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The operational matrices of two dimensional Bernstein polynomials for solving the hyperbolic partial differential equation with boundary conditions

Mohsen Mohamadi*, Amir Shahmari, Hassan Eshaghi Kenari

Department of Mathematics, Ayatollah Amoli Branch, Islamic Azad University, Amol, Iran

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Abstract

The wave equations are one of the most important equations in engineering and physics, which are usually formulated as hyperbolic partial differential equations with special boundary conditions. In this paper, a numerical method for solving these equations based on Bernstein polynomials is introduced. The properties of Bernstein polynomial operational matrices turn this differential equation and its boundary conditions into a system of algebraic equations. Some numerical examples illustrate the accuracy, validity, and applicability of the new technique.

Keywords: Bernstein polynomial, two dimensions Bernstein polynomial, Best approximation, Operational matrices, Kronecker products, Hyperbolic Partial differential equation
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1 Introduction

Second-order hyperbolic equations are one of the most widely used equations in mathematics, physics, and engineering. Some wave equations are modelled in the form of second-order hyperbolic equations. The application of these partial differential equations has caused their solution to be taken into consideration and different methods to solve them have been proposed, see [16, 22, 34, 39]. Methods to solve this problem which strongly depends on the boundary conditions of the problem. Hyperbolic partial differential equations with local and non-local boundary conditions are applied in the mathematical modelling of many physical and engineering problems. Therefore, recently, researchers have conducted many studies on the condition of the existence of the solution, the uniqueness of the solution, development literature, analysis, and implementation of exact methods for the numerical solution of time-dependent partial differential equations, see [17, 23, 30, 42, 43]. Consider the following second-order hyperbolic equation called the wave equation with the given boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = k(x,t) \qquad a < x < b \qquad 0 < t < T \tag{1.1}$$

with the initial conditions

$$u(x,0) = f(x) \qquad a < x < b \tag{1.2}$$

*Corresponding author
Email addresses: m.mohamadi2000@gmail.com or mohsen.mohamadi@iau.ac.ir (Mohsen Mohamadi), a.shahmari@gmail.com (Amir Shahmari), hkenari@yahoo.com (Hassan Eshaghi Kenari)

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$$u_t(x,0) = g(x) \qquad a < x < b \tag{1.3}$$

and local boundary conditions

$$u(a,t) = h(t) \qquad 0 < t < T \tag{1.4}$$

$$u(b,t) = r(t) \qquad 0 < t < T \tag{1.5}$$

where $k(x,t) \in l^2([a,b] \times [a,b])$ and $f(x), g(x), h(t), r(t) \in l^2([a,b])$ are known functions and $u(x,t)$ is an unknown function defined on $([a,b] \times [a,b])$ which be determined. The continuity of these functions is determined by the conditions of the problem. If these functions are piecewise continuous, numerical methods for solving these equations can be obtained by using multi-scaling functions. In this paper without reducing the generality of the problem let $[a,b] = [0,1]$, $T = 1$ and $c = 1$.

In recent decades, the development of numerical techniques for solving such problems has been an important research topic in many fields of applied science and engineering. There are various methods have been proposed by researchers in this regard. Such as finite difference methods ([10, 19]), the Galerkin techniques ([7, 9, 12, 38, 44]), finite element methods ([1, 16, 26, 40]), spectral techniques ([15, 47]), and so on (see [2, 4, 11, 13, 21, 24, 36] and their references). This paper is divided into the following sections. The second section includes a brief introduction to Bernstein functions and the function's approximation to them. The numerical schemes for the solution of (1.1) with the mentioned conditions (1.2) to (1.5) are also described in Section 3. Section 4 is dedicated to operational metrics and how to calculate them. The results of numerical experiments are given in Section 5. In Section 6, the method is implemented for non-local boundary conditions and an illustrative example confirms accuracy of the method . The last section consists of a brief conclusion.

2 Preliminaries

Bernstein polynomials are one of the most famous and important polynomials. These polynomials were defined by Sergei Natanovich Bernstein and used in the constructive proof of the Weierstrass approximation theorem [5]. This polynomial found many applications in other sciences and trends. Computer-aided design, scientific computing, probability distribution, interpolation, and approximation are a few examples of the use of these polynomials. These polynomials define as follows:

Definition 2.1. Let $m$be a positive integer number, ith Bernstein polynomial with degree m on the interval $[a,b]$ is defined as follows:

$$B_{i,m} = \binom{m}{i} \frac{(x-a)^i \, (b-x)^{m-i}}{(b-a)^m}$$

where $0 \le i \le m$ and $B_{i,m}(x) = 0$ for $i > m$.

