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Solutions of fractional functional integro-differential equations via Petryshyn's fixed point theorem

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Abstract

This article provides the presence of solutions to a fractional functional integro-differential equation via measures of non-compactness. We present and prove a novel theorem that guarantees the existence of solutions, employing Petryshyn's fixed point theorem in the space of continuous functions. These findings build upon previous studies by establishing the existence of results under less stringent conditions. Furthermore, we provide illustrative examples of such equations to showcase the efficacy of the obtained results.

Keywords: Measures of non-compactness, Fractional functional integro-differential equation, Fixed point theorem, Existence of solutions
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1 Introduction

Fractional differential equations with different forms of fractional derivatives play essential roles in various fields of applied science, including elasticity, fracture mechanics, and radiative equilibrium [29, 7, 21]. Many researchers applied various analytical studies and numerical methods to solve these types of equations [36, 3, 26]. For instance, the existence and the uniqueness of the solution of these equations were examined in [2, 13, 24].

The authors in [33] have solved a nonlinear fractional integro-differential equation of the Hammerstein type by converting it to the corresponding Volterra integral equation of the second kind.

In [5], Tau proposed a method that is based on the shifted Legendre polynomial to solve a class of fractional stochastic integro-differential equations.

Moreover, the technique of fixed point theorems has been employed to illustrate the existence of solutions to different types of problems such as in integral equations [16, 22], fractional differential equations [1, 6], and fractional integro-differential equations (FIDEs) [4, 25].

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The existence of the solution of the boundary value problem of fractional differential equations was discussed in [20, 30], utilizing the measure of non-compactness (MNC) with the Mönch fixed point theorem and with Darbo theorem in Banach space [8, 23].

In this paper, we provide and prove a new existence theorem for solving the following fractional functional integro-differential equation

$$
\begin{aligned}
&{}^{C}D^{\sigma}\left(\xi(\vartheta)+g\Big(\vartheta,\xi(\vartheta),\int_0^{\vartheta}h(\vartheta,\nu,\xi(\nu))d\nu\Big)\right) \qquad (1.1)\\
&\qquad = f(\vartheta,\xi(\alpha(\vartheta)))+F\left(\vartheta,\xi(\beta(\vartheta)),\xi(\theta(\vartheta)),\int_0^{\vartheta}k(\vartheta,\nu,\xi(\mu(\nu)))d\nu\right),\ \vartheta\in J_a=[0,a],
\end{aligned}
$$

with the initial conditions

$$\xi^{(i)}(0)=\xi_i,\quad \xi_i\in\mathbb{R}^+,\quad i=0,1,...,n-1, \qquad (1.2)$$

in the space of continuous functions $C([0,a],\mathbb{R})$, where ${}^{C}D^{\sigma}$ is the Caputo's fractional derivative and $\xi:J_a\to\mathbb{R}$ is the unknown function and $g:J_a\times\mathbb{R}^2\to\mathbb{R}$, $f:J_a\times\mathbb{R}\to\mathbb{R}$, $h,k:J_a^2\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

The present article is motivated by presenting a new existence theorem for solving the equation (1.1) by employing the technique of the MNC related to Petryshyn's fixed point theorem which performs a generalization of Darbo's and Schauder's fixed point theorems [9, 10, 11, 31]. Our assumptions are more simpler and general than the ones presented in the former studies such as we bypass the "sub-linear conditions" presented in [8]. Finally, we provide illustrative examples of such equations to showcase the efficacy of the obtained results.

2 Auxiliary facts and notations

Let $\mathbb{R}=(-\infty,+\infty)$, $J_a=[0,a]$, and $E=C(J_a)$ be the Banach space of continuous functions defined on J_a with the standard norm $\|.\|$. Denoted by $\bar{B}_\delta=\{z\in E:\| z\|\leq\delta\}$ the closed ball centered at the origin 0 of radius δ.

The symbol $\partial\bar{B}_\delta=\{z\in E:\| z\|=\delta\}$ represents a sphere in E around 0 with radius δ.

Definition 2.1. [18] The Riemann-Liouville fractional integral of order $\sigma>0$ of an integrable function ξ is defined as

$$I^{\sigma}\xi(\vartheta)=\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}(\vartheta-\nu)^{\sigma-1}\xi(\nu)d\nu,\quad \vartheta>0,$$

where $\Gamma(\sigma)=\int_0^{\infty}e^{-\nu}\nu^{\sigma-1}d\nu$.

Definition 2.2. [18] The Caputo derivative of fractional order σ for an absolutely continuous function ξ on J_a is defined by

$$\left({}^{C}D^{\sigma}\xi\right)(\vartheta)=\frac{1}{\Gamma(n-\sigma)}\int_0^{\vartheta}(\vartheta-\nu)^{n-\sigma-1}\xi^{(n)}(\nu)d\nu,$$

where $n=[\sigma]+1$ and $n-1<\sigma<n$.

Lemma 2.3. [18] Let $\sigma>0$ and $n=[\sigma]+1$. If $\xi(\vartheta)\in C^n[0,a]$, then

$$
\begin{aligned}
&(i)\quad \left(I^{\sigma}\,{}^{C}D^{\sigma}\xi\right)(\vartheta)=\xi(\vartheta)-\sum_{i=0}^{n-1}\frac{\xi^{(i)}(0)}{i!}\vartheta^{i},\\
&(ii)\quad \left({}^{C}D^{\sigma}I^{\sigma}\xi\right)(\vartheta)=\xi(\vartheta).
\end{aligned}
$$

Definition 2.4. [19] Let $P\subset E$, then $\alpha(P)$ refers to Kuratowski MNC, where

$$\alpha(P)=\inf\left\{\sigma>0:P=\bigcup_{i=1}^{n}P_i \text{ with diam }\quad P_i\leq\sigma,i=1,2,...,n\right\}.$$

Definition 2.5. [12] Let $P \subset E$, then $\mho(P)$ refers to Hausdorff MNC, where

$$\mho(P) = \inf\{\sigma > 0 : \text{P has a finite } \sigma\text{-net in } E\}.$$

Let $C[0,a]$ be the space of all real-valued continuous functions defined on J_a with the usual norm

$$\|\xi\| = \sup\{|\xi(s)| : s \in [0,a]\}.$$

The space $C[0,a]$ is also the structure of Banach algebra. The modulus of continuity of $\xi \in C[0,a]$ is defined as

$$\omega(\xi,\sigma) = \sup\{|\xi(s) - \xi(\bar{s})| : s, \bar{s} \in [0,a], |s - \bar{s}| \leq \sigma\}.$$

Theorem 2.6. [12] The Hausdorff MNC is similar to

$$\mho(P) = \lim_{\sigma\to 0} \sup_{\xi\in P} \omega(\xi,\sigma) \tag{2.1}$$

for all bounded sets $P \subset C[0,a]$.

