Int. J. Nonlinear Anal. Appl. 16 (2025) 6, 107–112
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.30132.4343

A generalization of Ekeland's variational principle by using the τ-distance and its application in equilibrium problem

Mahmood Ghobadi, Ali Farajzadeh*

Department of Mathematics, Razi University, Kermanshah, 67149, Iran

(Communicated by Mohammad Bagher Ghaemi)

Abstract

In this paper, a new version of Ekeland's variational principle by using the concept of τ-distance for bounded from below functions which are not necessarily lower semicontinuous is provided. This new version of Ekeland's variational principle will be applied to establish an existence theorem for a solution to the equilibrium problem in the setting of complete metric spaces.

Keywords: Lower semicontinuous Regularization, Ekeland's variational principle, Bounded from below, Equilibrium problem, τ-distance.
2020 MSC: 37C25

1 Introduction

To the best of our knowledge, the first appearance of equilibrium problems as we understand them now is due to Muu and Oettli [11] and it was further developed by Blum and Oettli [3]. They are conceptually connected to Ky Fan's minimax inequality [7] Which goes back to the equality result of von Neumann [16].

Ekeland's variational principle was first expressed by Ekeland [6] and developed by many authors and researchers [2, 4, 5, 8, 13]. Tataru in [14] defined the concept of Tataru's distance and using it proved the generalization of Ekeland's variational principle. Afterwards, in 1996, Kada in [10] stated the concept of w-distance and extended Ekeland's variational principle. The concept of τ-distance which is a generalization of w-distance and Tataru's distance was first introduced by Suzuki. He also improved the concept of Ekeland's variational principle ([12]).

The purpose of this paper is to study equilibrium problems to get some existing results. In fact, we recall the concept of τ-distance on a complete metric space and then a new version of Ekeland's variational principle by using the concept of τ-distance is proved and, by applying it, a version of the existence result of a solution for the equilibrium problem in compact domains in the setting of complete metric spaces are investigated. Finally, some examples to illustrate the results of this note are given. The results of this paper improve the corresponding results given in [2, 4, 5, 8].

*Corresponding author
Email addresses: ghobadimahmood83@gmail.com (Mahmood Ghobadi), farajzadehali@gmail.com (Ali Farajzadeh)

Received: March 2023 *Accepted:* January 2024

2 Preliminaries

In this section we introduce and remind tools that will be useful throughout the paper and we will use standard notations and terminology from real analysis.

By an equilibrium problem (EP)([15]) we understand the problem of finding

$$\bar{x} \in C \quad \text{such that} \quad f(\bar{x}, y) \geq 0 \quad \forall y \in C,$$

where C is a set and $f : C \times C \to \mathbb{R}$ is a bifunction.

Definition 2.1. Given a nonempty subset C of a topological space X, a function $h : C \to \mathbb{R}$ is said to be:

- **lower semicontinuous** if for each $x \in C$ and each $\lambda \in \mathbb{R}$ such that $h(x) > \lambda$, there exists a neighborhood V_x of x such that $h(y) > \lambda$, for all $y \in V_x \cap C$ or equivalently for any net $\{x_\alpha\} \subset C$, $x_\alpha \to x$ implies
$$h(x) \leq \liminf_{\alpha} h(x_\alpha).$$
- **upper semicontinuous** if for each $x \in C$, $-h$ is lower semicontinuous or equivalently for any net $\{x_\alpha\} \subset C$, $x_\alpha \to x$ implies
$$h(x) \geq \limsup_{\alpha} h(x_\alpha).$$

Also the epigraph of the function h is denoted by $\text{Epi}(h)$ and is defined by $\text{Epi}(h) := \{(x, \lambda) \in C \times \mathbb{R} : h(x) \leq \lambda\}$. The lower sub-level set at level λ of h is defined by $S_h(\lambda) := \{x \in C : h(x) \leq \lambda\}$. It is well-known that a function h is lower semicontinuous if and only if $\text{Epi}(h)$ is closed in $C \times \mathbb{R}$ or equivalently, if and only if $S_h(\lambda)$ is closed in C, for all $\lambda \in \mathbb{R}$.

