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A new subclass of Ma-Minda starlike functions associated with a heart-shaped curve

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Abstract

In this paper, we extend the q-derivative operator, which plays an essential role in quantum calculus. Indeed, by using the Hadamard product and generalized Koebe function we define the following (α, β, γ)-derivative operator

$$\mathrm{d}_{\alpha,\beta,\gamma} f(z) = \frac{1}{z} \left\{ f(z) * \mathfrak{L}_{\alpha,\beta,\gamma}(z) \right\},$$

where

$$\mathfrak{L}_{\alpha,\beta,\gamma}(z) = \frac{2(1-\gamma)z}{(1-\alpha z)(1-\beta z)},$$

and $\alpha \in [-1, 1]$, $\beta \in [-1, 1]$, $\alpha\beta \neq \pm 1$ and $\gamma \in [0, 1)$. Then by subordination relation, the operator $\mathrm{d}_{\alpha,\beta,\gamma} f(z)$, and a special function $\phi_\delta(z) = 1 + \delta z / \exp(\delta z)$ $(0 < \delta \leq 1)$, we define a new particular Ma-Minda class. We investigate some properties of this class, such as, radius problem and coefficient estimate.

Keywords: Unit disk, Analytic functions, Starlike function, Subordination, Radius problems, Coefficients problems
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1 Introduction

In 1992, Ma and Minda [15] introduced a certain class of starlike functions using the subordination relation "$\prec$" as follows:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z), \quad |z| < 1 \right\},$$

where φ is an analytic function with positive real part such that satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$. We note that the function φ maps the unit disk $|z| < 1$ onto a starlike domain with respect to $\varphi(0) = 1$ which is symmetric with respect to the real axis. Also, $\mathcal{A}$ denotes the family of all analytic and normalized functions in the unit disk.

During the past few decades there has been considerable interest in the study of Ma-Minda starlike functions. Many authors have studied the class $\mathcal{S}^*(\varphi)$ for special cases of φ. Here, we recall some of these cases:

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1. If we take $\varphi(z) = (1 + Az)/(1 + Bz)$, $-1 \le B < A \le 1$ then the class $\mathcal{S}^*(\varphi) \equiv \mathcal{S}^*[A, B]$ is called the Janowski starlike functions, see [8]. If we let $A = 1 - 2\alpha$ and $B = -1$, then $\mathcal{S}^*[1 - 2\alpha, -1]$ becomes the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, where $0 \le \alpha < 1$.
2. By taking $\varphi(z) = ((1 + z)/(1 - z))^\beta$, we obtain the class $\mathcal{SS}^*(\beta)$ of strongly starlike function of order β, where $\beta \in (0, 1]$.
3. The class $\mathcal{S}^*(\sqrt{1 + cz}) \equiv \mathcal{S}^*(q_c)$, $c \in (0, 1]$ was studied by Aouf et al. [3], while $\mathcal{S}^*(\sqrt{1 + z})$ was introduced by Sokół and Stankiewicz [23].
4. Raina and Sokół introduced and studied the class $\mathcal{S}^*(z + \sqrt{1 + z^2})$, see [20].
5. Letting $\varphi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\varphi)$ reduces the class $\mathcal{S}^*_{sin}$ which was defined by Cho et al. in [4].
6. The class $\mathcal{S}^*(1/(1 - z)^s) \equiv \mathcal{S}^*_s$, $c \in (0, 1]$ was introduced by Kanas et al. in [10] which is related to a domain bounded by a right branch of a hyperbola.
7. If we take $\varphi(z) = 1 + z/((1 - pz)(1 - qz))$, where $(p, q) \in [-1, 1] \times [-1, 1]$, then we get the class $\mathcal{S}^*_k(p, q)$ which is associated with the generalized Koebe function [11].
8. Masih et al. in [17] introduced and studied the class $\mathcal{S}^*((1 - z)^\lambda) \equiv \mathcal{S}^*_L(\lambda)$ $(0 < \lambda < 1)$.
9. The case $\varphi = 1 + 4z/3 + 2z^2/3$ was studied by Sharma et al. [21].
10. If we take $\varphi = e^z$, we get the class $\mathcal{S}^*_e$ which was introduced by Mendiratta et al. [16].

Some more special cases of Ma-Minda class can be found in [14, 21, 22]. Motivated by above works, in this paper we introduce a new class of Ma-Minda starlike functions.

The structure of this paper is as follows. In Section 2 we define a new derivative operator and a new subclass of analytic functions. In Section 3 we investigate radius problem and coefficient estimate.