Definition 2.7. [27] Let $\Gamma : E \to E$ be a continuous mapping so that $\forall P \subset E$ with P bounded, $\Gamma(P)$ is bounded and $\mho(\Gamma P) \leq \lambda\mho(P), \lambda \in (0,1)$. If

$$\mho(\Gamma P) < \mho(P), \text{for all } \mho(P) > 0,$$

then Γ is called condensing map.

Theorem 2.8. [28, 32] Let $\Gamma : \bar{B}_\delta \to E$ be a condensing mapping such that:

$$T(\xi) = \lambda\xi, \text{ for some } \xi \in \partial\bar{B}_\delta \text{ then } \lambda \leq 1.$$

Then T has at least one fixed point in $\bar{B}_\delta$.

3 Main Results

First, allow us to present the following assumptions:

(L1) $g \in C(J_a \times \mathbb{R}^2, \mathbb{R}), f \in C(J_a \times \mathbb{R}, \mathbb{R}), F \in C(J_a \times \mathbb{R}^3, \mathbb{R}), h, k \in C(J_a^2 \times \mathbb{R}, \mathbb{R})$ and, $\alpha, \beta, \theta, \mu : J_a \to J_a$ are continuous;

(L2) There exist non negative constants k_1, k_2, c_1, c_2, and c_3, where $k_1 < 1$ and

$$|g(\vartheta,\omega_1,\omega_2) - g(\vartheta,\varpi_1,\varpi_2)| \leq k_1|\omega_1 - \varpi_1| + k_2|\omega_2 - \varpi_2|;$$
$$|F(\vartheta,\omega_1,\omega_2,\omega_3) - F(\vartheta,\varpi_1,\varpi_2,\varpi_2)| \leq c_1|\omega_1 - \varpi_1| + c_2|\omega_2 - \varpi_2| + c_3|\omega_3 - \varpi_3|;$$

(L3) $\exists\ \delta_0 \geq 0$ such that

$$\sup\left\{L + A + \frac{M_1 a^\sigma}{\Gamma(1+\sigma)} + \frac{M_2 a^\sigma}{\Gamma(1+\sigma)}\right\} \leq \delta_0,$$

where

$$L = \sup\left\{\left|\sum_{i=0}^{n-1} \frac{\xi^{(i)}(0) + g^{(i)}(0,\xi_0,0)}{i!}\vartheta^i\right| : \forall\vartheta \in J_a, \forall\xi \in C[0,a]\right\},$$
$$A = \sup\{|g(\vartheta,\omega_1,\omega_2)| : \forall\vartheta \in J_a, \quad \text{and} \quad \omega_1 \in [-\delta_0,\delta_0], |\omega_2| \leq aB_1\},$$
$$B_1 = \sup\{|h(\vartheta,\nu,\omega_1)| : \forall\vartheta,\nu \in J_a, \quad \text{and} \quad \omega_1 \in [-\delta_0,\delta_0]\},$$
$$M_1 = \sup\{|f(\vartheta,\omega_1)| : \forall\vartheta \in J_a \quad \text{and} \quad \omega_1 \in [-\delta_0,\delta_0]\},$$
$$M_2 = \sup\{|F(\vartheta,\omega_1,\omega_2,\omega_3)| : \forall\vartheta \in J_a, \quad \omega_1,\omega_2 \in [-\delta_0,\delta_0], |\omega_3| \leq aB\}.,$$
$$B = \sup\{|k(\vartheta,\nu,\omega_1)| : \forall\vartheta,\nu \in J_a, \quad \text{and} \quad \omega_1 \in [-\delta_0,\delta_0]\},$$

Now, we show the equivalence between problem (1.1) with the initial conditions (1.2) and the fractional integral equation

$$\xi(\vartheta)=\sum_{i=0}^{n-1}\frac{\xi^{(i)}(0)+g^{(i)}(0,\xi_0,0)}{i!}\vartheta^i-g\Big(\vartheta,\xi(\vartheta),\int_0^{\vartheta}h(\vartheta,\nu,\xi(\nu))d\nu\Big)$$
$$+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta-\nu)^{1-\sigma}}d\nu+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta-\nu)^{1-\sigma}}d\nu, \tag{3.1}$$

where $(H\xi)(\vartheta)=\int_0^{\vartheta}k\big(\vartheta,\nu,\xi(\mu(\nu))\big)d\nu$.

Corollary 3.1. [18, Theorem 3.24] Under assumption (L1) and for $\sigma>0$, and $\xi\in C(J_a)$, then $\xi(\vartheta)$ verifies the problem (1.1) with the initial conditions (1.2) if, and only if, $\xi(\vartheta)$ fulfills the fractional integral equation (3.1).

Indeed, the proof can be easily done by applying the integral operator I^{σ} (2.1) to both sides of (1.1) and using Lemma (2.3) with the initial conditions (1.2) to get the integral equation (3.1). For more details see [8, 18]. Therefore, every solution of (3.1) is a solution of (1.1)–(1.2) and vice versa.

Theorem 3.2. With the conditions (L1)-(L3), Eq. (1.1) with the initial conditions (1.2) has at least one solution in $E=C(J_a)$.

Proof . We define the operator $T:B_{\delta_0}\to E$ as follows:

$$(T\xi)(\vartheta)=\sum_{i=0}^{n-1}\frac{\xi^{(i)}(0)+g^{(i)}(0,\xi_0,0)}{i!}\vartheta^i-g\Big(\vartheta,\xi(\vartheta)+\int_0^{\vartheta}h(\vartheta,\nu,\xi(\nu))d\nu\Big)+(T_1\xi)(\vartheta)+(T_2\xi)(\vartheta),$$

where

$$(T_1\xi)(\vartheta)=\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta-\nu)^{1-\sigma}}d\nu,\quad\text{and}\quad(T_2\xi)(\vartheta)=\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta}\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta-\nu)^{1-\sigma}}d\nu.$$

Step I, we need to demonstrate that $T:\ C(J_a)\to C(J_a)$. According to our assumptions, it suffices to demonstrate that for any function $\xi\in C(J_a)$ implies $T_1\xi$ and $T_2\xi$ are continuous on J_a.

