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A new generalization of the l_p spaces

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Abstract

In this work, using semidefinite matrices gives a new generalization of the l_p spaces and some inequalities containing lower bounds of some operators are proved. Also, by defining an inner product on the classes of an equivalence relation on operators, some inequalities similar to the well-known inequalities including the Copson, Cesaro, Hilbert and Hardy inequalities are obtained.

Keywords:
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1 Introduction

The Mahalanobis distance is a measure of the distance between a point P and a distribution D, introduced by P. C. Mahalanobis in 1936 [8]. The Mahalanobis distance between two random vectors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ of the same distribution with the covariance matrix Q is defined by

$$d(x, y) = \sqrt{(x - y)^t Q^{-1} (x - y)}.$$

The Mahalanobis distance is widely used in cluster analysis and classification techniques. The notion "lower bounds" of matrix operators at first, was introduced by R. Lyons [7] and then intensively studied for l_p spaces and its generalizations, e.g., [2, 3, 6].

If X is a Banach sequense space, we denote by $\delta(X)$ the set of decreasing, non-negative sequences in X. For a positive operator A on X, the lower bound of A is defined as

$$m_X(A) = \inf\{\|AX\| : \quad x \in \delta(X), \quad \|X\| = 1\}.$$

Let (w_n) be a decreasing, non-negative sequence, for $p \geq 1$,

$$l_p(w) = \{x = (x_n) : \quad \sum_{n=1}^{\infty} w_n |x_n|^p < \infty\}$$

is called weighted sequence space equipped with the norm $\|x\|_{p,w} = \left(\sum_{n=1}^{\infty} w_n |x_n|^p\right)^{\frac{1}{p}}$. If $\inf_{n\in\mathbb{N}} w(n) > 0$ then $l_p(w) = l_p$ with equivalent norms. So, we are mainly interested in the case when $\inf_{n\in\mathbb{N}} w(n) = 0$. Given a null

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sequence $x = (x_n)$, let (x_n^*) be the decreasing rearrangement of $|x_n|$. The Lorentz sequence space $d(w,p)$ is the space of null sequences x for which x^* is in $l_p(w)$. Denote by $\delta(w)$ the set of decreasing, non-negative sequences in $l_p(w)$, and define

$$\Delta_{p,w}(A) = \sup\{\|Ax\|_{p,w} : \quad x \in \delta(w) : \|x\|_{p,w} = 1\}.$$

Jameson and Lashkaripour [5] proved the following Lemma.

Lemma 1.1. Suppose that (w_n) is decreasing, that $aij \geq 0$ for all i, j, and A maps $\delta_p(w)$ into $l_p(w)$. Write $c_{m,j} = \sum_{i=1}^{\infty} a_{i,j}$. Suppose further that for each i $\lim_{j\to\infty} a_{i,j} = 0$ and either for each i, $a_{i,j}$ decreases with j or $a_{i,j}$ decreases with i for each j and $c_{m,j}$ decreases with j for each m. Then $\|A(x^*)\|_{d(w,p)} \geq \|A(x)\|_{d(w,p)}$ for non-negative elements x of $d(w,p)$. Hence $\|A\|_{d(w,p)} = \Delta_{p,w}(A)$.

In this work, inspired by the Mahalanobis distance, a new norm on sequence spaces is defined. Also some theorems and inequlities are proved.

2 Main Results

Definition 2.1. Let $X = \{x = (x_n)_{n=1}^{\infty}, \quad x_n \in \mathbb{C}\}$ and $Q \in L(X)$, be a matrix, we define

$$\|x\|_Q = (\overline{x}^t Q x)^{\frac{1}{2}},$$

where $\overline{x} = \{\overline{x_n}\}_{n=1}^{\infty}$, which $\overline{a}$ is the conjugate of a.

For positive semidefinite matrix Q there are A and $D = diag(\lambda_i)$ such that $Q = A^t DA$ and so

$$\begin{aligned}\|x\|_Q =&(\overline{x}^t A^t D A x)^{\frac{1}{2}}\\ =&\big((A\overline{x})^t D(Ax)\big)^{\frac{1}{2}}\\ =&\big(\sum_{i=1}^{\infty} \lambda_i |y_i|^2\big)^{\frac{1}{2}}\\ =&\|Ax\|_{2,w},\end{aligned}$$

where, $y_i = \sum_{j=1}^{\infty} a_{i,j} x_j$ and $w_i = \lambda_i$. In this case, $\|x\|_Q = 0$, iff. $x = 0$ and

$$\begin{aligned}\|x+y\|_Q =&\|A(x+y)\|_{2,w}\\ \leq&\|Ax\|_{2,w} + \|Ay\|_{2,w}\\ =&\|x\|_Q + \|y\|_Q.\end{aligned}$$