2 Preliminaries

Let us first introduce our notations. Throughout the paper we denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in the complex plane $\mathbb{C}$ and by $\partial\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$ the boundary of $\mathbb{D}$. Also, let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ and $\mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho,\ \rho > 0\}$. All analytic and normalized functions f $(f(0) = f'(0) - 1 = 0)$ in $\mathbb{D}$ having the form

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{D}), \tag{2.1}$$

with $a_n \in \mathbb{C}$ is denoted by $\mathcal{A}$. A subclass of $\mathcal{A}$ including of all univalent (one-to-one) functions in $\mathbb{D}$ is denoted by $\mathcal{U}$. For two analytic functions f_1 and f_2 in $\mathcal{A}$, we say that f_1 is subordinate to f_2, written as $f_1(z) \prec f_2(z)$ $(z \in \mathbb{D})$ or $f_1 \prec f_2$, if there exists a Schwarz function $w : \mathbb{D} \to \overline{\mathbb{D}}$ so that $f_1(z) = f_2(w(z))$ for all $z \in \mathbb{D}$. We have the following equivalence relation provided that f_2 is univalent in $\mathbb{D}$:

$$f_1(z) \prec f_2(z)\ (z \in \mathbb{D}) \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{D}) \subset f_2(\mathbb{D}).$$

Let $\alpha \in [0, 1)$. A function $f \in \mathcal{S}$ is said to be starlike of order α in Δ if, and only if,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathbb{D}).$$

Also, a function $f \in \mathcal{S}$ is a convex function of order α in Δ if, and only if,

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathbb{D}).$$

We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the class of starlike and convex functions of order α, respectively. By the Alexander theorem $f \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$. It should be remarked that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(\alpha) \equiv \mathcal{K}$ are, respectively, the classes of starlike and convex functions in $\mathbb{D}$. We say also that a function $f \in \mathcal{A}$ is strongly starlike function of order β, denoted by $\mathcal{SS}(\beta)$, if satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta, \quad (z \in \mathbb{D}).$$

We note that $\mathcal{SS}(1) \equiv \mathcal{S}^*$. The main objective of this paper is to introduce a special case of Ma-Minda starlike function class. Before, we introduce the q-calculus. Quantum calculus (q-calculus) was developed by Frank Hilton Jackson [6, 7] in the early twentieth century. A number of researchers were interested in this method of connecting mathematics and physics. Number theory, combinatorics, orthogonal polynomials, and basic hypergeometric functions are some of the mathematics areas where q-calculus finds application. We recall that the q-derivative for $q \in [0,1]$ was introduced and studied by Jackson as follows:

$$\mathrm{d}_q f(z) = \begin{cases} (f(qz) - f(z))/(qz - z), & \text{if} \quad z \neq 0,\ 0 \leq q < 1; \\ f'(0), & \text{if} \quad z = 0; \\ f'(z), & \text{if} \quad q = 1. \end{cases} \tag{2.2}$$

In the theory of basic hypergeometric series, the q-derivative operator plays an important role [2]. It is easy to see that if $f \in \mathcal{A}$, then

$$\mathrm{d}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^n, \quad \text{where} \quad [n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \quad (n = 2, 3, \ldots). \tag{2.3}$$

This is what we gain from convolution:

$$\mathrm{d}_q f(z) = \frac{1}{z}\{f(z) * h_q(z)\} = \frac{1}{z}\left\{f(z) * \frac{z}{(1-qz)(1-z)}\right\},$$

where $h_q(z) := z/(1-qz)(1-z)$. Very recently, Piejko and Sokół (see [19]) extended the q-operator $\mathrm{d}_q f(z)$ by replacing the real number q with the complex number ζ, where $|\zeta| \leq 1$. The aim of this paper is to extend the ζ-derivative operator $\mathrm{d}_\zeta f(z)$ by the following generalized Koebe function ([9])

$$\mathfrak{L}_{\alpha,\beta,\gamma}(z) = \frac{2(1-\gamma)z}{(1-\alpha z)(1-\beta z)}, \tag{2.4}$$

where $\alpha \in [-1,1]$, $\beta \in [-1,1]$, $\alpha\beta \neq \pm 1$ and $\gamma \in [0,1)$.

Definition 2.1. Let α, β be two complex numbers such that $|\alpha| \leq 1$ and $|\beta| \leq 1$. Also let γ be a real number so that $\gamma \in [0,1)$. We define the (α, β, γ)-derivative operator as

$$\mathrm{d}_{\alpha,\beta,\gamma} f(z) = \frac{1}{z}\left\{f(z) * \mathfrak{L}_{\alpha,\beta,\gamma}(z)\right\}, \tag{2.5}$$

where $\mathfrak{L}_{\alpha,\beta,\gamma}(z)$ is defined as in (2.4).