For this, take arbitrary $\vartheta_2,\vartheta_2,\in J_a$ and fix $\varepsilon>0$ with $|\vartheta_2-\vartheta_1|\leq\varepsilon$. Without loss of generality assume that $\vartheta_1\leq\vartheta_2$, then we obtain

$$\begin{aligned}|(T_1\xi)(\vartheta_2))-(T_1\xi)(\vartheta_1))|&=\left|\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_2}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}d\nu-\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_1-\nu)^{1-\sigma}}d\nu\right|\\
&\leq\frac{1}{\Gamma(\sigma)}\left|\int_0^{\vartheta_1}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}d\nu+\int_{\vartheta_1}^{\vartheta_2}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}d\nu-\int_0^{\vartheta_1}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_1-\nu)^{1-\sigma}}d\nu\right|\\
&\leq\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\left|\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}-\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_1-\nu)^{1-\sigma}}\right|d\nu+\frac{1}{\Gamma(\sigma)}\int_{\vartheta_1}^{\vartheta_2}\left|\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}\right|d\nu\\
&\leq\frac{M_1}{\Gamma(1+\sigma)}\{\vartheta_1^{\sigma}-\vartheta_2^{\sigma}+(\vartheta_2-\vartheta_1)^{\sigma}\}+\frac{M_1}{\Gamma(1+\sigma)}(\vartheta_2-\vartheta_1)^{\sigma}\\
&\leq\frac{3\varepsilon^{\sigma}M_1}{\Gamma(1+\sigma)}.\end{aligned}$$

The above inequality yields that the operator $T_1 : C(J_a) \to C(J_a)$. Also, for the operator T_2 we have

$$\begin{aligned}
|(T_2\xi)(\vartheta_2) - (T_2\xi)(\vartheta_1)| =& \left| \frac{1}{\Gamma(\sigma)} \int_0^{\vartheta_2} \frac{F(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu))}{(\vartheta_2 - \nu)^{1-\sigma}} d\nu - \frac{1}{\Gamma(\sigma)} \int_0^{\vartheta_1} \frac{F(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu))}{(\vartheta_1 - \nu)^{1-\sigma}} d\nu \right| \\
\le& \frac{1}{\Gamma(\sigma)} \left| \int_0^{\vartheta_1} \frac{F(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu))}{(\vartheta_2 - \nu)^{1-\sigma}} d\nu + \int_{\vartheta_1}^{\vartheta_2} \frac{F(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu))}{(\vartheta_2 - \nu)^{1-\sigma}} d\nu \right. \\
&\left. - \int_0^{\vartheta_1} \frac{F(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(s))}{(\vartheta_1 - \nu)^{1-\sigma}} d\nu \right| \\
\le& \frac{1}{\Gamma(\sigma)} \int_0^{\vartheta_1} \left| \frac{F(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu))}{(\vartheta_2 - \nu)^{1-\sigma}} - \frac{F(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu))}{(\vartheta_1 - \nu)^{1-\sigma}} \right| d\nu \\
&+ \frac{1}{\Gamma(\sigma)} \int_{\vartheta_1}^{\vartheta_2} \left| \frac{F(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu))}{(\vartheta_2 - \nu)^{1-\sigma}} \right| d\nu \\
\le& \frac{M_2}{\Gamma(1+\sigma)} \{\vartheta_1^\sigma - \vartheta_2^\sigma + (\vartheta_2 - \vartheta_1)^\sigma\} + \frac{M_2}{\Gamma(1+\sigma)} (\vartheta_2 - \vartheta_1)^\sigma \\
\le& \frac{3\varepsilon^\sigma M_2}{\Gamma(1+\sigma)}.
\end{aligned}$$

So, this yields that the operator $T_2 : C(J_0) \to C(J_0)$ and $T : C(J_0) \to C(J_0)$.

Step II, we will check that T is continuous on B_{δ_0}.

Considering $\varepsilon > 0$ and for arbitrary values $\xi, \eta \in B_{\delta_0}$ such that $\| \xi - \eta \| \le \varepsilon$, when $\vartheta \in J_a$ we will have

$$\begin{aligned}
\big|(T\xi)(\vartheta) - (\Gamma\eta)(\vartheta)\big| =& \left| \sum_{i=0}^{n-1} \frac{\xi^{(i)}(0) + g^{(i)}(0, \xi_0, 0)}{i!} \vartheta^i - g\Big(\vartheta, \xi(\vartheta) + \int_0^\vartheta h(\vartheta, \nu, \xi(\nu)) d\nu\Big) + (T_1\xi)(\vartheta) + (T_2\xi)(\vartheta) \right. \\
&\left. - \sum_{i=0}^{n-1} \frac{\eta^{(i)}(0) + g^{(i)}(0, \eta_0, 0)}{i!} \vartheta^i - g\Big(\vartheta, \eta(\vartheta) + \int_0^\vartheta h(\vartheta, \nu, \eta(\nu)) d\nu\Big) + (T_1\eta)(\vartheta) + (T_2\eta)(\vartheta) \right| \\
\le& \left| g\Big(\vartheta, \xi(\vartheta), \int_0^\vartheta h(\vartheta, \nu, \xi(\nu)) d\nu\Big) - g\Big(\vartheta, \eta(\vartheta), \int_0^\vartheta h(\vartheta, \nu, \eta(\nu)\Big) d\nu)\right| \\
&+ \frac{1}{\Gamma(\sigma)} \int_0^\vartheta \frac{|f(\nu, \xi(\alpha(\nu))) - f(\nu, \eta(\alpha(\nu)))|}{(\vartheta - \nu)^{1-\sigma}} d\nu \\
&+ \frac{1}{\Gamma(\sigma)} \int_0^\vartheta \frac{\Big|F\Big(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu)\Big) - F\Big(\nu, \eta(\beta(\nu)), \eta(\theta(\nu)), (H\eta)(\nu)\Big)\Big|}{(\vartheta - \nu)^{1-\sigma}} d\nu \\
\le& k_1 \| \xi - \eta \| + k_2 a\omega(h, \varepsilon) + \frac{\vartheta^\sigma}{\Gamma(1+\sigma)} \omega(f, \omega(\alpha, \varepsilon)) \\
&+ \frac{1}{\Gamma(\sigma)} \int_0^\vartheta \frac{\Big|F\Big(\nu, \xi(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu)\Big) - F\Big(\nu, \eta(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu)\Big)\Big|}{(\vartheta - \nu)^{1-\sigma}} d\nu \\
&+ \frac{1}{\Gamma(\sigma)} \int_0^\vartheta \frac{\Big|F\Big(\nu, \eta(\beta(\nu)), \xi(\theta(\nu)), (H\xi)(\nu)\Big) - F\Big(\nu, \eta(\beta(\nu)), \eta(\theta(\nu)), (H\xi)(\nu)\Big)\Big|}{(\vartheta - \nu)^{1-\sigma}} d\nu \\
&+ \frac{1}{\Gamma(\sigma)} \int_0^\vartheta \frac{\Big|F\Big(\nu, \eta(\beta(\nu)), \eta(\theta(\nu)), (H\xi)(\nu)\Big) - F\Big(\nu, \eta(\beta(\nu)), \eta(\theta(\nu)), (H\eta)(\nu)\Big)\Big|}{(\vartheta - \nu)^{1-\sigma}} d\nu \\
\le& k_1 \| \xi - \eta \| + k_2 a\omega(h, \varepsilon) + \frac{\vartheta^\sigma}{\Gamma(1+\sigma)} \omega(f, \omega(\alpha, \varepsilon)) + \frac{c_1 \vartheta^\sigma}{\Gamma(1+\sigma)} \| \xi - \eta \| \\
&+ \frac{c_2 \vartheta^\sigma}{\Gamma(1+\sigma)} \| \xi - \eta \| + \frac{c_3 a^2 \vartheta^\sigma}{\Gamma(1+\sigma)} \omega(k, \varepsilon),
\end{aligned}$$