Given a nonempty subset C of a topological space X, it is a basic fact from real analysis that every function $h : C \to \mathbb{R}$ (not necessarily lower semicontinuous) admits a lower semicontinuous regularization $\overline{h} : C \to \mathbb{R} \cup \{-\infty\}$ defined by $\text{Epi}(\overline{h}) := \overline{\text{Epi}(h)}$, the closure in $C \times \mathbb{R}$, equivalently by

$$\overline{h}(x) = \liminf_{y \to x} h(y) = \sup_{U} \inf_{y \in U \cap C} h(y),$$

where U runs all neighborhoods of x. It is well-known that for any $x \in C$ and any $\lambda \in \mathbb{R}$

1. $\overline{h}(x) = \inf\{\lambda \in \mathbb{R} : x \in \overline{S_h(\lambda)}\}$;
2. $\overline{h}(x) \leq h(x)$.

Example 2.2. Let $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) := \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases},$$

It is clear that h is not lower semicontinuous and its lower semicontinuous regularization is $\overline{h}(x) = x^2$.

Example 2.3. If $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(x) := [x]$, then it is easy to check that

$$\overline{h}(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{Z} \\ [x] & \text{if } x \notin \mathbb{Z} \end{cases}$$

Definition 2.4. ([8]) Let (X,d) be a metric space. Then a function $p : X \times X \to \mathbb{R}$ is called a τ-distance on X if

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for all $x, y, z \in X$, moreover, there exists a function $\eta : X \times \mathbb{R}^+ \to \mathbb{R}^+$ which is concave and continuous in its second argument and satisfying the following conditions:

(2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t \quad \forall x \in X, t \in \mathbb{R}^+$;

(3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply

$$p(w, x) \leq \liminf_{n} p(w, x_n) \quad \forall w \in X;$$

(4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

(5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Note that if d is a meter then it is τ-distance, by taking $p = d$ and $\eta(x,t) = t$.

Example 2.5. ([8]) Let (X,d) be a metric space. Then a function $p : X \times X \to [0,\infty)$ defined by $p(x,y) = |y|$, for all $x, y \in X$ is a τ-distance on X.

3 The Ekeland Variational Principle for equilibrium problems

In this section a new version of Ekeland's variational principle by using the concepts of τ-distance and relaxing the lower semicontinuity condition of the function is given. This new version of Ekeland's variational principle will be applied to establish an existence theorem for a solution of the equilibrium problem in the setting of complete metric spaces.

Theorem 3.1. (The Ekeland variational principle)([6]) Let C be a nonempty closed subset of a complete metric space (X,d), and $h : C \to \mathbb{R}$ be a lower semicontinuous function bounded from below. For every $\varepsilon > 0$, and for any $x_0 \in C$, there exists $\hat{x} \in C$ such that

$$h(\hat{x}) + \varepsilon d(x_0, \hat{x}) \leq h(x_0), \quad h(x) + \varepsilon d(x, \hat{x}) > h(\hat{x}),$$

for all $x \in C \setminus \{\hat{x}\}$.

The following theorem is a generalization of Ekeland's variational principle (Theorem 3.1) by replacing a meter by a τ-distance.

Theorem 3.2. ([12]) Let C be a nonempty closed subset of a complete metric space (X,d), and $h : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a τ-distance p on X. Then, for every $\varepsilon > 0$, and for any $u \in C$ with $p(u,u) = 0$, there exists $v \in C$ such that

$$h(v) + \varepsilon p(u,v) \leq h(u), \quad h(w) + \varepsilon p(v,w) > h(v),$$

for all $w \in C \setminus \{v\}$.

Proof . Let $C = \{x \in X : h(x) \leq h(u) - \varepsilon p(u,x)\}$. Then, it is obvious that C is nonempty and closed. Suppose that for every $x \in C$, there exists $w \in X$ such that $w \neq x$ and $h(w) \leq h(x) - \varepsilon p(x,w)$. Then

$$\begin{aligned} \varepsilon p(u,w) &\leq \varepsilon p(u,x) + \varepsilon p(x,w) \\ &\leq h(u) - h(x) + h(x) - h(w) \\ &= h(u) - h(w) \end{aligned}$$

and hence $w \in C$. By Theorem 5 in [12], there exists $x_0 \in C$ such that

$$h(x_0) = \inf_{z \in C} h(z).$$

This is a contradiction, for there exists $w_0 \in C$ with $h(w_0) < h(x_0)$. □

The next result is a new version of theorem 3.2 by relaxing the lower semicontinuity condition of the function h, and also it is and extension of the corresponding theorem given in [5] for τ-distance functions with a simpler proof than as given in [5].