We note that $\mathfrak{L}_{\alpha,\beta,\gamma}(z)$ is a generalization of Koebe function. We have

$$\mathfrak{L}_{\alpha,\beta,\gamma}(z) = \sum_{n=1}^{\infty} B_n z^n = \begin{cases} 2(1-\gamma)\sum_{n=1}^{\infty}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) z^n, & \alpha \neq \beta; \\ 2(1-\gamma)\sum_{n=1}^{\infty} n\alpha^{n-1} z^n, & \alpha = \beta, \end{cases}$$

It should be noted that if $\alpha = q \in [0,1]$ is a real number, $\beta = 1$ and $\gamma = 1/2$, then $\mathrm{d}_{\alpha,\beta,\gamma}$ reduces the Jackson operator while if $\alpha = \zeta$ is a complex number with $|\alpha| \leq 1$, $\beta = 1$ and $\gamma = 1/2$, then we have the ζ-derivative operator which is defined by Piejko and Sokół [19]. Thus, we understand $\mathrm{d}_{\alpha,\beta,\gamma}$ as the generalization of $\mathrm{d}_q f(z)$. If a function f belongs to the class $\mathcal{A}$ and $\alpha \neq \beta$, then

$$\begin{aligned} \mathrm{d}_{\alpha,\beta,\gamma} f(z) &= \frac{1}{z}\left\{f(z) * \mathfrak{L}_{\alpha,\beta,\gamma}(z)\right\} \\ &= \frac{1}{z}\left\{z + \sum_{n=2}^{\infty} a_n z^n * z + \sum_{n=1}^{\infty} 2(1-\gamma)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) z^n\right\} \\ &= \frac{1}{z}\left\{z + \sum_{n=2}^{\infty} 2(1-\gamma)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) a_n z^n\right\} \\ &= \frac{1}{z}\left\{z + \sum_{n=2}^{\infty} [n]_{\alpha,\beta,\gamma} a_n z^n\right\}, \end{aligned} \tag{2.6}$$

where

$$[n]_{\alpha,\beta,\gamma} := 2(1-\gamma)\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right), \quad (n=2,3,\ldots). \tag{2.7}$$

In the case $\alpha=\beta$, we have

$$\mathrm{d}_{\alpha,\alpha,\gamma}f(z) = \frac{1}{z}\left\{z+\sum_{n=2}^{\infty}[n]_{\alpha,\gamma}a_n z^n\right\}$$

where

$$[n]_{\alpha,\gamma} = 2n(1-\gamma)\alpha^{n-1}, \quad (n=2,3,\ldots).$$

By using the new derivative operator (2.5) we define a new subclass of analytic functions as follows:

Definition 2.2. Let α,β be two complex numbers with $|\alpha|\le 1$, $|\beta|\le 1$ and γ be a real number such that $\gamma\in[0,1)$. Let the function f belongs to the class $\mathcal{A}$ and $\delta\in(0,1]$. We say that f belongs to $\mathcal{S}^*(\alpha,\beta,\gamma,\delta)$ if it satisfies

$$\frac{z\mathrm{d}_{\alpha,\beta,\gamma}f(z)}{f(z)} \prec \phi_\delta(z), \quad (z\in\mathbb{D}), \tag{2.8}$$

where

$$\phi_\delta(z) := 1+\frac{\delta z}{e^{\delta z}}. \tag{2.9}$$

We note that $\mathcal{S}^*(1,1,1/2,\delta)$ is a special case of Ma-Minda starlike function class $\mathcal{S}^*(\varphi)$ with $\varphi(z)=1+\delta z/\exp(\delta z)$.

3 Radius Problems

As a result, we find the radius of convexity of $\phi_\delta(z)$.

Lemma 3.1. Let $\phi_\delta(z)$ be defined as in (2.9), where $\delta\in(0,1]$ and $z\in\mathbb{D}$. Then $\phi_\delta(z)$ is a convex univalent function in $|z|<r_c(\delta)$, where $r_c(\delta):=(3-\sqrt{5})/2\delta$.

Proof . Let $\delta\in(0,1]$. It follows from (2.9), by a simple calculation that

$$1+\frac{z\phi''_\delta(z)}{\phi'_\delta(z)} = 1-\left(\delta z+\frac{\delta z}{1-\delta z}\right), \quad (z\in\mathbb{D}).$$

By using the definition of convexity we obtain

$$\begin{aligned}\mathrm{Re}\left\{1+\frac{z\phi''_\delta(z)}{\phi'_\delta(z)}\right\} &= \mathrm{Re}\left\{1-\left(\delta z+\frac{\delta z}{1-\delta z}\right)\right\}\\ &\ge 1-\left|\delta z+\frac{\delta z}{1-\delta z}\right|\\ &\ge 1-\delta r-\frac{\delta r}{1-\delta r} =: h(r,\delta), \quad (|z|=r).\end{aligned}$$