where

$$\omega(f,\varepsilon)=\sup\{|f(\nu,\xi)-f(\nu,\eta)|:\nu\in J_a,\xi,\eta\in[-\delta_0,\delta_0],\|\xi-\eta\|\le\varepsilon\},$$
$$\omega(k,\varepsilon)=\sup\{|k(\vartheta,\nu,\xi)-k(\vartheta,\nu,\eta)|:\vartheta,\nu\in J_a,\xi,\eta\in[-\delta_0,\delta_0],\|\xi-\eta\|\le\varepsilon\},$$
$$\omega(h,\varepsilon)=\sup\{|h(\vartheta,\nu,\xi)-h(\vartheta,\nu,\eta)|:\vartheta,\nu\in J_a,\xi,\eta\in[-\delta_0,\delta_0],\|\xi-\eta\|\le\varepsilon\}.$$

Now, because the functions $f=f(\nu,\xi),h=h(\vartheta,\nu,\xi)$ and $k=k(\vartheta,\nu,\xi)$ are uniformly continuous on $J_a\times\mathbb{R}$ and $J_a^2\times\mathbb{R}$, respectively, we conclude $\omega(f,\omega(\alpha,\varepsilon))\to 0,\omega(h,\varepsilon)\to 0$ and $\omega(k,\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence, the continuity of T on B_{δ_0} results.

Step III, now, it is shown that T satisfies the densifying condition.

Let ε be an arbitrary positive constant. For $\xi\in P\subset E$ let $\vartheta_1,\vartheta_2\in J_a$ while $\vartheta_1\le\vartheta_2$ and $\vartheta_2-\vartheta_1\le\varepsilon$. Therefore we obtain

$$\begin{aligned}
|(T\xi)(\vartheta_2)-(T\xi)(\vartheta_1)|=&\Bigg|\sum_{i=0}^{n-1}\frac{\xi^{(i)}(0)+g^{(i)}(0,\xi_0,0)}{i!}\vartheta_2^i-g\Big(\vartheta_2,\xi(\vartheta_2),\int_0^{\vartheta_2}h(\vartheta_2,\nu,\xi(\nu))d\nu\Big)\\
&+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_2}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}d\nu+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_2}\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\vartheta\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_2-\nu)^{1-\sigma}}d\nu\\
&-\sum_{i=0}^{n-1}\frac{\xi^{(i)}(0)+g^{(i)}(0,\xi_0,0)}{i!}\vartheta_1^i+g\Big(\vartheta_1,\xi(\vartheta_1),\int_0^{\vartheta_1}h(\vartheta_1,\nu,\xi(\nu)\Big)d\nu)\\
&-\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_1-\nu)^{1-\sigma}}d\nu-\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_1-\nu)^{1-\sigma}}d\nu\Bigg|\\
\le&\left|\sum_{i=0}^{n-1}\frac{\xi^{(i)}(0)+g^{(i)}(0,\xi_0,0)}{i!}(\vartheta_2^i-\vartheta_1^i)\right|\\
&+\left|g\Big(\vartheta_1,\xi(\vartheta_1),\int_0^{\vartheta_1}h(\vartheta_1,\nu,\xi(\nu))d\nu\Big)-g\Big(\vartheta_1,\xi(\vartheta_1),\int_0^{\vartheta_2}h(\vartheta_2,\nu,\xi(\nu))d\nu\Big)\right|\\
&+\left|g\Big(\vartheta_1,\xi(\vartheta_1),\int_0^{\vartheta_2}h(\vartheta_2,\nu,\xi(\nu))d\nu\Big)-g\Big(\vartheta_1,\xi(\vartheta_2),\int_0^{\vartheta_2}h(\vartheta_2,\nu,\xi(\nu))d\nu\Big)\right|\\
&+\left|g\Big(\vartheta_1,\xi(\vartheta_2),\int_0^{\vartheta_2}h(\vartheta_2,\nu,\xi(\nu))d\nu\Big)-g\Big(\vartheta_2,\xi(\vartheta_2),\int_0^{\vartheta_2}h(\vartheta_2,\nu,\xi(\nu))d\nu\Big)\right|\\
&+\frac{1}{\Gamma(\sigma)}\left|\int_0^{\vartheta_1}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}d\nu+\int_{\vartheta_1}^{\vartheta_2}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}d\nu-\int_0^{\vartheta_1}\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_1-\nu)^{1-\sigma}}d\nu\right|\\
&+\frac{1}{\Gamma(\sigma)}|\int_0^{\vartheta_1}\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_2-\nu)^{1-\sigma}}d\nu+\int_{\vartheta_1}^{\vartheta_2}\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_2-\nu)^{1-\sigma}}d\nu\\
&-\int_0^{\vartheta_1}\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(s)\Big)}{(\vartheta_1-\nu)^{1-\sigma}}d\nu\Bigg|\\
\le&k_2\left|\int_0^{\vartheta_1}h(\vartheta_1,\nu,\xi(\nu))d\nu-\int_0^{\vartheta_2}h(\vartheta_2,\nu,\xi(\nu))d\nu\right|+k_1|\xi(\vartheta_1)-\xi(\vartheta_2)|+\omega_g(J_a,\varepsilon)\\
&+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\left|\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}-\frac{f(\nu,\xi(\alpha(\nu))}{(\vartheta_1-\nu)^{1-\sigma}}\right|d\nu+\frac{1}{\Gamma(\sigma)}\int_{\vartheta_1}^{\vartheta_2}\left|\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}\right|d\nu\\
&+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\left|\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_2-\nu)^{1-\sigma}}-\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu))\Big)}{(\vartheta_1-\nu)^{1-\sigma}}\right|d\nu\\
&+\frac{1}{\Gamma(\sigma)}\int_{\vartheta_1}^{\vartheta_2}\left|\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_2-\nu)^{1-\sigma}}\right|d\nu
\end{aligned}$$