For a lower triangular matrix $Q = [a_{i,j}]$ we have

$$\|x\|_Q = \big(\sum_{i=1}^{\infty} L_i |x_i|^2\big)^{\frac{1}{2}} = \|x\|_{2,w}, \quad (w_i = L_i)$$

where, $L_i = \sum_{j=1}^{\infty} a_{i,j}$ is the summation of the i-th row of Q. In special case, if for each i, $L_i = 1$ (for example, the Cesaro matrix) we have $\|x\|_Q = \|x\|_2$.
For a positive semidefinite matrix Q, we define

$$l_Q^2 = \{x = (x_n) : \quad \|x\|_Q < \infty\}.$$

Note that $\|x\|_Q^2 = x^t Q x = (x^t Q x)^t = x^t Q^t x = \|x\|_{Q^t}$ and so we have $l^2(Q) = l^2(Q^t)$. So for a positive semidefinite upper triangular matrix U we have $l^2(U) = l^2(U^t) = l_{2,w}$, $w_i = L_i(U^t) = C_i(U)$ where $L_i(A)$ and $C_i(A)$ respectively are the sum of rows and columns of A.

Definition 2.2. Suppose that $Q=(Q_i)_{i=1}^{\infty}$ is a sequence of positive semidefinite matrices. For $p \geq 1$ take

$$\|x\|_{p,Q} := \big(\sum_{i=1}^{\infty} \|x\|_{Q_i}^{p}\big)^{\frac{1}{p}}.$$

Note that for each i we have,

$$\|x+y\|_{Q_i}^{p} \leq \|x\|_{Q_i}\big(\|x+y\|_{Q_i}^{p-1}\big) + \|y\|_{Q_i}\big(\|x+y\|_{Q_i}^{p-1}\big).$$

On the other hand,

$$\sum_{i=1}^{\infty} \|x\|_{Q_i}\|x+y\|_{Q_i}^{p-1} \leq \big(\sum_{i=1}^{\infty} \|x\|_{Q_i}^{p}\big)^{\frac{1}{p}} \big(\sum_{i=1}^{\infty} \|x+y\|_{Q_i}^{q}\big)^{\frac{1}{q}}.$$

And so,

$$\|x+y\|_{p,Q}^{p} \leq \|x\|_{p,Q}\|x+y\|_{p,Q}^{\frac{p}{q}} + \|y\|_{p,Q}\|x+y\|_{p,Q}^{\frac{p}{q}}$$

therefore,

$$\|x+y\|_{p,Q} \leq \|x\|_{p,Q} + \|y\|_{p,Q}.$$

Now, the set

$$l_p(Q) := \{x=(x_n)_{n=1}^{\infty} : \quad \|x\|_{p,Q} < \infty\}$$

is a norm space. In special case, if Q_i is a matrix which $a_{i,i}=1$ and zero in otherwise then $l_p(Q)=l_p$. Therefore, the space $l_p(Q)$ is a generalization of the l_p space.

Theorem 2.3. Suppose that $P=(P_i)_{i=1}^{\infty}, Q=(Q_i)_{i=1}^{\infty}$ are two sequences of positive semidefinite matrices satisfying $P_i \leq Q_i$ for each $i \in \mathbb{N}$. Then, for $p \geq 1$, $l_p(P) \subseteq l_p(Q)$.

Proof . For each i we have $P_i \leq Q_i$ and so $x^t P_i x \leq x^t Q_i x$ which implies that $\|x\|_{P_i} \leq \|x\|_{Q_i}$. Therefore,

$$\|x\|_{p,P} = \big(\sum_{i=1}^{\infty} \|x\|_{P_i}\big)^{\frac{1}{p}} \leq \big(\sum_{i=1}^{\infty} \|x\|_{Q_i}\big)^{\frac{1}{p}} = \|x\|_{p,Q}.$$

So, we have $l_p(P) \subseteq l_p(Q)$. $\square$

For a matrix Q, we have $Q=L+D+U$ where L,U respectively are lower and upper triangular and D is diagonal. Suppose all of them are positive semidefinite matrices. We have

$$\|x\|_Q^2 = x^t Q x = x^t L x + x^t D x + x^t U x = \|x\|_l^2 + \|x\|_D^2 + \|x\|_U^2.$$