It is easy to check that $h(r,\delta)>0$ if and only if $r<(3-\sqrt{5})/2\delta$ which implies the result. □

It is easy to see that $\phi_\delta(z)$ have the Taylor series

$$\phi_\delta(z) = 1+\delta z-\delta^2z^2+\frac{\delta^3}{2}z^3-\frac{\delta^4}{6}z^4+\cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\delta^n}{(n-1)!}z^n. \tag{3.1}$$

Also, the function $\phi_\delta(z)$ is univalent, where $\delta\in(0,1]$. If we let δ tends to $1-$, then the image of $\mathbb{D}$ under $\phi_\delta(z)$ is bounded by a heart-shaped curve, see Figure 1(b) while tending $\delta\to 0+$ its range is bounded by a circle, see Figure 1(a). We note that for $\delta>1$, the function $\phi_\delta(z)$ is not univalent in $\mathbb{D}$, see Figure 1(c).

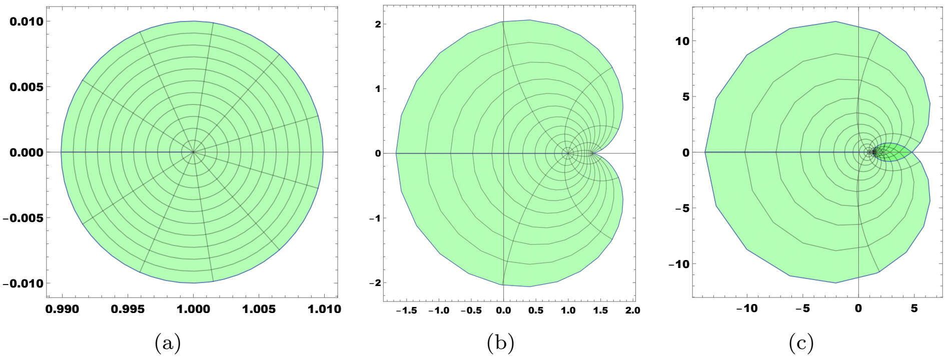

Figure 1: (a): The boundary curve of $\phi_{0.01}(\mathbb{D})$ (univalent) (b): The boundary curve of $\phi_{0.99}(\mathbb{D})$ (univalent) (c): The boundary curve of $\phi_2(\mathbb{D})$ (non-univalent)

Lemma 3.2. Let $\phi_\delta(z)$ be given by (2.9), where $\delta \in (0,1]$. Then

$$1-\delta e^{\delta} < \operatorname{Re}\{\phi_\delta(z)\} < 1+\delta e^{-\delta}.$$

Proof . We can easily check that

$$\operatorname{Re}\{\phi_\delta(z)\} = 1+\delta e^{-\delta\cos(\theta)}\left(\cos(\theta)\cos(\delta\sin(\theta))+\sin(\theta)\sin(\delta\sin(\theta))\right).$$

A simple calculation shows that $\operatorname{Re}\{\phi_\delta(z)\}$ gets its minimum at $\theta=\pi$ and its maximum at $\theta=0$. Therefore, the result follows. □

Corollary 3.3. Since $\phi_\delta(z)$ is a univalent function for all $\delta \in (0,1]$, therefore, if f belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma,\delta)$, then

$$1-\delta e^{\delta} < \operatorname{Re}\left(\frac{z\mathrm{d}_{\alpha,\beta,\gamma}f(z)}{f(z)}\right) < 1+\delta e^{-\delta}, \quad (z\in\mathbb{D}).$$

We continue this section by the following result.

Theorem 3.4. Let α,β be two complex numbers with $|\alpha|\le 1$, $|\beta|\le 1$ and γ be a real number such that $\gamma\in[0,1)$. Let $\delta_0 = 0.567143$ be the unique root of the equation $1-\delta e^{\delta}=0$, where $\delta\in(0,1]$. If a function $f\in\mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma,\delta)$ then

$$\operatorname{Re}\left(\frac{z\mathrm{d}_{\alpha,\beta,\gamma}f(z)}{f(z)}\right) > 0,$$

in the disk $\mathbb{D}_{r_s}$, where $r_s\in(0,\delta_0)$. The result is sharp.

