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Some inequalities for the generalized polar derivative of a polynomial

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Abstract

Recently Rather et al. [18] considered the generalized polar derivative and studied the relative position of zeros of the generalized polar derivative with respect to the zeros of the polynomial. In this paper, by taking into account the size of the constant term and the leading coefficient of the polynomial $P(z)$, we obtain some lower bound estimates for the generalized polar derivative of certain polynomials, which refine and generalize various results due to Aziz and Rather, Malik, Turán, Dubinin, Govil and others.

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1 Introduction

The problem concerning the extremal properties of polynomials attracted interests in the second half of 19^{th} century with some investigation of famous chemist Mendeleev who was interested to find the bound of the derivative of a special type of polynomial. It was Serge Bernstein, who formulated a result (for details see [20]) regarding the estimation of upper bound of the maximum modulus of the derived polynomial in terms of maximum modulus of the polynomial and proved that if $P(z)$ is polynomial of degree less than or equal to n, then

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

This excellent introduction to the topic of polynomial inequalities attracts many researchers to this field and motivates them to find refinements of the result for different types of polynomials. Let $\mathcal{P}_n$ denote the set of all polynomials of degree n over the field $\mathbb{C}$ of complex numbers, then concerning the estimation of the lower bound for the maximum modulus of derived polynomial in terms of maximum modulus of polynomial, Paul Turán [21] proved that if $P(z) \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.2}$$

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The inequality (1.2) is best possible and become equality for the polynomials having all its zeros on $|z| = 1$. Now concerning the estimation of the lower bounds of $Re\left(\frac{zP'(z)}{P(z)}\right)$ on $|z| = 1$, Dubinin [6] proved that

Theorem 1.1. If $P(z) \in P_n$ having all its zeros in $|z| \leq 1$, then for all z on $|z| = 1$, for which $P(z) \neq 0$,

$$Re\left(z\frac{P'(z)}{P(z)}\right) \geq \frac{n}{2}\left\{1 + \frac{1}{n}\left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\right\}. \tag{1.3}$$

As an application of this, Dubinin [6] obtained an interesting refinement of (1.2), by proving that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\max_{|z|=1}|P(z)|. \tag{1.4}$$

It was Malik [9] who extended the inequality (1.2) by proving that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k}\max_{|z|=1}|P(z)|. \tag{1.5}$$

Equality in (1.5) holds for the polynomial $P(z) = (z + k)^n$. Rather et al. [16] generalised Theorem 1.1 by proving the following result

Theorem 1.2. If $P(z) \in P_n$ having all its zeros in $|z| \leq k, k \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$

$$Re\left(z\frac{P'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right\}, \tag{1.6}$$

which in turn yield the following refinement of inequality (1.5) as well as generalization of inequality (1.4)

Theorem 1.3. If $P(z) \in P_n$ having all its zeros in $|z| \leq k, k \leq 1$, then for $|z| = 1$

$$|P'(z)| \geq \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right\}|P(z)|. \tag{1.7}$$

Let $D_\alpha[P](z)$ denote the polar differentiation (see [10]) of a polynomial $P(z)$ of degree n with respect to a complex number α, then

$$D_\alpha[P](z) = nP(z) + (\alpha - z)P'(z).$$

Note that the polynomial $D_\alpha[P](z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha[P](z)}{\alpha} = P'(z)$$

uniformly with respect z for $|z| \leq R, R > 0$. The Bernstein-type inequalities for the class of polynomials with ordinary derivative replaced by polar derivative have attracted number of mathematicians. In this direction, Aziz[2] was the first to establish inequalities concerning the polar derivative of a polynomial in terms of the modulus of the polynomial on the unit disk. As an extension of inequality (1.1) to the polar derivative, Aziz [2] proved the following result.

Theorem 1.4. If $P(z)$ is a polynomial of degree n, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, we have

$$|D_\alpha[P](z)| \leq n|\alpha z^{n-1}|\max_{|z|=1}|P(z)| \quad \text{for} \quad |z| \geq 1,$$

the result is best possible and equality in above inequality holds for $P(z) = cz^n$, $c \neq 0$.

Concerning the class of polynomials having all zeros in $|z| \leq k$, Aziz and Rather obtained several sharp results concerning the maximum modulus of $D_\alpha[P](z)$ on $|z| = 1$. Among other things, they [3] established the following extension of inequality (1.5) to the polar derivative of a polynomial.