Suppose that $Q=(Q_i)_{i=1}^{\infty}$ and $Q_i = A_i + D_i + B_i$. Also, all matrices are positive semidefinte and A_i's , B_i's respectively are lower and upper triangular. We have

$$\begin{aligned}
\|x\|_{p,Q}^{p} &= \sum_{i=1}^{\infty} \|x\|_{Q_i}^{p} \\
&= \sum_{i=1}^{\infty} \big(\|x\|_{A_i}^2 + \|x\|_{D_i}^2 + \|x\|_{B_i}^2\big)^{\frac{p}{2}} \\
&= \sum_{i=1}^{\infty} \big(\sum_{j=1}^{\infty} (L_i^j + \lambda_i^j + U_i^j)|x_j|^2\big)^{\frac{p}{2}} \\
&= \sum_{i=1}^{\infty} \big(\sum_{j=1}^{\infty} Q_i\rfloor_j |x_j|^2\big)^{\frac{p}{2}}
\end{aligned}$$

and so,

$$\|x\|_{p,Q} = \big(\sum_{i=1}^{\infty} \|x\|_{2,w_i}^{p}\big)^{\frac{1}{p}}, \quad w_i = (Q_i\rfloor_j)_{j=1}^{\infty}.$$

For a a matrix $A \in L(X)$ the symbol $A\rfloor_i$ stands for the i-th corner of A and is defined by

$$A\rfloor_i = \sum_{j=1}^{i} a_{i,j} + \sum_{k=1}^{i-1} a_{k,i}.$$

If $A, B \in L(X)$, we define

$$< A, B >= \sum_{i=1}^{\infty} A\rfloor_i \overline{B\rfloor_i}. \tag{2.1}$$

By taking, $a = (A\rfloor_1, A\rfloor_2, ...)$ and $b = (B\rfloor_1, B\rfloor_2, ...)$, then $< A, B >$ is the standard inner product. By defining an equivalence relation on $L(X)$ as following:

$$A \approx B \quad \text{if and only if for all } i, A\rfloor_i = B\rfloor_i,$$

we may define a norm on the classes of the equivalences relation as $\|A\|_\rfloor = \sqrt{< A, A >}$.

Also, the Cauchy-Schwarz inequality is established $| < A, B > | \leq \|A\|_\rfloor \|B\|_\rfloor$.

Theorem 2.4. Suppose that $Q = (Q_i)_{i=1}^{\infty}$ is a sequence of positive semidefinite matrices and $C = \sum_{i=1}^{\infty} Q_i$ then $l_2(Q) = l^2(C)$. Also, $l_2(Q)^* = l^2((\tilde{C})^{-1})$ where $(\tilde{C})^{-1} = diag(C\rfloor_i)_{i=1}^{\infty}$.

Proof . We have

$$\|x\|_C^2 = x^t C x = \sum_{i=1}^{\infty} x^t Q_i x = \sum_{i=1}^{\infty} \|x\|_{Q_i}^2$$

which implies that $\|x\|_C = \|x\|_{2,Q}$ and so $l_2(Q) = l^2(C) = l^2(w)$ with $w = (C\rfloor_i)_{i=1}^{\infty}$. Now we have $l_2^*(Q) = (l^2(w))^* = l^2(v) = l^2((\tilde{C})^{-1})$ where $v = (\frac{1}{C\rfloor_i})_{i=1}^{\infty}$. Note that , if p, q are conjugate then, $l_p^*(w) = l^q(v)$ with $v_i = w_i^{-\frac{q}{p}}$. □

Theorem 2.5. Suppose that $Q = (Q_i)_{i=1}^{\infty}$ is a sequence of positive semidefinite matrices. Then

$$\|x\|_{p,Q} \leq \left(\sum_{i=1}^{\infty} \|Q_i\|_\rfloor^{\frac{p}{2}} \right)^{\frac{1}{p}} \|x\|_4.$$

Therefore,

$$\sum_{i=1}^{\infty} \|x\|_{2,w_i}^p \leq (\sum_{i=1}^{\infty} \|Q_i\|_\rfloor^{\frac{p}{2}} \|x\|_4^p, \tag{2.2}$$

where, $w_i = (Q_i\rfloor_j)_{j=1}^{\infty}$.