Proof . Let the function $f\in\mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma,\delta)$, where $\delta\in(0,1]$. Then by definition there exists a Schwarz function w such that

$$\frac{z\mathrm{d}_{\alpha,\beta,\gamma}f(z)}{f(z)} = 1+\frac{\delta w(z)}{e^{\delta w(z)}}, \quad (z\in\mathbb{D}). \tag{3.2}$$

It follows from

$$e^{-|z|}\le|e^z|\le e^{|z|} \quad \text{and} \quad |w(z)|\le|z| \tag{3.3}$$

that for all $z\in\mathbb{D}$

$$\operatorname{Re}\left(\frac{z\mathrm{d}_{\alpha,\beta,\gamma}f(z)}{f(z)}\right) = \operatorname{Re}\left(1+\frac{\delta w(z)}{e^{\delta w(z)}}\right) \ge 1-\left|\frac{\delta w(z)}{e^{\delta w(z)}}\right| \ge 1-\frac{\delta|z|}{e^{-\delta|z|}} = 1-\delta r e^{\delta r} =: h(r),$$

where $|z|=r<1$. Since $h'(r)<0$ for all $r\in(0,1)$, we conclude that h is strictly decreasing function on the interval $[0,1]$ and it decreases from $h(0)=1>0$ to the value $h(1)=1-\delta e^{\delta}<0$, where $\delta\in(0.57,1]$. Therefore, the equation $h(r)=0$ has only one root in the interval $(0,1)$. We conclude that $h(r)>0$ if and only if $0<r<\delta_0$. Thus the proof is completed. □

If we take $\alpha=\beta=1$ and $\gamma=1/2$ in Theorem 3.4, we get the following result.

Example 3.5. note function

$$f_\delta(z) = ze^{(-e^{-\delta z}+1)} = z + z^2 - \frac{1}{6}z^4 + \frac{z^5}{24} + O(z^6) + \cdots = z\sum_{n=0}^{n=\infty}\frac{(1-exp^{-z})^n}{n!} \tag{3.4}$$

from the fact that $f_\delta(z) \in \mathcal{A}$

$$\frac{zf'_\delta(z)}{f_\delta(z)} = \frac{z\left(e^{(-e^{-\delta z}+1)} + z\delta e^{-\delta z}e^{(-e^{-\delta z}+1)}\right)}{ze^{(-e^{-\delta z}+1)}} = \frac{e^{(-e^{-\delta z}+1)}(1+z\delta e^{-\delta z})}{e^{(-e^{-\delta z}+1)}} = 1 + z\delta e^{-\delta z} = \phi_\delta(z) \tag{3.5}$$

so $f_\delta(z) \in S^*(1,1,\frac{1}{2},\delta)$ because $\delta \in (0,1]$ so various δ makes different examples. also $f_\delta(z)$ satisfies in theorem 3.4 because we proved in lemma (3.2) taht

$$1 - \delta e^{\delta} < \mathrm{Re}\{\phi_\delta(z)\} < 1 + \delta e^{-\delta}.$$

so if $1 - \delta e^{\delta} = 0$ thenRe$\{\phi_\delta(z)\} > 0$ but in theorem 3.4 we proved for $\delta_0 = 0.56$ which is unique root of equation $1 - \delta e^{\delta} = 0$, Re$\{\phi_{\delta_0}(z)\} > 0$ therefore from 3.5 we conclude that Re$\{\frac{zf'_{\delta_0}(z)}{f_{\delta_0}(z)}\} > 0$

Corollary 3.6. Let δ_0 be defined as in Theorem 3.4. If a function $f \in \mathcal{A}$ satisfies the following subordination relation

$$\frac{zf'(z)}{f(z)} \prec \phi_\delta(z), \quad (z \in \mathbb{D}) \tag{3.6}$$

where $\phi_\delta(z)$ is defined as in (2.9), then f is a starlike univalent function in $\mathbb{D}_{r_s}$, where $r_s \in (0, \delta_0)$. The result is sharp for the function

$$f_1(z) = z\exp\left(e^{-\delta z} - 1\right) = z - \delta z^2 + \delta^2 z^3 - \frac{5}{6}\delta^3 z^4 + o(z^5), \quad (z \in \mathbb{D}).$$

The following due to Nehari [18] will be useful.

Lemma 3.7. Let $w(z)$ be analytic in $\mathbb{D}$ and satisfying $|w(z)| \le 1$ for all $z \in \mathbb{D}$. Then

$$|w'(z)| \le \frac{1 - |w(z)|^2}{1 - |z|^2}. \tag{3.7}$$

Theorem 3.8. The radius of convexity of the class $\mathcal{S}^*(1,1,1/2,\delta)$ is $r \in (0, 0.17)$ for all $\delta \in (0,1]$.