It is obvious there are $m+1$ polynomials with degree m. Namely for unit interval $B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}$. Bernstein polynomials have many interesting properties but it is not necessary to express them in this article. For example, they have non-negativity and partition-of-unity properties. The avid reader can see [27, 31, 32, 33, 35, 41, 45, 46, 47]. These polynomials are linearly independent and any given polynomial can be expanded with Bernstein polynomials. Also, these polynomials for two positive integers numbers, m and n, can be defined on two-dimensional real space like the following definition. The following definition is used for approximation of any two variables function. In this paper, our variables are length and time, which are shown by x and t, respectively.

Definition 2.2. 2-dimensional Bernstein polynomials are defined on $[a,b] \times [a,b]$ as follows:

$$\beta_{(i,m),(j,n)}(x,y) = \binom{m}{i}\binom{n}{j} \frac{(x-a)^i (y-a)^j (b-x)^{m-i} (b-y)^{n-j}}{(b-a)^{m+n}},$$

where $i = 0, 1, \cdots, m$ and $j = 0, 1, \cdots, n$. If $m = n$ Bernstein polynomial denote by $\beta_{(i,j)}(x,y)$. Similar to the one-dimensional case, there are some properties. For example, the positivity property, disjointness, and partition-of-unity are held. In other words:

1)$\beta_{(i,m)(j,n)}(x,y) \geqslant 0$.

2)$\beta_{(i,m)(j,n)}(x,y) = B_{i,m}(x)B_{j,n}(y)$.

3)$\displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n}\beta_{(i,m)(j,n)}(x,y) = 1$.

Indeed, many properties can be generalized to two-dimensional. For more details see [6, 8, 14, 18, 28, 29, 37]. Let $X = l^2([a,b])$ and $Y = Span\{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\}$ be a subspace of X. Kreyszig proved some lemmas and theorems to show there exists a unique best approximation for any $f \in X$from Y. Also, if y_0 denotes the unique best approximation, the following important result is shown in [20]:

$$\forall f \in X : \langle f - y_0, \varphi\rangle = 0$$

where $\varphi \in Y$ and $\langle f, g\rangle = \int_a^b f(t)g^T(t)dt$. So, it is easy to see:

$$\langle f, \varphi\rangle = \langle y_0, \varphi\rangle.$$

Since $\varphi \in Y$, there are unique coefficients like $c_0, c_1, \dots, c_m$ such that:

$$f \simeq \sum_{i=0}^{m} c_i B_{i,m}(x) = C^T\Phi \tag{2.1}$$

where $C = \begin{bmatrix} c_0 & c_1 & \cdots & c_m\end{bmatrix}^T$ and $\Phi = \begin{bmatrix} B_{0,m} & B_{1,m} & \cdots & B_{m,m}\end{bmatrix}^T$. For determining $c_0, c_1, \dots, c_m$ notice:

$$\langle f, \Phi\rangle = \left\langle C^T\Phi, \Phi\right\rangle = C^T\langle\Phi, \Phi\rangle = C^TQ$$

where $Q = \langle\Phi, \Phi\rangle$ is an $(m+1)\times(m+1)$ symmetric matrix which named **dual** matrix. Therefore, $C^T = Q^{-1}\langle f, \Phi\rangle$. Now, consider a two-dimensional space $l^2([a,b]\times[a,b])$, for $f(x,y), g(x,y) \in l^2([a,b]\times[a,b])$, inner products are defined as follows:

$$\langle f(x,y), g(x,y)\rangle = \int_a^b\int_a^b f(x,y)g(x,y)dxdy.$$

Suppose that $k(x,y)$ be an arbitrary function in $l^2([a,b]\times[a,b])$, then it can be expanded in terms of two-dimensional Bernstein polynomials as follows:

$$k(x,y) \simeq \sum_{i=0}^{m}\sum_{j=0}^{n} K_{i,j}\psi_{i,j} = K^T\Psi(x,y) \tag{2.2}$$

where K and $\Psi(x,y)$ are two $(m+1)(n+1)\times 1$ vectors given by

$$K(x,y) = \begin{bmatrix} K_{00} & \dots & K_{0n} & \cdots & K_{m1} & \dots & K_{mn}\end{bmatrix}^T$$

and

$$\Psi(x,y) = \Phi(x)\otimes\Phi(y) = \begin{bmatrix}\psi_{00}(x,y) & \psi_{01}(x,y) & \dots & \psi_{0n}(x,y) & \cdots & \psi_{m0}(x,y) & \psi_{m1}(x,y) & \dots & \psi_{mn}(x,y)\end{bmatrix}^T,$$

where $\otimes$ denotes the Kronecker product and $\psi_{i,j} = \beta_{(i,m),(j,n)}$. For a two-dimensional function like $k(x,t)$, the unique expansion computed as follows:

$$\langle k, \Psi\rangle = \left\langle K^T\Psi, \Psi\right\rangle = K^T\langle\Psi, \Psi\rangle = K^T\bar{Q},$$

where $\bar{Q} = \langle\Psi, \Psi\rangle$ is an $(m+1)(n+1)\times(m+1)(n+1)$ matrix which named Dual matrix. Now, to check the convergence of these expansions, another form of this approximation is presented. Considering the importance and age of these polynomials, various reviews can be found regarding all the features of these convergences[20]. For a given function f on $[0,1]$, the Bernstein polynomial, for each positive integer m, is also obtained from the following equation:

$$B_m(f;x) = \sum_{i=0}^{m}\binom{m}{i}f(\frac{i}{m})x^i(1-x)^{m-i}.$$

The following theorem says for a continuous function$f$ on $[0,1]$ the Bernstein polynomials $B_m(f;x)$ converges uniformly to$f$on $[0,1]$.

Theorem 2.3 ([6, 27, 35, 41]). suppose $f \in C[0,1]$ and for any$\epsilon > 0$, there exists an integer N such that

$$\forall n \geq N \qquad 0 < x < 1 \qquad |f(x) - B_m(f;x)| < \epsilon$$

Theorem 2.4 ([6, 27, 35, 41]). If $f \in C^p[0,1]$, for some integer $p \geq 0$, then $B_m^{(p)}(f;x)$ converges uniformly to $f^{(p)}(x)$ on $[0,1]$.

As we have just seen, not only the $B_m(f;x)$ converges to f, but its derivatives converge to derivatives of f. For the function $f(x,y) \in l^2([0,1] \times [0,1])$, the two-dimensional Bernstein polynomial of degree (m,n), corresponding to the function f, is defined by

$$B_{m,n}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} f(\frac{i}{m},\frac{j}{n})\beta_{(i,m),(j,n)}(x,y) == \sum_{i=0}^{m}\sum_{j=0}^{n} f(\frac{i}{m},\frac{j}{n})B_{i,m}(x)B_{j,n}(y).$$

Lemma 2.5. ([18, 28]) if f is continuous in $l^2([0,1] \times [0,1])$, and $B_{m,n}f$ is the Bernstein polynomial of f, then

$$|B_{m,n}(f;x,y) - f(x,y)| \leq \frac{3}{2}\left[\omega^{(1)}(f;n^{-\frac{1}{2}} + \omega^{(2)}(f;m^{-\frac{1}{2})}\right] \leq \frac{3}{2}\omega(f;n^{-\frac{1}{2}},m^{-\frac{1}{2}}),$$

where partial moduli of continuity of f are denoted by $\omega^{(i)}(f;\delta)$, that means:

$$\omega^{(1)}(f;\delta) = \underset{y}{Sup} \underset{|x_1-x_2|\leq\delta}{Sup} |f(x_1,y) - f(x_2,y)|$$

$$\omega^{(2)}(f;\delta) = \underset{x}{Sup} \underset{|y_1-y_2|\leq\delta}{Sup} |f(x,y_1) - f(x,y_2)|$$

and $\omega(f;\delta,\epsilon)$ is the complete modulus of continuity of f.

Lemma 2.6 ([18, 28]). If all the partial derivatives of $f(x,y)$ of order less than p exist and are continuous in $[0,1] \times [0,1]$, then $\frac{\partial^p}{\partial x^q \partial y^{p-q}} B_{m,n}(f;x,y)$ uniformly converges to$\frac{\partial^p}{\partial x^q \partial y^{p-q}} f(x,y)$ in $[0,1] \times [0,1]$ as n and m approach infinity in any manner whatever.

3 Operational matrices

In this section, operational matrices of Bernstein polynomials of the 1-dimensional and 2-dimensional are studied and reviewed. For one-dimensional functions, there are articles in which researchers have provided accurate or approximate matrices. We mention these matrices briefly, you can refer to sources for how to find and calculate them.

Lemma 3.1. [31, 32, 33, 45, 46] Operational matrices of integration is an $(m+1) \times (m+1)$ matrix such that

$$\int_0^x \Phi(t)dt = P\Phi(x).$$

In the mentioned references, this matrix has been calculated in two different ways.

Corollary 3.2. $\int_0^1 \Phi(t)dt = P\Phi(1)$.

Lemma 3.3. [31, 32, 33, 45, 46] operational matrix of the derivative is an $(m+1) \times (m+1)$ matrix which is denoted by D and defines as follows:

$$\frac{d}{dx}\Phi(x) = D\Phi(x).$$

Corollary 3.4. $\frac{d^n}{d^n}\Phi(x) = D^n\Phi(x)$.