Proof . For a sequence $x = (x_n)_{n=1}^{\infty}$ we put $X = diag(|x_i|^2)_{i=1}^{\infty}$. Now we have

$$\begin{aligned}
\|x\|_{p,Q} =& \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} Q_i\rfloor_j |x_j|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
=& \left(\sum_{i=1}^{\infty} \left(< Q_i, X > \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
\leq& \left(\sum_{i=1}^{\infty} \left(\|Q_i\|_\rfloor \|X\|_\rfloor \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
=& \left(\sum_{i=1}^{\infty} \|Q_i\|_\rfloor^{\frac{p}{2}} \right)^{\frac{1}{p}} \|X\|_\rfloor^{\frac{1}{2}} \\
=& \left(\sum_{i=1}^{\infty} \|Q_i\|_\rfloor^{\frac{p}{2}} \right)^{\frac{1}{p}} \|x\|_4.
\end{aligned}$$

□

By taking $Q_i\rfloor_j = \frac{1}{i}$ for $j \leq i$ and zero otherwise and $x_j = \sqrt{a_j}$ in (2.2) we have

$$\sum_{i=1}^{\infty} (\frac{\sum_{j=1}^{i} a_j}{j})^p \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{p}{2}}} \|a\|_2^p = \zeta(\frac{p}{2})\|a\|_2^p. \tag{2.3}$$

Hence, we prove that

$$\|Ca\|_p \leq \zeta^{\frac{1}{p}}(\frac{p}{2})\|a\|_2. \tag{2.4}$$

Bennett in [3] (Theorem 4) for every decreasing non-negative sequence $a \in l^p$ prove that

$$\|Ca\|_p \geq \zeta^{\frac{1}{p}}(p)\|a\|_p.$$

The Copson matrix B is an upper triangular matrix which is defined by $b_{i,j} = \frac{1}{j}$ for $1 \leq i \leq j$ and zero otherwise. In fact $B = C^t$, transpose of the Cesaro matrix which is defined by $c_{i,j} = \frac{1}{i}$ for $j \leq i$ and zero otherwise. Copson in [4] proved that

$$\left(\sum_{n=1}^{\infty} (\sum_{k=n}^{\infty} \frac{|x_k|}{k})^p\right)^{\frac{1}{p}} \leq p(\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}. \tag{2.5}$$

By taking $Q_i\rfloor_j = \frac{1}{j}$ for $1 \leq i \leq j$ and zero otherwise and $x_j = \sqrt{a_j}$ in (2.2) we have

$$\sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} \frac{a_j}{j})^p \leq \sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} \frac{1}{j^2})^{\frac{p}{2}} \|a\|_2^p. \tag{2.6}$$

Applying (2.5) we have

$$\sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} \frac{1}{j^2})^{\frac{p}{2}} = \sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} \frac{\frac{1}{j}}{j})^{\frac{p}{2}} \leq (\frac{p}{2})^{\frac{p}{2}} \zeta(\frac{p}{2}).$$

Therefore, we have

$$\sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} \frac{a_j}{j})^p \leq (\frac{p}{2})^{\frac{p}{2}} \zeta(\frac{p}{2})\|a\|_2^p.$$

As an another application of the above theorem, by taking $Q_i = diag\big(\frac{1}{(i+j-1)}\big)_{j=1}^{\infty}$ and $x = (\sqrt{a_j})_{j=1}^{\infty}$ we have

$$\begin{aligned}
\sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} \frac{a_j}{i+j-1})^{\frac{p}{2}} &\leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} (\frac{1}{i+j-1})^2\right)^{\frac{p}{2}} \|a\|_2^p \\
&= \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} (\frac{1}{j})^2\right)^{\frac{p}{2}} \|a\|_2^p \\
&\leq (\frac{p}{2})^{\frac{p}{2}} \zeta(\frac{p}{2})\|a\|_2^p.
\end{aligned}$$

Suppose that $Q = (Q_i)$ is a sequence of positive semidefinite matrices. Denote by $\delta_p(Q)$ the set of decreasing non-negative sequences in $l_p(Q)$ and define

$$\Delta_{p,Q}(T) = \sup\{\|Tx\|_{p,Q} : \quad x \in \delta_p(Q) : \|x\|_{p,Q} = 1\},$$

where, $T = [t_{i,j}]$ is a linear operator on $l_p(Q)$. We assume that $t_{i,j} \geq 0$ for all i, j which implies that the norm is determined by the action of T on non-negative sequences. At the following we prove the analogous form of Lemma 1.1 for $l_p(Q)$. In fact, the following theorem establishes conditions insuring that $\|T\|_{l_p(Q)}$ is determined by decreasing, non-negative sequences.