$$
\begin{aligned}
\leq\ & k_2\Big|\int_0^{\vartheta_1}\Big(h(\vartheta_1,\nu,\xi(\nu))-h(\vartheta_2,\nu,\xi(\nu))\Big)d\nu-\int_{\vartheta_1}^{\vartheta_2}h(\vartheta_2,\nu,\xi(\nu))d\nu\Big|+k_1\omega(\xi,\varepsilon)+\omega_g(J_a,\varepsilon)\\
&+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\left|\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}-\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_1-\nu)^{1-\sigma}}\right|d\nu+\frac{1}{\Gamma(\sigma)}\int_{\vartheta_1}^{\vartheta_2}\left|\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta_2-\nu)^{1-\sigma}}\right|d\nu\\
&+\frac{1}{\Gamma(\sigma)}\int_0^{\vartheta_1}\left|\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_2-\nu)^{1-\sigma}}-\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_1-\nu)^{1-\sigma}}\right|d\nu\\
&+\frac{1}{\Gamma(\sigma)}\int_{\vartheta_1}^{\vartheta_2}\left|\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta_2-\nu)^{1-\sigma}}\right|d\nu.
\end{aligned}
$$

For simplicity, we use the following notation:

$$
\begin{aligned}
&\omega_g(J_a,\varepsilon)=\sup\{|g(\vartheta,\omega_1,\omega_2)-g(\bar{\vartheta},\omega_1,\omega_2)|:|\vartheta-\bar{\vartheta}|\leq\varepsilon,\quad \vartheta\in J_a,\quad \omega_1\in[-\delta_0,\delta_0],\quad |\omega_2|\leq aC\},\\
&\omega_h(J_a,\varepsilon)=\sup\{|h(\vartheta,\nu,\omega_1)-h(\bar{\vartheta},\nu,\omega_1)|:|\vartheta-\bar{\vartheta}|\leq\varepsilon,\quad \vartheta,\nu\in J_a,\quad \omega_1\in[-\delta_0,\delta_0]\}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|(T\xi)(\vartheta_2)-(T\xi)(\vartheta_1)|\ \leq\ & k_2a\omega_h(J_a,\varepsilon)+k_2\varepsilon B_1+k_1\omega(\xi,\varepsilon)+\omega_g(J_a,\varepsilon)\\
+\ & \frac{M_1}{\Gamma(1+\sigma)}\{\vartheta_1^\sigma-\vartheta_2^\sigma+(\vartheta_2-\vartheta_1)^\sigma\}+\frac{M_1}{\Gamma(1+\sigma)}(\vartheta_2-\vartheta_1)^\sigma\\
+\ & \frac{M_2}{\Gamma(1+\sigma)}\{\vartheta_1^\sigma-\vartheta_2^\sigma+(\vartheta_2-\vartheta_1)^\sigma\}+\frac{M_2}{\Gamma(1+\sigma)}(\vartheta_2-\vartheta_1)^\sigma\\
\leq\ & k_2a\omega_h(J_a,\varepsilon)+k_2\varepsilon B_1+k_1\omega(\xi,\varepsilon)+\omega_g(J_a,\varepsilon)+\frac{3\varepsilon^\sigma M_1}{\Gamma(1+\sigma)}+\frac{3\varepsilon^\sigma M_2}{\Gamma(1+\sigma)}.
\end{aligned}
$$

This yields the following estimate:

$$\omega(T\xi,\varepsilon)\leq k_1\omega(\xi,\varepsilon),\qquad \xi\in P.$$

Thus, taking the supremum in P, then the limit as $\varepsilon\to 0$ we obtain

$$\mho(TP)\leq k_1\mho(P).$$

Hence T is a condensing map.

Step IV, finally, let $\xi\in\partial\bar{B}_{\delta_0}$. If $T\xi=\lambda\xi$, then we have $\|T\xi\|=\lambda\|\xi\|=\lambda\delta_0$ and with the condition (L3), we get

$$
\begin{aligned}
|T\xi(\vartheta)|=\Bigg|&\sum_{i=0}^{n-1}\frac{\xi^{(i)}(0)+g^{(i)}(0,\xi_0,0)}{i!}\vartheta^i-g\Big(\vartheta,\xi(t),\int_0^\vartheta h(\vartheta,\nu,\xi(\nu))d\nu\Big)\\
&+\frac{1}{\Gamma(\sigma)}\int_0^\vartheta\frac{f(\nu,\xi(\alpha(\nu)))}{(\vartheta-\nu)^{1-\sigma}}d\nu+\frac{1}{\Gamma(\sigma)}\int_0^\vartheta\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta-\nu)^{1-\sigma}}d\nu\Bigg|\leq\delta_0,\quad \vartheta\in J_a,
\end{aligned}
$$

hence $\|T\xi\|\leq\delta_0$, this means $\lambda\leq 1$. □

4 Particular cases and examples

Now, we will extend and discuss the results presented in ([8], Theorem 3) as a particular case of our results which discuss the following equation

$$
{}^CD^\gamma\Big(x(\vartheta)+g(\vartheta,x(\vartheta)\Big)=f(\vartheta,x(\vartheta))+F\Big(\vartheta,x(\vartheta),\int_0^\vartheta k(\vartheta,\nu)H\big(x\big(\mu(\nu)\big)\big)d\nu\Big),\ \vartheta\in J_a, \tag{4.1}
$$

with the initial conditions

$$
x^{(i)}(0)=x_i,\quad i=0,1,...,n-1. \tag{4.2}
$$

Corollary 4.1. [8] Suppose

(M1) $g, f \in C(J_a \times \mathbb{R}, \mathbb{R})$ and there exist the functions $l_1, l_2 : J_a \to J_a$ being continuous such that

$$|g(\vartheta, \omega_1) - g(\vartheta, \varpi_1)| \le l_1(\vartheta)|\omega_1 - \varpi_1|, \quad B = \sup\{g(\vartheta, 0), \vartheta \in J_a\};$$
$$|f(\vartheta, \omega_1) - f(\vartheta, \varpi_1)| \le l_2(\vartheta)|\omega_1 - \varpi_1|, \quad B_1 = \sup\{f(\vartheta, 0), \vartheta \in J_a\}.$$

(M2) $F \in C(J_a \times \mathbb{R}^2, \mathbb{R})$, and there exists a continuous function $l_3 : J_a \to J_a$ so that

$$|F(\vartheta, \omega_1, \omega_2) - F(\vartheta, \varpi_1, \varpi_2)| \le l_3(\vartheta)(|\omega_1 - \varpi_1| + |\omega_2 - \varpi_2|), \quad \forall \omega_1, \omega_2, \varpi_1, \varpi_2 \in \mathbb{R},$$
$$D = \sup\{F(\vartheta, 0, 0), \vartheta \in J_a\}.$$

Let $l = \max_i\{|l_i(\vartheta)| : \vartheta \in J_a\}$, $i = 1, 2, 3$ and $0 \le l < 1$. Also, let there exists a constant $E > 0$ such that $B, B_1, D \le E$.