Lemma 3.5. [31, 32, 33, 45, 46] Let C be an $(m+1)$- vector, then:

$$\Phi(x)\Phi^T(x)C = \hat{C}\Phi(x),$$

where $\hat{C}$ is an $(m+1)\times(m+1)$ matrix.

Lemma 3.6. [31, 32, 33, 45, 46]consider an $(m+1)\times(m+1)$ square matrix M, It can be proved that there exists $\tilde{M}$, an $(m+1)$–vector, such that

$$\Phi^T(x)M\Phi(x) = \tilde{M}^T\Phi(x),$$

Now, consider $\Psi(x,y) = \Phi(x)\otimes\Phi(y)$ as a two-dimensional Bernstein polynomial. We want to compute operational matrices in two-dimensional cases. At first, consider $\Psi(x,y)$ and its partial derivative.

Theorem 3.7. $\frac{\partial}{\partial x}\Psi(x,y) = \bar{D}\Psi(x,y)$.

Proof . In this proof, we will use Lemma 3.3. Also, [1] is denoted a 1×1 matrix.

$$\begin{aligned}\frac{\partial}{\partial x}\Psi(x,y) &= \frac{\partial}{\partial x}(\Phi(x)\otimes\Phi(y))\\ &= \frac{\partial}{\partial x}\Phi(x)\otimes\Phi(y)\\ &= D\Phi(x)\otimes\Phi(y)\\ &= (D\otimes[1])(\Phi(x)\otimes\Phi(y))\\ &= \bar{D}\Psi(x,y).\end{aligned}$$

So, $\bar{D} = (D\otimes[1])$ and the proof is completed. □

Theorem 3.8. $\frac{\partial}{\partial y}\Psi(x,y) = \bar{\bar{D}}\Psi(x,y)$.

Proof . The proof is similar to the theorem 3.7.

$$\begin{aligned}\frac{\partial}{\partial y}\Psi(x,y) &= \frac{\partial}{\partial y}(\Phi(x)\otimes\Phi(y))\\ &= (\Phi(x)\otimes\frac{\partial}{\partial y}\Phi(y))\\ &= (\Phi(x)\otimes D\Phi(y))\\ &= ([1]\otimes D)(\Phi(x)\otimes\Phi(y))\\ &= \bar{\bar{D}}\Psi(x,y).\end{aligned}$$

Therefore, $\bar{\bar{D}} = [1]\otimes D$. The proof is completed. □

Theorem 3.9. $\int_0^x \Psi(t,y)dt = \bar{P}\Psi(x,y)$.

Proof . in this proof lemma 3.1 are applied.

$$\begin{aligned}\int_0^x \Psi(t,y)dt &= \int_0^x (\Phi(t)\otimes\Phi(y))dt\\ &= \int_0^x (\Phi(t)dt)\otimes\Phi(y)\\ &= P\Phi(x)\otimes\Phi(y)\\ &= (P\otimes[1])(\Phi(x)\otimes\Phi(y))\\ &= \bar{P}\Psi(x,y).\end{aligned}$$

Therefore, $\bar{P} = P\otimes[1]$, where P is a Bernstein operational matrix of integration and proof is completed. □

Theorem 3.10. $\int_0^y \Psi(x,t)dt = \bar{\bar{P}}\Psi(x,y)$.

Proof . Proof is similar to theorem 3.9.

$$\begin{aligned}\int_0^y \Psi(x,t)dt &= (\Phi(x) \otimes \int_0^y \Phi(t)dt) \\ &= \Phi(x) \otimes (\int_0^y \Phi(t)dt) \\ &= \Phi(x) \otimes P\Phi(y) \\ &= ([1] \otimes P)(\Phi(x) \otimes \Phi(y)) \\ &= ([1] \otimes P)(\Phi(x) \otimes \Phi(y)) \\ &= \bar{\bar{P}}\Psi(x,y).\end{aligned}$$

Therefore, $\bar{\bar{P}} = [1] \otimes P$, where P is a Bernstein operational matrix of integration and proof is completed. □

In the next theorem, we use Kronecker decomposition for an arbitrary matrix. Kronecker decomposition is so important in Image processing and there are several approximate methods to do this [3, 25].

Theorem 3.11. Suppose u is an $(m+1)^2 \times 1$ vector so that there are two $(m+1)$-vector like u_1, u_2 such that $u = u_1 \otimes u_2$. Now, $\Psi(x,y)\Psi^T(x,y)u = \tilde{u}\Psi(x,y)$, where $\tilde{u}$ is an $(m+1) \times (m+1)$ matrix.