Theorem 1.5. If all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \le k$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \ge k \le 1$.

$$\max_{|z|=1}|D_\alpha[P](z)| \ge \frac{n}{1+k}(|\alpha|-k)\max_{|z|=1}|P(z)|. \tag{1.8}$$

Rather et al. [16] refined this inequality by proving the following result

Theorem 1.6. If $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k, k \le 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$

$$\max_{|z|=1}|D_\alpha P(z)| \ge n\left(\frac{|\alpha|-k}{1+k}\right)\left\{1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}\max_{|z|=1}|P(z)|. \tag{1.9}$$

In literature, there exist several generalizations and refinements of these inequalities (For reference see [8],[13]-[11]). Given a polynomial $P \in \mathcal{P}_n$ of the form $P(z) = c(z-z_1)(z-z_2)(z-z_3)...(z-z_n)$ and $\mathbb{R}^n_+$ be the set of all n-tuples $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ of non-negative real numbers (not all zeros) with $\gamma_1 + \gamma_2 + \cdots + \gamma_n = \wedge$, Sz-Nagy [19] introduced a generalised derivative of P(z) defined by

$$P^\gamma(z) := P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j} = \sum_{j=1}^{n}\gamma_j P_k(z), \tag{1.10}$$

where $P_j(z) = c\prod_{\substack{i=1\\ i\ne j}}^{n}(z-z_i)$ for $1 \le j \le n$. Noting that for $\gamma = (1,1,1,\dots,1)$, $P^\gamma(z) = P'(z)$, which is the reason we call it generalized derivative of polynomial $P(z)$. Recently, F. A. Bhat et al.[5], extended inequality (1.5) to Nagy derivatives by proving that

Theorem 1.7. if $P(z) \in P_n$ having all its zeros in $|z| \le k \le 1$ then

$$\max_{|z|=1}|P^\gamma(z)| \ge \frac{\wedge}{(1+k)}\max_{|z|=1}|P(z)|. \tag{1.11}$$

Next we define generalized polar derivative of $P(z)$ as

$$D^\gamma_\alpha[P](z) := \wedge P(z) + (\alpha - z)P^\gamma(z),$$

where $\wedge = \sum_{j=1}^{n}\gamma_j$. Noting that for $\gamma = (1,1,1,\dots,1)$, $D^\gamma_\alpha[P](z) = D_\alpha P(z)$. N. A. Rather et al. [12], extended Theorem 1.7 to the generalised polar derivatives and proved that

Theorem 1.8. If all the zeros of a polynomial $P(z) \in \mathcal{P}_n$ lie in $|z| \le k$ where $k \le 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$,

$$\max_{|z|=1}|D^\gamma_\alpha[P](z)| \ge \frac{\wedge}{1+k}(|\alpha|-k)\max_{|z|=1}|P(z)|. \tag{1.12}$$

2 Main Results

In this section prove various results which extend Theorem 1.2, Theorem 1.3, to Nagys derivatives. Besides this we also prove refinements of inequality (1.11) and inequality (1.12). We begin by proving the following extension of Theorem 1.2 to Nagy derivative

Theorem 2.1. If $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a polynomial of degree n having all zeros in a disc $|z| \le k, k \le 1$, then for all z on $|z| = 1$ for which $p(z) \ne 0$,

$$Re\left(\frac{zP^\gamma(z)}{P(z)}\right) \ge \frac{k}{1+k}\left\{\frac{\wedge}{k} + \gamma_m\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}, \tag{2.1}$$

where $\gamma_m = \min\{\gamma_1, \gamma_2, ..., \gamma_n\}$.

Remark 2.2. For $\gamma = (1,1,\cdots,1)$, the above theorem reduces to Theorem 1.2, which for $k = 1$ yields Theorem 1.1.

Now concerning the estimation of the lower bounds of $|P^{\gamma}(z)|$ on $|z|=1$, for the polynomials having all its zeros in the disc $|z|\leq k,\ k\leq 1$, we prove the following extension of Theorem 1.3 to generalised Nagy derivative, which is a refinement of Theorem 1.7 as well.

Theorem 2.3. If $P(z)\in P_n$ has all its zeros in $|z|\leq k, k\leq 1$, then for $|z|=1$

$$|P^{\gamma}(z)|\geq \frac{k}{1+k}\left\{\frac{\wedge}{k}+\gamma_m\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}|P(z)|, \tag{2.2}$$

where $\gamma_m=\min\{\gamma_1,\gamma_2,...,\gamma_n\}$ and the result is best possible as shown by the polynomial $P(z)=(z+k)^n$.

Remark 2.4. If we take $\gamma=(1,1,\cdots,1)$ the above theorem reduces to Theorem 1.3.