Proof . If the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(1,1,1/2,\delta)$, then (3.6) holds true. It means that there is Schwarz function $w(z)$ such that

$$\frac{zf'(z)}{f(z)} = \phi_\delta(w(z)) = 1 + \frac{\delta w(z)}{e^{\delta w(z)}}, \quad (z \in \mathbb{D}). \tag{3.8}$$

By taking the logarithmic differential of (3.8), we obtain

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= \frac{zf'(z)}{f(z)} + \frac{z(\delta w'(z)e^{\delta w(z)} + \delta w'(z))}{e^{\delta w(z)} + \delta w(z)} - \delta z w'(z) \\ &= 1 + \frac{\delta w(z)}{e^{\delta w(z)}} + \left(\frac{e^{\delta w(z)} + 1}{e^{\delta w(z)} + \delta w(z)} - 1\right)\delta z w'(z), \quad (z \in \mathbb{D}). \end{aligned}$$

Moreover, by applying (3.3) and Lemma 3.7 we obtain

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \mathrm{Re}\left(1 + \frac{\delta w(z)}{e^{\delta w(z)}} + \left(\frac{e^{\delta w(z)} + 1}{e^{\delta w(z)} + \delta w(z)} - 1\right)\delta z w'(z)\right) \\ &\ge 1 - \left|\frac{\delta w(z)}{e^{\delta w(z)}} + \left(\frac{e^{\delta w(z)} + 1}{e^{\delta w(z)} + \delta w(z)} - 1\right)\delta z w'(z)\right| \\ &\ge 1 - \frac{\delta|z|}{e^{-\delta|z|}} - \left(\frac{e^{|z|} + 1}{e^{-\delta|z|} - \delta|z|} + 1\right)\delta|zw'(z)| \\ &\ge 1 - \frac{\delta r}{e^{-\delta r}} - \left(\frac{e^{r} + 1}{e^{-\delta r} - \delta r} + 1\right)\frac{\delta r}{1 - r^2} =: h(r,\delta), \quad (z \in \mathbb{D}). \end{aligned}$$

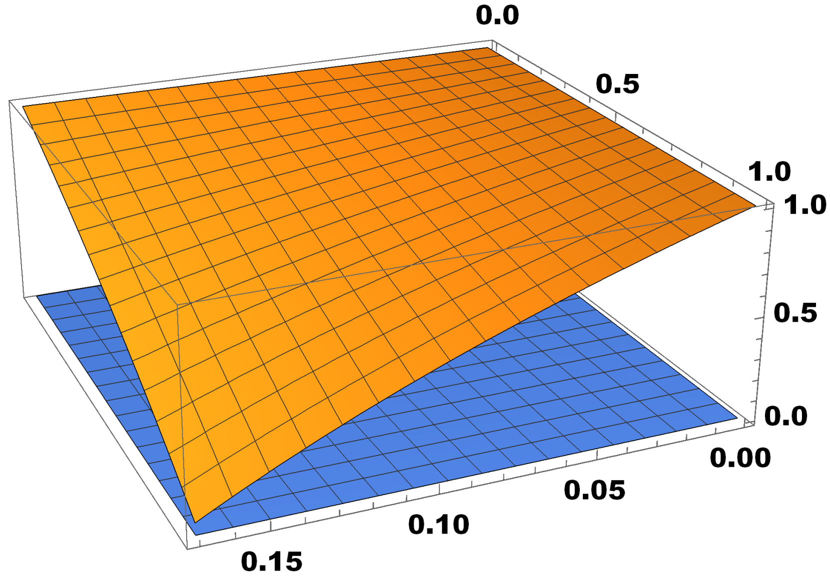

Figure 2: The 3D graph of the function $w = h(r, \delta)$ (orange) and $w = 0$ (blue)

Computer experiment shows that $h(r, \delta) > 0$ for all $r \in (0, 0.17)$, and $\delta \in (0, 1]$, see Figure 2. The proof now is complete.

Notation: Because $f_\delta(z)$ belongs to $S^*(1, 1, \frac{1}{2}, \delta)$ so convexity for this function in $r \in (0, 0.17)$ is trivial. □

4 On coefficients

We need the following lemmas.

Lemma 4.1. ([18, p. 172]) Assume that w is a Schwarz function so that $w(z) = \sum_{n=1}^{\infty} w_n z^n$. Then

$$|w_1| \le 1 \quad and \quad |w_n| \le 1 - |w_1|^2, \quad (n = 2, 3, \dots).$$

Lemma 4.2. ([1, Lemma 1]) If $w(z) = \sum_{n=1}^{\infty} w_n z^n$ is a Schwarz function, then

$$|w_2 - tw_1^2| \le \begin{cases} -t, & t \le -1; \\ 1, & -1 \le t \le 1; \\ t, & t \ge 1. \end{cases}$$

All inequalities are sharp.