Proof . The definition of $\Psi(x,y)$ and Kronecker products features imply:

$$\begin{aligned}\Psi(x,y)\Psi^T(x,y)u &= (\Phi(x) \otimes \Phi(y))(\Phi^T(x) \otimes \Phi^T(y))(u_1 \otimes u_2) \\ &= (\Phi(x) \otimes \Phi(y))(\Phi^T(x)u_1) \otimes (\Phi^T(y)u_2) \\ &= (\Phi(x)\Phi^T(x)u_1 \otimes (\Phi(y)\Phi^T(y)u_2),\end{aligned}$$

Now, Lemma 3.5 implies $= (\tilde{u_1} \otimes \tilde{u_2})(\Phi(x) \otimes \Phi(y))$. It is enough to let $\tilde{u} = \tilde{u_1} \otimes \tilde{u_2}$ and so proof is completed. □

Theorem 3.12. Suppose M is an $(m+1)^2 \times (m+1)^2$ matrix, Then there is an $(m+1)^2$–vector like $\tilde{M}$ such that:

$$\Psi^T(x,y)M\Psi(x,y) = \tilde{M}^T\Psi(x,y).$$

Proof . The proof has the same procedure as the previous theorem.

$$\begin{aligned}\Psi^T(x,y)M\Psi(x,y) =&(\Phi^T(x) \otimes \Phi^T(y))M(\Phi(x) \otimes \Phi(y)) \\ =&(\Phi^T(x) \otimes \Phi^T(y))M_1.M_2(\Phi(x) \otimes \Phi(y)),\end{aligned}$$

which $M = M_1.M_2$ is an arbitrary decomposition. Now according to the properties of Kronecker matrix product and using lemma 3.6

$$\begin{aligned}&=(\Phi^T(x)M_1 \otimes \Phi^T(y)M_2)(\Phi(x) \otimes \Phi(y)) \\ &=((\Phi^T(x)M_1\Phi(x)) \otimes (\Phi^T(y)M_2 \otimes \Phi(y)) \\ &=(\tilde{M_1}^T\Phi(x)) \otimes (\tilde{M_2}^T\Phi(y)) \\ &=(\tilde{M_1}^T \otimes \tilde{M_2}^T)(\Phi(x)) \otimes \Phi(y)) \\ &=\tilde{M}^T\Psi(x,y).\end{aligned}$$

where $\tilde{M} = \tilde{M_1} \otimes \tilde{M_2}$, and proof is completed. □

4 Implementation of method

In this section, consider the main equation and its initial boundary conditions. We try to convert this equation to an integrodifferential equation involving initial boundary conditions. Without reducing the generality of the problem and for simplicity, suppose $a = 0, b = 1, c = 1$. Consider (1.1) as follows:

$$u_{xx}(x,t) = u_{tt}(x,t) - k(x,t)$$

so with an integration:

$$u_x(x,t) = u_x(0,t) + \int_0^x (u_{tt}(s,t) - k(s,t))ds. \tag{4.1}$$

Repeating the same action gives results:

$$u(x,t) = u(0,t) + \int_0^x u_x(s,t)ds = h(t) + xu_x(0,t) + \int_0^x \int_0^s (u_{tt}(x,t) - k(s,t))dsds$$

put $x = 1$ and obtain:

$$u_x(0,t) = r(t) - h(t) - \int_0^1 \int_0^s (u_{tt}(s,t) - k(s,t))dsds. \tag{4.2}$$

Now, replacing (4.2) in (4.1) gives:

$$u(x,t) = (1-x)h(t) + xr(t) - x\int_0^1 \int_0^s (u_{tt}(s,t) - k(s,t))dsds + \int_0^x \int_0^s (u_{tt}(s,t) - k(s,t))dsds \tag{4.3}$$

differentiation twice both sides of (4.3), implies

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial t^2}\left[(1-x)h(t) + xr(t) - x\int_0^1 \int_0^s (u_{tt}(s,t) - k(s,t))dsds + \int_0^x \int_0^s (u_{tt}(s,t) - k(s,t))dsds\right]. \tag{4.4}$$

Now, integration of (4.4) with respect to t, gives

$$\frac{\partial}{\partial t}u(x,t) = g(x) + \int_0^t \left(\frac{\partial^2}{\partial t^2}\left[(1-x)h(t) + xr(t) - x\int_0^1 \int_0^s (u_{tt}(s,t) - k(s,t))dsds + \int_0^x \int_0^s (u_{tt}(s,t) - k(s,t))dsds\right]\right) dt. \tag{4.5}$$

Again, integration of (4.5) with respect to t, results

$$u(x,t) = f(x) + tg(x) \tag{4.6}$$

$$+ \int_0^t \int_0^t \frac{\partial^2}{\partial t^2}\left((1-x)h(t) + xr(t)\right) dtdt \tag{4.7}$$

$$- \int_0^t \int_0^t \frac{\partial^2}{\partial t^2}\left[x\int_0^1 \int_0^s (u_{tt}(s,t) - k(s,t))dsds\right] dtdt \tag{4.8}$$

$$+ \int_0^t \int_0^t \frac{\partial^2}{\partial t^2}\left[\int_0^x \int_0^s (u_{tt}(s,t) - k(s,t))dsds\right] dtdt. \tag{4.9}$$

