathrm{cl}_1', \mathrm{cl}_2')$ be two bi-closure spaces and $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl}_1', \mathrm{cl}_2')$ be a bi-continuous function. If the bi-isotonic space X is pairwise compact, then $f(X)$ is a pairwise compact subset in Y.

Proof . Let us consider any pairwise closed family $\{F_j |\, F_j \subset f(X)\}_{j\in J}$ of $\mathrm{cl}_1'-$closed or $\mathrm{cl}_2'-$closed subsets F_j of $f(X)$ satisfying $\bigcap\limits_{j\in I} F_j = \varnothing$. From Proposition 2.7, $\mathrm{cl}_i\left(f^{-1}(F_j)\right) \subseteq f^{-1}(\mathrm{cl'}_i(F_j))$ for each $i \in \{1, 2\}$ and $j \in J$ since f is bi-continuous. By virtue of $\mathrm{cl}_i'(F_j) = F_j$ for each $i \in \{1, 2\}$ and $j \in J$, it is found that $\mathrm{cl}_i\left(f^{-1}(F_j)\right) \subseteq f^{-1}(F_j)$. Moreover, the inverse image of $\bigcap\limits_{j\in I} F_j = \varnothing$ gives us

$$f^{-1}\left(\bigcap_{j\in I} F_j\right) = \bigcap_{j\in I} f^{-1}(F_j) = \varnothing.$$

This shows $\bigcap\limits_{j\in I} \mathrm{cl}_i\left(f^{-1}(F_j)\right) = \varnothing$. In the bi-closure space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, the subsets $\mathrm{cl}_i\left(f^{-1}(F_j)\right)$ are cl_1-closed or cl_2-closed and the family

$$\left\{ \mathrm{cl}_i\left(f^{-1}(F_j)\right) \middle|\, f^{-1}(F_j) \subset X \right\}_{j\in J}$$

of these subsets has a finite subfamily

$$\left\{\mathrm{cl}_i\left(f^{-1}\left(F_{j_1}\right)\right), \mathrm{cl}_i\left(f^{-1}\left(F_{j_2}\right)\right), \ldots, \mathrm{cl}_i\left(f^{-1}\left(F_{j_n}\right)\right)\right\}$$

satisfying $\bigcap_{k=1}^{n} \mathrm{cl}_i\left(f^{-1}\left(F_{j_k}\right)\right) = \varnothing$ since X is pairwise compact. From Proposition 2.10, we get

$$f\left(\mathrm{cl}_i\left(f^{-1}\left(F_{j_k}\right)\right)\right) \subseteq \mathrm{cl}'_i\left(f\left(f^{-1}\left(F_{j_k}\right)\right)\right) = \mathrm{cl}'_i\left(F_{j_k}\right) = F_{j_k}$$

since f is a bi-continuous function and each subsets $f^{-1}\left(F_{j_k}\right) \subset X$ where $k \in \{1, 2, \ldots, n\}$. Consequently,

$$f\left(\bigcap_{k=1}^{n} \mathrm{cl}_i\left(f^{-1}\left(F_{j_k}\right)\right)\right) \subseteq \bigcap_{k=1}^{n} f\left(\mathrm{cl}_i\left(f^{-1}\left(F_{j_k}\right)\right)\right) \subseteq \bigcap_{k=1}^{n} \mathrm{cl}'{}_i\left(F_{j_k}\right) = \bigcap_{k=1}^{n} F_{j_k}$$

gives $\varnothing = \bigcap_{k=1}^{n} F_{j_k}$, and the proof is completed. □

The following corollaries are direct results of Theorem 3.31.

Corollary 3.32. Let $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to \left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ be a bi-continuous surjective function. If the bi-closure space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is pairwise compact, then the bi-closure space $\left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ is pairwise compact.

Corollary 3.33. Let $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to \left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ be a bi-homeomorphism. The bi-closure space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is pairwise compact if and only if the bi-closure space $\left(Y, \mathrm{cl}'_1, \mathrm{cl}'_2\right)$ is pairwise compact.

4 Conclusion

In conclusion, our work significantly contributes to the pairwise compactness in the extended forms of bitopological space, thereby supporting the advancement of research within generalized topological structures. Since our approach allows us to consider the set of points close to a subset in two different ways simultaneously, we improve an understanding of this particular type of space and pairwise compactness, closedness, and Hausdorffness relations with the following findings:

- In bi-isotonic spaces, pairwise compactness has been expressed by the finite intersection property and pairwise open cover.
- Pairwise compactness in bi-isotonic subspaces has been defined by both reduced closure functions and interior functions that are duals of these functions.
- Pairwise compactness has been characterized by the concept of neighborhood, and it was seen that working in bi-isotonic spaces is not sufficient in this process. Therefore, although many characterizations related to compactness have been extended to bi-isotonic spaces in a conventional way, some characterizations have occasionally been given in bi-closure or bi-neighborhood spaces.
- Interesting results have been obtained by investigating the pairwise Hausdorffness of separation axioms and pairwise compactness relations in bi-closure space.
- It has also been observed that similar cross-relationships exist for closed subsets of pairwise compact spaces.
- Finally, it has been observed that the pairwise compactness is preserved under bi-continuity between bi-closure spaces.

The obtained results will represent a crucial step towards a more in-depth examination of the compactness properties of bi-isotonic spaces, leading to discoveries and advancements in both theoretical and applied mathematics. As our findings may resonate across mathematics, we anticipate our work will inspire further investigations on countable, sequential, or local pairwise compactness in bi-isotonic spaces.

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