Theorem 3.3. Let C be a nonempty closed subset of a complete metric space (X,d), and $h : C \to \mathbb{R}$ be a function bounded from below. Assume that there exists a τ-distance p on X. For every $\varepsilon > 0$, and for any $x_0 \in C$ with $p(x_0, x_0) = 0$, there exists $\hat{x} \in C$ such that

$$\overline{h}(\hat{x}) + \varepsilon p(x_0, \hat{x}) \leq h(x_0), \quad h(x) + \varepsilon p(x, \hat{x}) > \overline{h}(\hat{x}),$$

for all $x \in C \setminus \{\hat{x}\}$.

Proof . We consider the lower semicontinuous regularization $\overline{h}(x)$ which is defined as follows,

$$\overline{h}(x) = \liminf_{y \to x} h(y) = \sup_{U} \inf_{y \in U \cap C} h(y),$$

where U runs over all neighborhoods of x. Since h is bounded below, there exists $\alpha \in \mathbb{R}$ such that $h(y) \geq \alpha$, for all $y \in U \cap C$. Thus

$$\inf_{y \in U \cap C} h(y) \geq \alpha,$$

this implies $\overline{h}(x) \geq \alpha$. Therefore $\overline{h}$ is bounded below and it satisfies all the hypotheses of Theorem 3.2. Using Theorem 3.2 we have $\overline{h}(\hat{x}) + \varepsilon p(x_0, \hat{x}) \leq \overline{h}(x_0)$ and $\overline{h}(x) + \varepsilon p(x, \hat{x}) > \overline{h}(\hat{x})$, for all $x \in C \setminus \{\hat{x}\}$. Since $\overline{h}(x) \leq h(x)$, for all $x \in C$, we have $\overline{h}(\hat{x}) + \varepsilon p(x_0, \hat{x}) \leq h(x_0)$ and $h(x) + \varepsilon p(x, \hat{x}) > \overline{h}(\hat{x})$, for all $x \in C \setminus \{\hat{x}\}$. This completes the proof. □

The following example illustrates Theorem 3.3.

Example 3.4. Let $X = \mathbb{R}$, $C = [0, \infty)$ and d be the Euclidean metric on X. The function $h : C \to \mathbb{R}$ is defined by $h(x) := [x]$, for all $x \in C$ (Example 2.3) is not lower semicontinuous. Obviously, h satisfies all conditions of Theorem 3.3 and so if we take $x_0 = 1$ and $0 < \varepsilon \leq 1$ then $\hat{x} = 0$ is a candidate which fulfills in the conclusion of Theorem 3.3.

The next theorem is a direct consequence of Theorem 3.3 for bifunctions. Moreover, it provides sufficient conditions for non-emptiness of the solution set of (EP).

Theorem 3.5. Let C be a nonempty closed subset of a complete metric space (X,d), and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that there exists a τ-distance p on X. Assume that the following conditions hold

(i) f is bounded from below and lower semicontinuous with respect to its second argument;

(ii) $f(x, x) = 0$, for all $x \in C$;

(iii) f satisfies the triangle inequality property

$$f(x, y) \leq f(x, z) + f(z, y) \quad \forall x, y, z \in C.$$

Then, for all $\varepsilon > 0$ and all $x_0 \in C$ with $p(x_0, x_0) = 0$, there exists $\hat{x} \in C$ such that

$$f(x_0, \hat{x}) + \varepsilon p(x_0, \hat{x}) \leq 0, \quad f(\hat{x}, x) + \varepsilon p(x, \hat{x}) > 0,$$

for all $x \in C \setminus \{\hat{x}\}$. Furthermore if C be a compact set, and f and p be upper semicontinuous with respect to its first argument, then the solution set of (EP) is nonempty and compact.