Now, we use the approximation of functions by using the Bernstein polynomials. (2.1) and (2.2) imply

$$u(x,t) = U^T\Psi(x,t) \tag{4.10}$$

$$k(x,t) = K^T\Psi(x,t) \tag{4.11}$$

$$f(x) = F^T\Psi(x,t) \tag{4.12}$$

$$tg(x) = G^T\Psi(x,t) \tag{4.13}$$

$$(1-x)h(t) = H^T\Psi(x,t) \tag{4.14}$$

$$xr(t) = R^T\Psi(x,t) \tag{4.15}$$

$$x = \phi^T(x)X \tag{4.16}$$

where U is an unknown $(m+1)^2$ vector which must be determined and F, K, G, H, R are $(m+1)^2-$ vectors and X is an $(m+1)-$ vector. By (4.10) and (4.11) and using theorem 3.8 conclude:

$$u_{tt}(x,t) - k(x,t) = U^T \frac{\partial^2}{\partial t^2}\Psi(x,t) - K^T\Psi(x,t) = (U^T\bar{\bar{D}}^2 - K^T)\Psi(x,t). \tag{4.17}$$

For convenience, put

$$N_U^T = (U^T\bar{\bar{D}}^2 - K^T). \tag{4.18}$$

Consider (4.7), theorems 3.8, 3.10, and equations (4.14) and (4.15) give

$$\begin{aligned}\int_0^t\int_0^t \frac{\partial^2}{\partial t^2}((1-x)h(t) + xr(t))dtdt &= (H^T + R^T)\int_0^t\int_0^t \frac{\partial^2}{\partial t^2}\Psi(x,t)dtdt \\ &= (H^T + R^T)\bar{\bar{D}}^2\int_0^t\int_0^t \Psi(x,t)dtdt \\ &= (H^T + R^T)\bar{\bar{D}}^2\bar{\bar{P}}^2\Psi(x,t).\end{aligned} \tag{4.19}$$

For the approximation of (4.8), at first, we notice (4.17) and (4.18)

$$u_{tt}(x,t) - k(x,t) = N_U^T\Psi(x,t) = \phi^T(x)N\phi(t),$$

where N_U is the vectorization operator applied on N. Next,

$$\begin{aligned}\int_0^t\int_0^t \frac{\partial^2}{\partial t^2}\left[x\int_0^1\int_0^s \phi^T(x)N\phi(t)dsds\right]dtdt &= \int_0^t\int_0^t \frac{\partial^2}{\partial t^2}\left[\phi^T(x)XP^2N\phi(1)N\phi(t)\right]dtdt \\ &= \int_0^t\int_0^t \frac{\partial^2}{\partial t^2}\left[Ver(XP^2N\phi(1))^T.\Psi(x,t)\right]dtdt \\ &= Ver(XP^2N\phi(1))^T\bar{\bar{D}}^2\bar{\bar{P}}^2\Psi(x,t)\end{aligned} \tag{4.20}$$

Finally, for the (4.9), applying theorems 3.9, 3.8 and 3.10 implies:

$$\begin{aligned}\int_0^t\int_0^t \frac{\partial^2}{\partial t^2}\left[\int_0^x\int_0^x N_U^T\Psi(x,t)dsds\right]dtdt &= \int_0^t\int_0^t \frac{\partial^2}{\partial t^2}N_U^T\bar{P}^2\Psi(x,t)dtdt \\ &= \int_0^t\int_0^t N_U^T\bar{P}^2\bar{\bar{D}}^2\Psi(x,t)dtdt \\ &= N_U^T\bar{P}^2\bar{\bar{D}}^2\bar{\bar{P}}^2\Psi(x,t)\end{aligned} \tag{4.21}$$

Now equations (4.12), (4.13), (4.19), (4.20), and (4.21) give:

$$U^T = F^T + G^T + H^T\bar{\bar{D}}^2\bar{\bar{P}}^2 + R^T\bar{\bar{D}}^2\bar{\bar{P}}^2 + Ver(XP^2N\phi(1))^T\bar{\bar{D}}^2\bar{\bar{P}}^2 + N_U^T\bar{P}^2\bar{\bar{D}}^2\bar{\bar{P}}^2. \tag{4.22}$$

To simplify let's

$$A^T = F^T + G^T + H^T\bar{\bar{D}}^2\bar{\bar{P}}^2 + R^T\bar{\bar{D}}^2\bar{\bar{P}}^2, \qquad B_U^T = Ver(XP^2N\phi(1))^T\bar{\bar{D}}^2\bar{\bar{P}}^2 + N_U^T\bar{P}^2\bar{\bar{D}}^2\bar{\bar{P}}^2.$$

Therefore the final algebraic equation is $U - B_U = A$.