Theorem 2.6. Suppose that $T = [t_{i,j}], t_{i,j} \geq 0$ maps $\delta_p(Q)$ into $l_p(Q)$. Write $c_{m,j} = \sum_{i=1}^{m} t_{i,j}$. Suppose further that:

(i) $\lim_{j\to\infty} t_{i,j} = 0$ for each i, and either

(ii) $a_{i,j}$ decreases with j for each i, or

(iii) $a_{i,j}$ decreases with i for each j and $c_{m,j}$ decreases with j for each m also, $Q_i\rfloor_j$ decreases with j for each i.

Then, $\|T(x^*)\|_{p,Q} \geq \|T(x)\|_{p,Q}$ for non-negative elements x of $l_p(Q)$. Hence, $\|T\|_{l_pQ} = \Delta_{p,Q}(T)$.

Proof . Let $z = Tx$ and $z' = Tx^*$. Write $X_j = x_1 + \cdots + x_j,$. By Abel summation and condition (ii), we have

$$z_i = \sum_{j=1}^{\infty} t_{i,j} x_j = \sum_{j=1}^{\infty} (t_{i,j} - t_{i,j+1}) X_j,$$

$$z'_i = \sum_{j=1}^{\infty} t_{i,j} x_j^* = \sum_{j=1}^{\infty} (t_{i,j} - t_{i,j+1}) X_j^*.$$

Since $X_j \le X_j^*$ for all j, we have $z_i \le z'_i$ for all i. Now, we have

$$\begin{aligned} \|z\|_{p,Q}^p &= \sum_{i=1}^{\infty} \big(\sum_{j=1}^{\infty} Q_i\rfloor_j |z_j|^2 \big)^{\frac{p}{2}} \\ &\le \sum_{i=1}^{\infty} \big(\sum_{j=1}^{\infty} Q_i\rfloor_j |z'_j|^2 \big)^{\frac{p}{2}} = \|z'\|_{p,Q}^p, \end{aligned}$$

which implies that $\|z\|_{p,Q} \le \|z'\|_{p,Q}$. Now, assume (iii). Then z_i and z'_i decrease with i, and

$$Z_m = \sum_{i=1}^{m} \sum_{j=1}^{\infty} t_{i,j} x_j = \sum_{j=1}^{\infty} c_{m,j} x_j = \sum_{j=1}^{\infty} (c_{m,j} - c_{m,j+1}) X_j,$$

$$Z'_m = \sum_{i=1}^{m} \sum_{j=1}^{\infty} t_{i,j} x_j^* = \sum_{j=1}^{\infty} c_{m,j} x_j^* = \sum_{j=1}^{\infty} (c_{m,j} - c_{m,j+1}) X_j^*.$$

Hence, $Z_m \le Z'_m$ for all m. By the majorization principle [1],

$$\sum_{k=1}^{m} z_k^2 \le \sum_{k=1}^{m} z_k'^2.$$

Also, since $Q_i\rfloor_j$ decreases with j for each i, by the Abel summuation we have

$$\begin{aligned} \sum_{j=1}^{\infty} Q_i\rfloor_j |z_j|^2 &= \sum_{j=1}^{\infty} \big(Q_i\rfloor_j - Q_i\rfloor_{j+1} \big) \sum_{k=1}^{j} |z_k|^2 \\ &\le \sum_{j=1}^{\infty} \big(Q_i\rfloor_j - Q_i\rfloor_{j+1} \big) \sum_{k=1}^{j} |z'_k|^2 \\ &= \sum_{j=1}^{\infty} Q_i\rfloor_j |z'_j|^2 \end{aligned}$$

and hence $\|z\|_{p,Q} \le \|z'\|_{p,Q}$. $\square$

By taking $Q_1 = I$ the identity matrix and $Q_i = diag\big(\frac{1}{i+j}\big)_{j=1}^{\infty}, i \ge 2$ and $T = C$ the cesaro operator , all conditions of the above theorem are established. Now we have

$$\begin{aligned} \|x\|_{p,Q}^p &= \sum_{i=1}^{\infty} \big(\sum_{j=1}^{\infty} Q_i\rfloor_j |x_j|^2 \big)^{\frac{p}{2}} \\ &= \sum_{i=1}^{\infty} \big(\sum_{j=1}^{\infty} \frac{|x_j|^2}{i+j} \big)^{\frac{p}{2}}. \end{aligned}$$