(M3) For $H : C(J_a) \to C(J_a)$, there exist a constant $N > 0$ so that

$$|(Hx)(\vartheta) - (Hy)(\vartheta)| \le N|x(\vartheta) - y(\vartheta)|$$

for any $\vartheta \in J_a$ and for all $x, y \in C(J_a)$.

(M4) There exists a function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ being nondecreasing such that

$$\|Hx\| \le \Lambda(\|x\|),$$

for all $x \in C(J_a)$.

(M5) If $|\sum_{i=0}^{n-1} \frac{x^{(i)}(0)+g^{(i)}(0,x_0)}{i!}\vartheta^i| \le J$, then there exists $\delta \ge 0$ of the inequality

$$J + E + l\delta + \frac{a^\sigma}{\Gamma(\sigma+1)}[2l\delta + 2E + la\|k\|\Lambda(\delta)] \le \delta.$$

Then Eq. (4.1) with the initial conditions (4.2) has at least a solution in J_a.

Proof . It is clear that Eq. (4.1) is a particular case of Eq. (1.1). Here $\alpha(\vartheta) = \beta(\vartheta) = \theta(\vartheta) = \vartheta$, $k(\vartheta, \nu, x(\mu(\nu))) = k(\vartheta, \nu)H(x(\mu(\nu)))$, $g(\vartheta, \omega_1, \omega_2) = g(\vartheta, \omega_1)$, and $F(\vartheta, \omega_1, \omega_2, \omega_3) = F(\vartheta, \omega_1, \omega_3)$.
By employing Riemann-Liouville fractional integrating and Lemma 2.3, Eq. (4.1) takes the form

$$\begin{aligned} x(t) = &\sum_{i=0}^{n-1} \frac{x^{(i)}(0) + g^{(i)}(0, x_0)}{i!}\vartheta^i - g(\vartheta, x(\vartheta)) + \frac{1}{\Gamma(\sigma)}\int_0^\vartheta \frac{f(\nu, x(\nu))}{(\vartheta-\nu)^{1-\sigma}}d\nu \\ &+ \frac{1}{\Gamma(\sigma)}\int_0^\vartheta \frac{F(\nu, x(\nu), (Hx)(\nu))}{(\vartheta-\nu)^{1-\sigma}}d\nu. \end{aligned}$$

(M2) and (M4) imply that the assumption (L2) is satisfied. It suffices to show that (L3) also holds. We have

$$\begin{aligned} |x(\vartheta)| \le &\left|\sum_{i=0}^{n-1} \frac{x^{(i)}(0) + g^{(i)}(0, x_0, 0)}{i!}\vartheta^i\right| + |g(\vartheta, x(\vartheta)) - g(\vartheta, 0)| + |g(\vartheta, 0)| + \frac{1}{\Gamma(\sigma)}\int_0^\vartheta \frac{|f(\nu, x(\nu)) - f(\nu, 0)| + |f(\nu, 0)|}{(\vartheta-\nu)^{1-\sigma}}d\nu \\ &+ \frac{1}{\Gamma(\sigma)}\int_0^\vartheta \frac{|F\Big(\nu, x(\nu), (Hx)(\nu)\Big) - F(\nu, 0, 0)| + |F(\nu, 0, 0)|}{(\vartheta-\nu)^{1-\sigma}}d\nu \\ &\le |\sum_{i=0}^{n-1} \frac{x^{(i)}(0) + g^{(i)}(0, x_0)}{i!}\vartheta^i| + l\|x\| + B + \frac{l\|x\| + B_1}{\Gamma(\sigma+1)}a^\sigma + \frac{l\|x\| + la\|k\|\Lambda(\|x\|) + D}{\Gamma(\beta+1)}a^\sigma \\ &\le J + E + l\delta + \frac{a^\sigma}{\Gamma(\sigma+1)}[2l\delta + 2E + la\|k\|\Lambda(\delta)] \end{aligned} \tag{4.3}$$

for all $\vartheta \in J_a$. From the estimates (4.3) and assumption (M5), we conclude that there exists $\delta_0 = \delta > 0$ such that

$$\sup_{\vartheta \in J_a} |x(\vartheta)| \le \delta_0.$$

Finally, Theorem 3.2 gives the desired result. □

Remark 4.2. The above corollary is the main result of [8], which has five conditions (M1-M5), and its expression and proof are like Theorem 3.1 using Petryshyn's theorem, where we reduce them to the conditions (L1-L3). The advantage of using Petryshyn's theorem is that we omit its statement and proof.

Now, we provide more particular cases of our outcomes.

1. If $g(\vartheta,\omega_1,\omega_2)=0$, $f(\vartheta,\omega_1)=q(\vartheta)$, $F(\vartheta,\omega_1,\omega_2,\omega_3)=\omega_3$, and $\mu(\nu)=\nu$ then Eq. (1.1) reduced to the equation

$$^{C}D^{\gamma}(\xi(\vartheta))=q(\vartheta)+\int_0^{\vartheta}k(\vartheta,\nu,\xi(\nu))d\nu,\ \vartheta\in J_a,$$

with

$$\xi^{(i)}(0)=\delta_i, i=0,1,...,n-1,$$

which has been examined in [33].

2. For $g(\vartheta,\omega_1,\omega_2)=0$, $\alpha(\vartheta)=\vartheta$, $F(\vartheta,\omega_1,\omega_2,\omega_3)=\omega_3$, and $\mu(\nu)=\nu$ we get the following nonlinear fractional Volterra integro-differential equations of the Hammerstein type studied in [15]

$$^{C}D^{\gamma}(\xi(\vartheta))=f(\vartheta,\xi(\vartheta))+\int_0^{\vartheta}k(\vartheta,\nu,\xi(\nu))d\nu,\ \vartheta\in J_a.$$

3. For $g(\vartheta,\omega_1,\omega_2)=\rho(\vartheta,\omega_1)$, $\alpha(\vartheta)=\vartheta$, $F(\vartheta,\omega_1,\omega_2,\omega_3)=q(\vartheta,\omega_1,\omega_3)$, $\beta(\vartheta)=\vartheta$, $k(\vartheta,\nu,\xi)=p(\vartheta,\nu)H(\xi)$, and $\mu(\nu)=\nu$ we get the following equations [8].