Proof . Define $h(x) = f(x_0, x)$, for all $x \in C$. It follows from Theorem 3.3 that there exist lower semicontinuous regularization $\overline{h}$ and $\hat{x} \in C$ such that

$$\overline{h}(\hat{x}) + \varepsilon p(x_0, \hat{x}) \leq h(x_0), \quad h(x) + \varepsilon p(x, \hat{x}) > \overline{h}(\hat{x}),$$

for all $x \in C \setminus \{\hat{x}\}$. Thus, it follows from the definition of $\overline{h}(x)$, (i) and (ii) that

$$\begin{aligned} f(x_0, \hat{x}) + \varepsilon p(x_0, \hat{x}) &\leq \liminf_{y \to \hat{x}} f(x_0, y) + \varepsilon p(x_0, \hat{x}) \\ &= \liminf_{y \to \hat{x}} h(y) + \varepsilon p(x_0, \hat{x}) \\ &= \overline{h}(\hat{x}) + \varepsilon p(x_0, \hat{x}) \\ &\leq h(x_0) = f(x_0, x_0) = 0, \end{aligned}$$

and also, it follows from the definition of $\overline{h}(x)$, (i) and (iii) that

$$\begin{aligned} f(x_0, \hat{x}) + f(\hat{x}, x) + \varepsilon p(x, \hat{x}) &\geq f(x_0, x) + \varepsilon p(x, \hat{x}) \\ &= h(x) + \varepsilon p(x, \hat{x}) \\ &> \overline{h}(\hat{x}) \\ &= \liminf_{y \to \hat{x}} h(y) = \liminf_{y \to \hat{x}} f(x_0, y) \\ &\geq f(x_0, \hat{x}), \end{aligned}$$

for all $x \in C \setminus \{\hat{x}\}$. This completes the proof of the first part of the assertion. Since $f(\hat{x}, x) + \varepsilon p(x, \hat{x}) > 0$, for all $x \in C \setminus \{\hat{x}\}$, then there exists $\{\hat{x}_n\} \in C$ such that

$$f(\hat{x}_n, x) + \frac{1}{n} p(\hat{x}_n, x) > 0$$

The compactness of C implies existence of a subsequence $\{\hat{x}_{n_k}\}$ of $\{\hat{x}_n\}$ such that $\hat{x}_{n_k} \to \hat{x}$ as $n \to \infty$. Hence, $f(\hat{x}_{n_k}, x) + \frac{1}{n_k} p(\hat{x}_{n_k}, x) > 0$. Then,

$$\limsup_{k\to\infty} f(\hat{x}_{n_k}, x) + \limsup_{k\to\infty} \frac{1}{n_k} p(\hat{x}_{n_k}, x) \geq 0.$$

It follows from the upper semicontinuity f and p in the first argument that

$$f(\hat{x}, x) \geq \limsup_{k\to\infty} f(\hat{x}_{n_k}, x) > 0, \quad \forall x \in C.$$

This means that $\hat{x}$ is a solution of (EP). The compactness of the solution set (EP) follows from the upper semicontinuity f in the first argument. □

Example 3.6. Let $X = \mathbb{R}$, $C = [-1, 1]$, d the Euclidean metric on X and $f : C \times C \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} x^2 + x + y & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

It is easy to verify the solution set (EP) equals to $[\frac{\sqrt{5}-1}{2}, 1]$.

Example 3.7. Let $X = C = \mathbb{R}$ and d be the Euclidean metric on X. Consider the function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = |y|$, for all $x, y \in X$. The function p is a τ-distance. Define the function $h : C \to \mathbb{R}$ by

$$h(x) := \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

for all $x \in C$ (Example 2.2). Obviously, h satisfies all conditions of Theorem 3.3 and so if we take $x_0 = 0$ then $\hat{x} = 0$ is a candidate which fullfils in the conclusion of Theorem 3.3.

Example 3.8. ([8]) Let $X = \mathbb{R}$, $C = [0, \infty)$ and d be the Euclidean metric on X. Consider the function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = |y|$, for all $x, y \in X$. The function p is a τ-distance (see Example 2.5). Define the function $f : C \times C \to \mathbb{R}$ by $f(x, y) = \sin(y - x)$, for all $x, y \in C$. Obviously, f satisfies all conditions of Theorem 3.5 and so if we take $x_0 = 0$ then $\hat{x} = 0$ is a candidate which fullfils in the conclusion of Theorem 3.5.

Let C be a nonempty subset of a topological space X, and $f, g : C \times C \to \mathbb{R}$ be two bifunctions. the following inequality holds: $f \geq g$, that means $f(x, y) \geq g(x, y)$, for all $x, y \in C$. As a direct consequence of Theorem 3.5 we have the following Theorem.