5 Numerical examples

In this section, we present several different numerical examples to express the accuracy and efficiency of the method. It is tried to select the examples in such a way that their exact or analytical answer is clear.

Example 5.1. Consider the following homogenous wave equation with the exact solution $u(x,t) = x^2 + t^2$.

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, && 0 < x < 1, t > 0\\
&u(x,0) = x^2, && 0 < x < 1\\
&u_t(x,0) = 0, && 0 < x < 1\\
&u(0,t) = t^2, && t > 0\\
&u(1,t) = t^2 + 1, && t > 0.
\end{aligned}$$

For $m = 3$ the solution is exact.

Example 5.2. The exact solution of the following equation is $u(x,t) = \sin(xt)$.

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + (x^2 - t^2)\sin(xt), && 0 < x < 1, t > 0\\
&u(x,0) = 0, && 0 < x < 1\\
&u_t(x,0) = x, && 0 < x < 1\\
&u(0,t) = 0, && t > 0\\
&u(1,t) = \sin t, && t > 0.
\end{aligned}$$

Table 1: Absolute error in some points for $m = 1, 2, 3$.

(x_0, y_0)	$E_1(x_0, t_0)$	$E_2(x_0, t_0)$	$E_3(x_0, t_0)$
(0.1,0.1)	0.001736416786	0.000160330466	0.000258390165732311
(0.2,0.2)	0.00059992569	0.00224914910	0.00154544635840923
(0.3,0.3)	0.00331951992	0.00586763159	0.00347681411315022
(0.4,0.4)	0.0096726483	0.0100148802	0.00432352017280749
(0.5,0.5)	0.0175551123	0.0133836809	0.00163465523755768
(0.6,0.6)	0.0251053380	0.0147626546	0.00653777723474636
(0.7,0.7)	0.0290201849	0.0135205969	0.0198360528449574
(0.8,0.8)	0.0240361705	0.0101252540	0.0332637102946646
(0.9,0.9)	0.0024575763	0.0066152159	0.0344036463896897

As can be seen in the table, the absolute error has been calculated at some arbitrary points. These values listed in the table indicate that due to the small degree of the polynomials, the answers have acceptable accuracy. By increasing the value of m, more favorable answers can be obtained. This is shown in the next example. Figure 1 shows absolute error for $m = 2$.

Example 5.3. $u(x,t) = e^{-(x/10+t/9)}$ is the exact solution of the

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{19}{8100}e^{-(\frac{x}{10}+\frac{t}{9})}, 0 < x < 1, t > 0 &&\\
&u(x,0) = e^{\frac{-x}{10}}, && 0 < x < 1\\
&u_t(x,0) = \frac{-1}{9}e^{\frac{-x}{10}}, && 0 < x < 1\\
&u(0,t) = e^{-\frac{t}{9}}, && t > 0\\
&u(1,t) = e^{\frac{-t}{9}-\frac{1}{10}}, && t > 0.
\end{aligned}$$

In the Table 2, the absolute error at some specific points for two different values of the degree of polynomials is calculated. In Figure 2, the absolute error for $m = 2$ are shown.

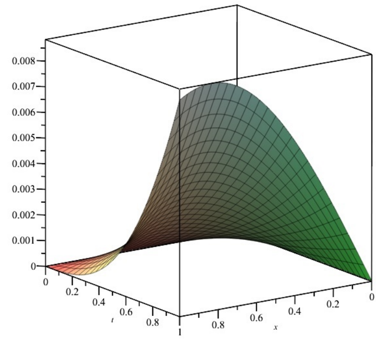

Figure 1: Absolut error for $m = 2$.

Table 2: Absolute errors in some points

(x_0, t_0)	$E_2(x_0, t_0)$	$E_8(x_0, t_0)$
(0.1,0.1)	4.6885×10^{-6}	$4.04833855 \times 10^{-11}$
(0.2,0.2)	3.0379×10^{-6}	$4.04833855 \times 10^{-11}$
(0.3,0.3)	3.7753×10^{-6}	$6.548123013 \times 10^{-11}$
(0.4,0.4)	1.1108×10^{-6}	$9.666058332 \times 10^{-11}$
(0.5,0.5)	9.4812×10^{-6}	$1.03958177 \times 10^{-11}$
(0.6,0.6)	1.84194×10^{-6}	$8.206985246 \times 10^{-11}$
(0.7,0.7)	2.42276×10^{-6}	$4.091960184 \times 10^{-11}$
(0.8,0.8)	2.24229×10^{-6}	$2.00247131 \times 10^{-11}$
(0.9,0.9)	7.7339×10^{-6}	$1.1948086 \times 10^{-11}$