Next we extend Theorem 1.6 to the class of generalized polar derivatives of the polynomial by proving the following result:

Theorem 2.5. If all the zeros of a polynomial $P(z)\in\mathcal{P}_n$ lie in $|z|\leq k$ where $k\leq 1$, then for $\alpha\in\mathbb{C}$ with $|\alpha|\geq k$,

$$\underset{|z|=1}{max}|D_\alpha^{\gamma}[P](z)|\geq\frac{|\alpha|-k}{1+k}\left\{\wedge+k\gamma_m\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}\underset{|z|=1}{max}|P(z)|, \tag{2.3}$$

where $\gamma_m=\min\{\gamma_1,\gamma_2,...,\gamma_n\}$ and the result is best possible as shown by the polynomial $P(z)=(z+k)^n$.

Remark 2.6. It is easy to verify that Theorem 2.5 is refinement of Theorem 1.8.

Remark 2.7. For the n-tuple $\gamma=(1,1,1,...,1)$, the inequality (2.3) reduces to the inequality (1.9).

Remark 2.8. If we divide both sides of inequality (2.3) by $|\alpha|$ and let $|\alpha|\to\infty$, inequality (2.3) reduces to inequality (2.2).

For $k=1$ Theorem 2.5 reduces to the following result.

Corollary 2.9. If all the zeros of a polynomial $P(z)\in\mathcal{P}_n$ lie in $|z|\leq 1$, then for $\alpha\in\mathbb{C}$ with $|\alpha|\geq 1$,

$$\underset{|z|=1}{max}|D_\alpha^{\gamma}[P](z)|\geq\frac{|\alpha|-1}{2}\left\{\wedge+\gamma_m\left(\frac{|a_n|-|a_0|}{|a_n|+|a_0|}\right)\right\}\underset{|z|=1}{max}|P(z)|, \tag{2.4}$$

where $\gamma_m=\min\{\gamma_1,\gamma_2,...,\gamma_n\}$ and the result is best possible as shown by the polynomial $P(z)=(z+1)^n$.

Remark 2.10. If we divide both sides of inequality (2.4) by $|\alpha|$ and let $|\alpha|\to\infty$, and for n-tuple $\gamma=(1,1,1,...,1)$, the inequality results to inequality (1.4).

3 Lemmas

For the proof of theorems we need following lemmas, the first lemma is due to Rather et al., [16].

Lemma 3.1. If $0\leq x_j\leq 1, \quad j=1,2,\vdots,n$, then

$$\sum_{j=1}^{n}\frac{1-x_j}{1+x_j}\geq\frac{1-\prod_{j=1}^{n}x_j}{1+\prod_{j=1}^{n}x_j}\quad \forall n\in N. \tag{3.1}$$

Next two lemmas are due to Rather et al., [12].

Lemma 3.2. If $P(z)$ is a polynomial of degree n and $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$, then $for \quad |z|=1$

$$|Q^{\gamma}(z)|=|\wedge P(z)-zP^{\gamma}(z)| \quad\text{and}\quad |P^{\gamma}(z)|=|\wedge Q(z)-zQ^{\gamma}(z)|.$$

Lemma 3.3. If $P(z)$ is a polynomial of degree n having all zeros in $|z|\leq k$ where $k\leq 1$, then

$$k|P^{\gamma}(z)|\geq|Q^{\gamma}(z)| \quad\text{for}\quad |z|=1,$$

where $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$.

4 Proof of the Theorems

Proof .[**Proof of Theorem 2.1**] Since $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ where $|z_j| \le k \le 1$.
By definition

$$P^{\gamma}(z) = P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z - z_j},$$

so that for the points $e^{i\theta}$, $0 \le \theta < 2\pi$ with $p(e^{i\theta}) \ne 0$, we have

$$\left(e^{i\theta}\frac{P^{\gamma}(e^{i\theta})}{P(e^{i\theta})}\right) = \sum_{j=1}^{n}\left(\frac{\gamma_j e^{i\theta}}{e^{i\theta} - k_j e^{i\theta_j}}\right), \qquad z_j = k_j e^{i\theta_j}, k \le 1.$$

Therefore, for the points $e^{i\theta}$, $0 \le \theta < 2\pi$ with $p(e^{i\theta}) \ne 0$, we have

$$\begin{aligned} Re\left(e^{i\theta}\frac{P^{\gamma}(e^{i\theta})}{P(e^{i\theta})}\right) &= \sum_{j=1}^{n} Re\left(\frac{\gamma_j e^{i\theta}}{e^{i\theta} - k_j e^{i\theta_j}}\right) \\ &\ge \sum_{j=1}^{n}\frac{\gamma_j}{1 + |z_j|}. \end{aligned}$$