$$^{C}D^{\gamma}\Big(\xi(\vartheta)+\rho(\vartheta,\xi(\vartheta))\Big)=h(\vartheta,\xi(\vartheta))+q\Big(\vartheta,\xi(\vartheta),\int_0^{\vartheta}p(\vartheta,\nu)H(\xi(\nu)))d\nu\Big),\ \vartheta\in J_a,$$

with

$$\xi^{(i)}(0)=\xi_i,\quad i=0,1,...,n-1.$$

4. Let $g=0, f(\vartheta,\xi)=a(\vartheta)$, and $F(\vartheta,\omega_1,\omega_2,\omega_3)=\omega_3$, then we have the following nonlinear integro-differential equations that studied in [34]

$$D^{\sigma}u(\vartheta)=a(\vartheta)+\int_0^{\vartheta}h(\vartheta,\nu)u(\lambda u(\nu))d\nu,$$
$$u(0)=u_0,$$

where

$$D^{\sigma}\xi(\vartheta)=\frac{1}{\Gamma(1-\sigma)}\int_0^{\vartheta}(\vartheta-\mu)^{-\sigma}\xi'(\mu)d\mu,\quad \vartheta>0,\quad o<\sigma<1.$$

5. Let $g(\vartheta,\omega_1,\omega_2)=\lambda\vartheta^{\sigma}$ and $F=0$,then we have the following nonlinear integro-differential equations that studied in [35, 14].

$$D^{\sigma}(u(\vartheta)-\lambda\vartheta^{\sigma})=f(\vartheta,u(u(\vartheta))),$$
$$u(0)=u_0,$$

Example 4.3. Consider

$$^{C}D^{1.25}\Big(\xi(\vartheta)+\frac{1}{5}\sin(\xi(\vartheta))+\frac{1}{3}\int_0^{\vartheta}\frac{\nu^2e^{-2\vartheta}\sqrt{\xi(\nu)}}{1+\vartheta}d\nu\Big)=\frac{1}{\sqrt{16+\vartheta}}e^{-\vartheta}+\frac{\xi(t^3)}{1+|\xi(\vartheta^3)|}e^{-\ln(1+\vartheta^2)}$$
$$+\frac{\vartheta^2}{3+3\vartheta^2}\ln\left(1+\frac{|\xi(\vartheta^2)|+|\xi(\sqrt{\vartheta})|}{2}\right)+\frac{\vartheta}{3+3\vartheta}\int_0^{\vartheta}\frac{\vartheta e^{-3\nu}}{1+\vartheta^2}\left(\frac{1}{2}+\int_0^{\nu}\zeta\sin\zeta\sqrt{1+\xi(\zeta)}d\zeta\right)d\nu,\ \vartheta\in J_a \tag{4.4}$$

with

$$\xi^{(i)}(0)=\xi_i,\quad i=0,1. \tag{4.5}$$

Here, Eq. (4.4) is a particular case of Eq. (1.1) with $\sigma = 1.25, \quad n = 2, \quad a = 1,$

$$g(\vartheta,\omega_1,\omega_2) = \frac{1}{5}\sin(\omega_1) + \frac{1}{3}\omega_2, \quad \omega_2 = \int_0^{\vartheta} \frac{\nu^2 e^{-2\vartheta}\sqrt{\xi(\nu)}}{1+\vartheta}d\nu,$$

$$f(\vartheta,\xi(\alpha(\vartheta))) = \frac{1}{\sqrt{16+\vartheta}}e^{-\vartheta} + \frac{\xi(\vartheta^3)}{1+|\xi(\vartheta^3)|}e^{-\ln(1+\vartheta^2)}, \quad \xi(\mu(\vartheta)) = \frac{1}{2} + \int_0^{\vartheta} \zeta\sin\zeta\sqrt{1+\xi(\zeta)}d\zeta,$$

$$F(\vartheta,\omega_1,\omega_2,\omega_3) = \frac{\vartheta^2}{3+3\vartheta^2}\ln\left(1+\frac{\omega_1+\omega_2}{2}\right) + \frac{\vartheta}{3+3\vartheta}\omega_3,$$

$$\omega_3 = \int_0^{\vartheta} \frac{\vartheta e^{-3\nu}}{1+\vartheta^2}\left(\frac{1}{2} + \int_0^{\nu} \zeta\sin\zeta\sqrt{1+\xi(\zeta)}\,d\zeta\right)d\nu.$$

It is clear that (L1) holds. Also, conditions (L2) and (L3) are satisfied. we have

$$|g(\vartheta,\omega_1,\omega_2) - g(\vartheta,\varpi_1,\varpi_2)| \le \frac{1}{5}|\omega_1-\varpi_1| + \frac{1}{3}|\omega_2-\varpi_2|$$

and

$$\begin{aligned}
&|F(\vartheta,\omega_1,\omega_2,\omega_3) - F(\vartheta,\varpi_1,\varpi_2,\varpi_3)| \\
&\quad = \left|\frac{\vartheta^2}{3+3\vartheta^2}\ln\left(1+\frac{\omega_1+\omega_2}{2}\right) + \frac{\vartheta}{3+3\vartheta}\omega_3 - \frac{\vartheta^2}{3+3\vartheta^2}\ln\left(1+\frac{\varpi_1+\varpi_2}{2}\right) - \frac{\vartheta}{3+3\vartheta}\varpi_3\right| \\
&\quad \le \frac{\vartheta^2}{3+3\vartheta^2}\left|\frac{\omega_1+\omega_2}{2} - \frac{\varpi_1+\varpi_2}{2}\right| + \frac{\vartheta}{3+3\vartheta}|\omega_3-\varpi_3| \\
&\quad \le \frac{1}{12}(|\omega_1-\varpi_1| + |\omega_2-\varpi_2|) + \frac{1}{6}|\omega_3-\varpi_3|.
\end{aligned}$$

Here $k_1 = \frac{1}{5} < 1, k_2 = \frac{1}{3}, c_1 = \frac{1}{12}, c_2 = \frac{1}{12}, c_3 = \frac{1}{6}$.
Also, suppose that $\|\xi\| \le \delta_0, \delta_0 > 0$ and $\xi_0 = 0, \xi_1 = 1$, then we have

$$\begin{aligned}
|\xi(\vartheta)| &= \Bigg|\sum_{i=0}^{1} \frac{\xi^{(i)}(0) + g^{(i)}(0,\xi_0,0)}{i!}\vartheta^i - \left(\frac{1}{5}\sin(\xi(\vartheta)) + \frac{1}{3}\int_0^{\vartheta} \frac{\nu^2 e^{-2\vartheta}\sqrt{\xi(\nu)}}{1+\vartheta}d\nu\right) \\
&\quad + \frac{1}{\Gamma(1.25)}\int_0^{\vartheta} \frac{f(\nu,\xi(\alpha(\nu))}{(\vartheta-\nu)^{0.25}}d\nu + \frac{1}{\Gamma(1.25)}\int_0^{\vartheta} \frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta-\nu)^{0.25}}d\nu\Bigg| \\
&\le \frac{7}{5} + \frac{\sqrt{\delta_0}}{3} + \frac{1}{\Gamma(2.25)}(\frac{1}{4}+1) + \frac{1}{\Gamma(2.25)}\left(\frac{\delta_0}{6} + \frac{1}{12}(\frac{1}{2} + \sqrt{1+\delta_0})\right), \quad \forall\vartheta \in J_a.
\end{aligned}$$