Theorem 4.3. Let the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha, \beta, \gamma, \delta)$ such that α, β, γ satisfy the assumption of Theorem 3.4, and $0 < \delta \le 1$. Then

$$|a_2| \le \frac{\delta}{[2]_{\alpha,\beta,\gamma} - 1}, \quad and \quad |a_3| \le \frac{\delta([2]_{\alpha,\beta,\gamma} - 1) + \delta^2}{([2]_{\alpha,\beta,\gamma} - 1)([3]_{\alpha,\beta,\gamma} - 1)}, \tag{4.1}$$

where $[\cdot]_{\alpha,\beta,\gamma}$ is defined as in (2.7). The result is sharp.

Proof . If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha, \beta, \gamma, \delta)$, then there exists a Schwarz function $w(z) = w_1 z + w_2 z^2 + \cdots$ such that

$$\frac{z\mathrm{d}_{\alpha,\beta,\gamma} f(z)}{f(z)} = \phi_\delta(w(z)), \quad (z \in \mathbb{D}), \tag{4.2}$$

holds true. By using the Taylor series of (2.6) we obtain

$$\frac{z\mathrm{d}_{\alpha,\beta,\gamma}f(z)}{f(z)} = 1 + ([2]_{\alpha,\beta,\gamma} - 1)a_2 z + \left(([3]_{\alpha,\beta,\gamma} - 1)\, a_3 - ([2]_{\alpha,\beta,\gamma} - 1)\, a_2^2\right) z^2 + \cdots . \tag{4.3}$$

On the other hand,

$$\begin{aligned}\phi_\delta(w(z)) &= 1 + \frac{\delta w(z)}{e^{\delta w(z)}} = 1 + \delta w(z) - \delta^2 w^2(z) + \frac{\delta^3}{2} w^3(z) - \cdots \\ &= 1 + \delta(w_1 z + w_2 z^2 + \cdots) + \delta^2 (w_1 z + w_2 z^2 + \cdots)^2 + \cdots \\ &= 1 + \delta w_1 z + (\delta w_2 - \delta^2 w_1^2) z^2 + (\delta^3 w_1^3/2 + \delta w_3 - 2\delta^2 w_1 w_2) z^3 + \cdots .\end{aligned} \tag{4.4}$$

Equating the corresponding coefficients of (4.3) and (4.4) we get

$$([2]_{\alpha,\beta,\gamma} - 1)a_2 = \delta w_1, \text{ and } ([3]_{\alpha,\beta,\gamma} - 1)\, a_3 - ([2]_{\alpha,\beta,\gamma} - 1)\, a_2^2 = \delta w_2 - \delta^2 w_1^2. \tag{4.5}$$

If we apply Lemma 4.1, then $|a_2| \leq \delta/([2]_{\alpha,\beta,\gamma} - 1)$ which implies the first inequality of (4.1). It follows from both equalities of (4.5) that

$$a_3 = \frac{\delta(w_2 - \delta w_1^2)([2]_{\alpha,\beta,\gamma} - 1) + \delta^2 w_1^2}{([3]_{\alpha,\beta,\gamma} - 1)([2]_{\alpha,\beta,\gamma} - 1)}.$$

The estimation a_3 follows from Lemma 4.1 and Lemma 4.2. The proof is now complete. □

Example 4.4. if we put $\delta = 1$ in inequations 4.3 in paper considering to 3.4 we see that $1 = |a_2| \leq 1 and 0 = |a_3| < 1$ so function $f_\delta(z)$ satisfies in theorem 4.3

Let P be the family of holomorphic function $p(z)$ in $\mathbb{D}$ such that $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$. In order to prove the next result we need the following lemma.

Lemma 4.5. [15, Lemma 1] Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be in P. Then

$$|p_2 - \mu p_1^2| \leq \begin{cases} -4\mu + 2, & \mu \leq 0; \\ 2, & 0 \leq \mu \leq 1; \\ 4\mu - 2, & \mu \geq 1. \end{cases}$$

All inequalities are sharp.

Theorem 4.6. Let $\delta \in (0, 1]$, and α, β, γ satisfy the assumption of Theorem 3.4. If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha, \beta, \gamma, \delta)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} -4\mu' + 2, & \mu \leq \mu_1; \\ 2, & \mu_1 \leq \mu \leq \mu_2; \\ 4\mu' - 2, & \mu \geq \mu_2, \end{cases} \tag{4.6}$$

where

$$\mu' := \frac{\mu\delta([3]_{\alpha\beta\gamma} - 1) - [(1+\delta)(1 - [2]_{\alpha\beta\gamma}) + \delta]([2]_{\alpha\beta\gamma} - 1)}{2([2]_{\alpha\beta\gamma} - 1)^2}, \tag{4.7}$$

$$\mu_1 := \frac{[(1+\delta)(1 - [2]_{\alpha\beta\gamma}) + \delta]([2]_{\alpha\beta\gamma} - 1)}{\delta([3]_{\alpha\beta\gamma} - 1)}, \tag{4.8}$$

and

$$\mu_2 := \frac{[(1+\delta)(1 - [2]_{\alpha\beta\gamma}) + \delta]([2]_{\alpha\beta\gamma} - 1) + 2([2]_{\alpha\beta\gamma} - 1)^2}{\delta([3]_{\alpha\beta\gamma} - 1)}. \tag{4.9}$$

The result is sharp.