Theorem 3.9. Let C be a nonempty closed subset of a complete metric space (X,d), and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that there exists a τ-distance p on X. Assume that there exists a bifunction $g : C \times C \to \mathbb{R}$ such that

(i) $f \geq g$;

(ii) g is bounded from below and lower semicontinuous with respect to its second argument;

(iii) $g(x, x) = 0$, for all $x \in C$;

(iv) g satisfies the triangle inequality property $g(x, y) \leq g(x, z) + g(z, y)$, for all $x, y, z \in C$.

Then, for all $\varepsilon > 0$ and all $x_0 \in C$ with $p(x_0, x_0) = 0$, there exists $\hat{x} \in C$ such that

$$g(x_0, \hat{x}) + \varepsilon p(x_0, \hat{x}) \leq 0, \quad f(\hat{x}, x) + \varepsilon p(x, \hat{x}) > 0,$$

for all $x \in C \setminus \{\hat{x}\}$. The conclusion of Theorem 3.9 is similar to the one in (Corollary 3.1 in [5]), where we replacing a meter by a τ-distance.

4 Conclusion

A new form of Ekeland's variational principle for functions which are not necessarily lower semicontinuous in the setting of τ-distance spaces. By using this result Ekeland's variational principle is extended for bifunctions which satisfy the triangle inequality property. Then, this form of Ekeland's variational principle is applied for proving and existence result of the solution for the equilibrium problem. This result is an improvement of Theorem 3.2 in [8] and similar results for equilibrium versions of Ekeland's variational principle were obtained by Alfuraidan and Khamsi (Theorem 2.1 in [1]) and Feng, Xie and Wu (Theorem 3, Theorem 4 and Theorem 5 and Corollary 1 in [9]) and also Bianchi, Kassay and Pini (Theorem 2.1 by replacing a norm by a meter in [2]).

References

[1] M.R. Alfuraidan and M.A. Khamsi, *Graphical Ekeland's principle for equilibrium problems,* Proc. Amer. Math. Soc. Ser B. **9** (2022), 33–40

[2] M. Bianchi, G. Kassay, and R. Pini, *Existence of equilibria via Ekeland's principle,* J. Math. Anal. **305** (2005), 502–512.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Student **63** (1994), 123–145.

[4] M. Castellani and G. Massimiliano, *Ekeland's principle for cyclically antimonotone equilibrium problems,* Nonlinear Anal.: Real World Appl. **32** (2016), 213–228.

[5] J. Cotrina, M. Thera, and J. Zuniga, *An existence result for quasi-equilibrium problems via Ekeland's variational principle,* J. Optim. Theory Appl. **187** (2020), 336–355.

[6] I. Ekeland, *On the variational principle,* J. Math. Anal. Appl. **47** (1974), 324–353.

[7] K. Fan, *A minimax inequality and applications,* Inequalities III (Proc. Third Sympos., Univ. California, Los Angeles, Calif, 1969; dedicated to the memory of Theodore S. Motzkin), 1972, pp. 103–113.

[8] A.P. Farajzadeh, S. Plubtieng, and A. Hoseinpour, *A generalization of Ekeland variational principle by using the τ-distance with its applications,* J. Inequal. Appl. **2017** (2017), 181.

[9] Y. Feng, J. Xie, and B. Wu, *A new equilibrium version of Ekeland variational principle and its applications,* Axioms **11** (2022), no. 2, 68.

[10] O. Kada, T. Suzuki, and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces,* Math. Jpn. **44** (1996), 381–391.

[11] L. Muu and W. Oettli, *Convergence of an adaptive penalty scheme for finding constrained equilibria,* Nonlinear Anal. Theory Meth. Appl. **18** (1992), 1159–1166.

[12] T. Suzuki, *Generalized distance and existence theorems in complete metric spaces,* J. Math. Anal. Appl. **253** (2001), 440–458.

[13] W. Takahashi, *Nonlinear Functional Analysis,* Yokohama Publishers, 2002.

[14] D. Tataru, *Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms,* J. Math. Anal. Appl. **163** (1992), 345–392.

[15] M. Thera, *Old and new results on equilibrium and quasi-equilibrium problems,* Variation. Anal. Optim. Webinars, 2020, 1–46.

[16] J. von Neumann, *Zur theorie der gesellschaftsspiele,* Math. Ann. **100** (1928), 295–320.