That is, for all z on $|z| = 1$ for which $P(z) \ne 0$, we have

$$\begin{aligned} Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) &\ge \sum_{j=1}^{n}\frac{\gamma_j}{1+k} + \sum_{j=1}^{n}\gamma_j\left(\frac{k - |z_j|}{(1 + |z_j|)(1 + k}\right) \\ &\ge \frac{\wedge}{(1+k)} + \frac{k\gamma_m}{1+k}\sum_{j=1}^{n}\frac{k - |z_j|}{k + k\,|z_j|}, \end{aligned}$$

where $\gamma_m = \min\{\gamma_1, \gamma_2, ..., \gamma_n\}$. Since $k \le 1$, implies $k + k\,|z_j| \le k + |z_j|$. Hence, for all z on $|z| = 1$ for which $P(z) \ne 0$, we have

$$Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) \ge \frac{\wedge}{(1+k)} + \frac{\gamma_m k}{1+k}\sum_{j=1}^{n}\frac{k - |z_j|}{k + |z_j|} = \frac{\wedge}{(1+k)} + \frac{\gamma_m k}{1+k}\sum_{j=1}^{n}\frac{1 - \frac{|z_j|}{k}}{1 + \frac{|z_j|}{k}}.$$

Using lemma 3.1, we have for all z on $|z| = 1$, for which $P(z) \ne 0$,

$$\begin{aligned} Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) &\ge \frac{\wedge}{(1+k)} + \frac{\gamma_m k}{1+k}\left[\frac{1 - \prod_{j=1}^{n}\frac{|z_j|}{k}}{1 + \prod_{j=1}^{n}\frac{|z_j|}{k}}\right] \\ &= \frac{\wedge}{(1+k)} + \frac{\gamma_m k}{(1+k)}\left[\frac{1 - \frac{|a_0|}{k^n |a_n|}}{1 + \frac{|a_0|}{k^n |a_n|}}\right] \\ &= \frac{\wedge}{(1+k)} + \frac{\gamma_m k}{(1+k)}\left[\frac{k^n\,|a_n| - |a_0|}{k^n\,|a_n| + |a_0|}\right]. \end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof .[**Proof of Theorem 2.3**] For points z on $|z| = 1$ with $P(z) \ne 0$, we have

$$\left|\frac{zP^{\gamma}(z)}{P(z)}\right| \ge Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right). \tag{4.1}$$

Since $P(z) \in P_n$ has all its zeros in $|z| \le k \le 1$, therefore in view of inequalities 2.1 and 4.1, we have for the points z on $|z| = 1$ for which $P(z) \ne 0$,

$$\left|\frac{zP^{\gamma}(z)}{P(z)}\right| \ge Re\left(\frac{zP^{\gamma}(z)}{P(z)}\right) \ge \frac{k}{1+k}\left\{\frac{\wedge}{k} + \gamma_m\left(\frac{k^n\,|a_n| - |a_0|}{k^n\,|a_n| + |a_0|}\right)\right\}.$$

That is,

$$|P^{\gamma}(z)| \geq \frac{k}{1+k}\left\{\frac{\wedge}{k}+\gamma_m\left(\frac{k^n\,|a_n|-|a_0|}{k^n\,|a_n|+|a_0|}\right)\right\}|P(z)| \qquad \text{for} \quad |z|=1. \tag{4.2}$$

This completes the proof of Theorem 2.3. □

Proof .[Proof of Theorem 2.5] By using triangle's inequality, we get

$$\begin{aligned}|D_{\alpha}^{\gamma}[P](z)| &= |\wedge P(z)+(\alpha-z)P^{\gamma}(z)|\\ &= |\wedge P(z)+\alpha P^{\gamma}(z)-zP^{\gamma}(z)|\\ &\geq |\alpha||P^{\gamma}(z)|-|\wedge P(z)-zP^{\gamma}(z)|.\end{aligned}$$

On using lemma (3.2) and lemma (3.3) this implies, for $|z|=1$

$$\begin{aligned}|D_{\alpha}^{\gamma}[P](z)| &\geq |\alpha||P^{\gamma}(z)|-|Q^{\gamma}(z)|\\ &\geq |\alpha||P^{\gamma}(z)|-k|P^{\gamma}(z)|\\ &= (|\alpha|-k)|P^{\gamma}(z)|.\end{aligned}$$

This in view of inequality (4.2) yields

$$\max_{|z|=1}|D_{\alpha}^{\gamma}[P](z)| \geq \frac{|\alpha|-k}{1+k}\left\{\wedge+k\gamma_m\left(\frac{k^n\,|a_n|-|a_0|}{k^n\,|a_n|+|a_0|}\right)\right\}\max_{|z|=1}|P(z)|.$$

That completes the proof of Theorem 2.5. □

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