Proof . If a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\alpha,\beta,\gamma,\delta)$, then there exists a Schwarz function such that (3.2) holds true. We define

$$C(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots . \tag{4.10}$$

Then

$$w(z) = \frac{C(z)-1}{C(z)+1} = 1 + \frac{1}{2}c_1 z + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \cdots . \tag{4.11}$$

Taking the above $w(z)$ into account, we get

$$\phi_\delta(w(z)) = 1 + \frac{1}{2}\delta c_1 z + \frac{1}{2}\delta\left(c_2 - \frac{1}{2}(1+\delta)c_1^2\right)z^2 + \cdots . \tag{4.12}$$

Equating the corresponding coefficients (4.3) and (4.12) gives

$$([2]_{\alpha,\beta,\gamma} - 1)a_2 = \frac{1}{2}\delta c_1 \tag{4.13}$$

and

$$([3]_{\alpha,\beta,\gamma} - 1)\, a_3 - ([2]_{\alpha,\beta,\gamma} - 1)\, a_2^2 = \frac{1}{2}\delta\left(c_2 - \frac{1}{2}(1+\delta)c_1^2\right). \tag{4.14}$$

It follow from both (4.13) and (4.14) that

$$a_2 = \frac{\delta c_1}{2([2]_{\alpha\beta\gamma} - 1)} \tag{4.15}$$

and

$$a_3 = \frac{\left[\delta(1+\delta)(1-[2]_{\alpha\beta\gamma}) + \delta^2\right] c_1^2 + 2\delta c_2([2]_{\alpha\beta\gamma} - 1)}{4([2]_{\alpha\beta\gamma} - 1)([3]_{\alpha\beta\gamma} - 1)}. \tag{4.16}$$

Now from (4.15) and (4.16) for the complex number μ we get

$$a_3 - \mu a_2^2 = \frac{\delta}{2([3]_{\alpha\beta\gamma} - 1)}\left(c_2 - \frac{\mu\delta([3]_{\alpha\beta\gamma} - 1) - [(1+\delta)(1-[2]_{\alpha\beta\gamma}) + \delta]([2]_{\alpha\beta\gamma} - 1)}{2([2]_{\alpha\beta\gamma} - 1)^2}c_1^2\right).$$

Letting

$$\mu' := \frac{\mu\delta([3]_{\alpha\beta\gamma} - 1) - [(1+\delta)(1-[2]_{\alpha\beta\gamma}) + \delta]([2]_{\alpha\beta\gamma} - 1)}{2([2]_{\alpha\beta\gamma} - 1)^2} \tag{4.17}$$

and applying Lemma 4.5 we get the desired result. □

Example 4.7. for the function $f_\delta(z)$ from 3.4 we obtain $|a_3 - \mu a_2^2| = |\mu|$ so if we put $|\mu| = 1$ then

$$\begin{cases} \mu = 1 \\ or \\ \mu = -1 \end{cases}$$

on the other hand if we put $\delta = 1$ in 4.8 and 4.9 and 4.17 equations in paper we will have $\mu_1 = \frac{-1}{2}$ and $\mu_2 = \frac{1}{2}$ and $\mu' = \frac{2\mu+1}{2}$ and for $\mu = -1$, $\mu' = \frac{-1}{2}$ and for $\mu = 1$, $\mu' = \frac{3}{2}$ so if we put these values in theorem 4.6 inequality we will have

$$|a_3 - \mu a_2^2| = |\mu| = 1 < \begin{cases} -4 \times (\frac{-1}{2}) + 2 = 4, & -1 = \mu < \mu_1 = \frac{-1}{2}; \\ \\ 4 \times (\frac{3}{2}) - 2 = 4, & 1 = \mu > \mu_2 = \frac{1}{2}, \end{cases} \tag{4.18}$$

and for second inequality we put $\mu = \frac{1}{4}$ so $|a_3 - \mu a_2^2| = |\mu| = |\frac{1}{4}| < 2 \quad when \frac{-1}{2} = \mu_1 < \frac{1}{4} = \mu < \mu_2 = \frac{1}{2}$ if $\mu_1 = \mu = \mu_2 = 1$ then inequality 4.6 is trivial.

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