Now, by the Hilbert's Inequality which asserts that

$$\sum_{n=1}^{\infty} \big(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \big)^p < \big(\frac{\pi}{\sin(\frac{\pi}{p})} \big)^p \sum_{n=1}^{\infty} a_n^p, \quad (p > 1)$$

for $p > 2$ we have

$$\|x\|_{p,Q} < \big(\frac{\pi}{\sin(\frac{2\pi}{p})}\big)^{\frac{1}{2}}\|x\|_p.$$

Also,

$$\begin{aligned}\|Cx\|_{p,Q}^p =& \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{X_j^2}{j^2(i+j)}\right)^{\frac{p}{2}}\\ &< \left(\frac{\pi}{\sin(\frac{2\pi}{p})}\right)^{\frac{p}{2}}\sum_{j=1}^{\infty}\left(\frac{X_j}{j}\right)^p.\end{aligned}$$

Applying the Hardy's inequality which asserts

$$\sum_{n=1}^{\infty}\big(\frac{1}{n}\sum_{m=1}^{n}a_m\big)^p \le \big(\frac{p}{p-1}\big)^p\sum_{n=1}^{\infty}a_n^p, \quad (p>1)$$

we have

$$\begin{aligned}\|Cx\|_{p,Q}^p <&\big(\frac{\pi}{\sin(\frac{2\pi}{p})}\big)^{\frac{p}{2}}\sum_{j=1}^{\infty}\big(\frac{X_j}{j}\big)^p\\ \le&\big(\frac{\pi}{\sin(\frac{2\pi}{p})}\big)^{\frac{p}{2}}\big(\frac{p}{p-1}\big)^p\|x\|_p^p.\end{aligned}$$

So,

$$\|Cx\|_{p,Q} < \big(\frac{\pi}{\sin(\frac{2\pi}{p})}\big)^{\frac{1}{2}}\big(\frac{p}{p-1}\big)\|x\|_p.$$

By taking $Q_i = diag\big(\frac{1}{i+j-1}\big)_{j=1}^{\infty}$, by the same manner one may obtain

$$\|Cx\|_{p,Q} < \big(\frac{\pi}{\sin(\frac{2\pi}{p})}\big)^{\frac{1}{2}}\big(\frac{p}{p-1}\big)\|x\|_p.$$

On the other hand by applying Propositin 1 of [3] for $p \ge 1$ we have

$$\begin{aligned}\|Cx\|_{p,Q}^p =&\sum_{i=1}^{\infty}\big(\sum_{j=1}^{\infty}\frac{X_j^2}{j^2(i+j-1)}\big)^{\frac{p}{2}}\\ \ge&\sum_{i=1}^{\infty}\sum_{r=1}^{\infty}\big(\sum_{j=1}^{r}\frac{X_j^2}{j^2}\big)^{\frac{p}{2}}\big[(i+r-1)^{-\frac{p}{2}}-(i+r)^{-\frac{p}{2}}\big]\\ =&\sum_{r=1}^{\infty}\big(\sum_{j=1}^{r}\big(\frac{X_j}{j}\big)^{\frac{p}{2}}\sum_{i=1}^{\infty}(i+r-1)^{-\frac{p}{2}}-(i+r)^{-\frac{p}{2}}\big)\\ =&\sum_{r=1}^{\infty}\big(\frac{\sum_{j=1}^{r}(\frac{X_j}{j})^2}{r}\big)^{\frac{p}{2}}\\ =&\|C^2x\|_{\frac{p}{2}}^{\frac{p}{2}}.\end{aligned}$$

So for $p > 2$ we have

$$\|C^2x\|_{\frac{p}{2}}^{\frac{1}{2}} \le \|Cx\|_{p,Q} < \big(\frac{\pi}{\sin(\frac{2\pi}{p})}\big)^{\frac{1}{2}}\big(\frac{p}{p-1}\big)\|x\|_p,$$

specially, by choosing $x = (1,0,0,\cdots)$, we have

$$\sum_{i=1}^{\infty}\big(\frac{1+\frac{1}{2}+\cdots+\frac{1}{i}}{i}\big)^{\frac{p}{2}} \le \sum_{i=1}^{\infty}\big(\sum_{j=1}^{\infty}\frac{1}{j^2(i+j-1)}\big)^{\frac{p}{2}} < \big(\frac{\pi}{\sin(\frac{2\pi}{p})}\big)^{\frac{1}{2p}}\big(\frac{p}{p-1}\big)^{\frac{1}{p}}.$$

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