So, condition (L3) holds if $\frac{7}{5} + \frac{\sqrt{\delta_0}}{3} + \frac{5}{4\Gamma(2.25)} + \frac{1}{\Gamma(2.25)}\left(\frac{\delta_0}{6} + \frac{1}{12}(\frac{1}{2} + \sqrt{1+\delta_0})\right) \le \delta_0$. This shows that $\delta_0 = 3.947$ is a solution of the above inequality. In view of Theorem 3.2, every problems (4.4)-(4.5) has at least one solution defined on $[0,1]$.

Example 4.4. Consider

$$\begin{aligned}
{}^{C}D^{0.5}\Big(\xi(\vartheta) + \frac{1+\ln(1+|\xi(\vartheta)|)}{(\vartheta+2)^2}\Big) &= \frac{1}{3}\vartheta e^{-(\vartheta+1)} + \frac{2\sin(\vartheta)\xi(\sqrt[3]{\vartheta})}{9(1+\sqrt{1+\vartheta})} + \frac{1}{5}\sin\left(\frac{3\xi(\vartheta^2)}{1+\vartheta^3}\right) + \frac{1}{(4+\vartheta)}\frac{|\xi(\vartheta)|}{1+|\xi(\vartheta)|} \\
&\quad + \frac{1}{3}e^{-\vartheta}\int_0^{\vartheta} \frac{e^{\vartheta}\cos(\xi(1-\nu))}{\sqrt{1+2\vartheta)}}\left(\frac{\vartheta\sin(\xi(\vartheta))}{4} + \frac{1}{3}\int_0^{\nu} \vartheta\arctan\left(\frac{|\xi(1-\zeta)|}{1+|\xi(1-\zeta)|}\right)d\zeta\right)d\nu,
\end{aligned} \tag{4.6}$$

with

$$\xi(0) = \xi_0 = 0, \tag{4.7}$$

for $\vartheta \in [0,1]$. In view of Eq. (1.1), we have $\sigma = 0.5, \quad n = a = 1$,

$$g(\vartheta,\xi(\vartheta)) = \frac{1+\ln(1+|\xi(\vartheta)|)}{(\vartheta+2)^2}, f(\vartheta,\xi(\alpha(\vartheta))) = \frac{1}{3}\vartheta e^{-(\vartheta+1)} + \frac{2\sin(\vartheta)\xi(\sqrt[3]{\vartheta})}{9(1+\sqrt{1+\vartheta})},$$

$$\xi(\mu(\vartheta)) = \frac{\vartheta\sin(\xi(\vartheta))}{4} + \frac{1}{3}\int_0^{\vartheta} \vartheta\arctan\left(\frac{|\xi(1-\eta)|}{1+|\xi(1-\eta)|}\right)d\eta,$$

$$F(t,\omega_1,\omega_2,\omega_3) = \frac{1}{5}\sin\left(\frac{3\omega_1}{1+\vartheta^3}\right) + \frac{1}{(4+\vartheta)}\frac{|\omega_2|}{1+|\omega_2|} + \frac{1}{3}e^{-\vartheta}\omega_3,$$

$$\omega_3 = \int_0^{\vartheta} \frac{e^{\vartheta}\cos(\xi(1-\nu))}{\sqrt{1+2\vartheta)}}\left(\frac{\nu\sin(\xi(\nu))}{4} + \frac{1}{3}\int_0^{\nu}\nu\arctan\left(\frac{|\xi(1-\zeta)|}{1+|\xi(1-\zeta)|}\right)d\zeta\right)d\nu.$$

Observe that (L1) holds. We show that conditions (L2) and (L3) are satisfied. we have

$$|g(\vartheta,\omega_1) - g(\vartheta,\varpi_1)| \le \frac{1}{4}|\omega_1 - \varpi_1|$$

and

$$|F(\vartheta,\omega_1,\omega_2,\omega_3) - F(\vartheta,\varpi_1,\varpi_2,\varpi_3)| \le \frac{3}{5}|\omega_1-\varpi_1| + \frac{1}{4}|\omega_2-\varpi_2| + \frac{1}{3}|\omega_3-\varpi_3|.$$

Here $k_1 = \frac{1}{4} < 1, c_1 = \frac{3}{5}, c_2 = \frac{1}{4}, c_3 = \frac{1}{3}$. Also, suppose that $\|\xi\| \le \delta_0, \delta_0 > 0$ and $\xi_0 = 0$, then we have

$$|\xi(\vartheta)| = \left|\xi(0) + g(0,\xi_0) - g(\vartheta,\xi(\vartheta)) + \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\vartheta}\frac{f(\nu,\xi(\alpha(\nu))}{(\vartheta-\nu)^{\frac{1}{2}}}d\nu + \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\vartheta}\frac{F\Big(\nu,\xi(\beta(\nu)),\xi(\theta(\nu)),(H\xi)(\nu)\Big)}{(\vartheta-\nu)^{\frac{1}{2}}}d\nu\right|$$
$$\le \frac{1}{4} + \frac{1+\delta_0}{4} + \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{1}{3}(1+\frac{\delta_0}{3})\right) + \frac{1}{\Gamma(\frac{3}{2})}\left(\frac{1}{5} + \frac{\delta_0}{4} + \frac{1}{3}(\frac{1}{4} + \frac{\delta_0}{3})\right) < \delta_0, \quad \vartheta \in J_a.$$

This shows $\delta_0 = 5.507$ is a solution of the above inequality. In view of Theorem 3.2, every problems (4.6)- (4.7) has at least one solution defined on $[0,1]$.

5 Conclusion and Perspective

The current study presents the existence of the solution to some fractional functional integro-differential equations, which is based on a more general form of the non-linear FIDEs and involves some other relevant works as well. In the proposed method, Petryshyn's fixed point theorem and the concept of MNC with more limited conditions were applied. The interested authors may examine and extend these results in different function spaces, such as Lebesgue, Hölder, Orlicz, or Sobolev spaces.

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