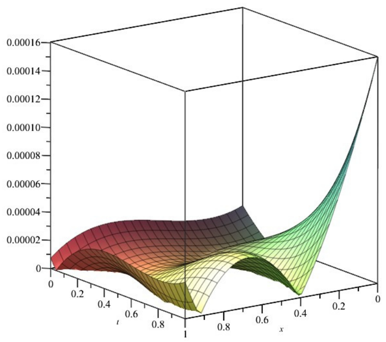

Figure 2: Absolut error for $m = 2$.

6 Nonlocal Boundary conditions

In the simulation of some partial differential equations, boundary conditions appear in the form of integral equations. The method presented in this article can solve these problems. In nonlocal boundary conditions, equations (1.2) and (1.3) convert to

$$u(0,t) = \int_0^1 \rho(x)u(x,t)dx \tag{6.1}$$

$$u(1,t) = \int_0^1 \omega(x)u(x,t)dx. \tag{6.2}$$

First, we need to find a relationship between the expansion of one-variable functions in one-variable and two-variables spaces.

Lemma 6.1. Let $f(x) \in l^2([0,1])$ is an arbitrary function with unique expansion $f(x) = F^T\phi(x)$. Also, this function has a unique expansion with respect to $\Psi(x,y)$ as follows:

$$f(x) = F_1^T \Psi(x,y),$$

where $F_1 = F \otimes O_{m+1}$ and O_{m+1} is an $(m+1)$-vector whose all elements are equal to one.

Proof . With respect to (2.1), we have

$$\begin{aligned} f(x) =& F^T\Phi(x) \\ =& F^T\Phi(x) \otimes [1] \\ =& F^T\Phi(x) \otimes O_{m+1}\Phi(y) \\ =& (F^T \otimes O_{m+1})(\Phi(x) \otimes \Phi(y)) \\ =& F_1^T\Psi(x,y). \end{aligned}$$

□

According to the expansion of the functions can be written:

$$\rho(x) = \Lambda^T\Psi(x,t) \qquad \text{and} \qquad \omega(x) = \Omega^T\Psi(x,t)$$

where Λ, Ω are two $(m+1)^2$-vectors. Now consider (6.1), Kronecker product properties imply:

$$u(0,t) = \int_0^1 \rho(x)u(x,t)dx \quad = \textstyle\int_0^1 \Lambda^T\Phi(x)\Phi^T(x)U\Phi(t)dx \quad = \Lambda^T(\displaystyle\int_0^1 \Phi(x)\Phi^T(x)dx)U\Phi(t).$$

Now, Dual matrix and Lemma 6.1 give:

$$\Lambda^T QU\Phi(t) = ((U^T Q\Lambda) \otimes O_{m+1})\Psi(x,t).$$

With the same argument, a similar result holds for the equation (6.2):

$$u(1,t) = \textstyle\int_0^1 \omega(x)u(x,t)dx = ((U^T Q\Omega) \otimes O_{m+1})\Psi(x,t)$$

Now, by substituting the obtained results in place of $f(x)$ and $g(x)$ in (4.12) and (4.13) respectively, the solution of the equation can be obtained for non-local boundary conditions.

Example 6.2. Consider the following equation with exact solution $u(x,t) = x^2t^2$.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 2(x^2 - t^2)$$

$$\begin{cases} u(x,0) = 0 \\ u_t(x,0) = 0 \\ u(0,t) = 0 \\ u(1,t) = \int_0^1 3u(x,t)dx. \end{cases}$$

Table 3, shows the absolute error in some points for $m = 1,2,3$. Considering the amount of calculated error in some points, the accuracy of the method is acceptable. Figure 6 and Figure 7 show estimate solution and exact solution for $m = 2$, respectively.

Table 3: Absolute errors in some points

(x_0, t_0)	$E_1(x_0, t_0)$	$E_2(x_0, t_0)$	$E_3(x_0, t_0)$
(0.1,0.1)	0.0000676666667	0.00001200000000	0.000002337502040
(0.2,0.2)	0.0000506666666	0.0001066666667	0.00001672394016
(0.3,0.3)	0.0003189999999	0.0003360000001	0.00004895588570
(0.4,0.4)	0.0006773333332	0.0007040000002	0.00009584639990
(0.5,0.5)	0.001041666666	0.001166666667	0.0001452380952
(0.6,0.6)	0.001304000000	0.001632000001	0.0001789784815
(0.7,0.7)	0.001332333333	0.001960000001	0.0001788575998
(0.8,0.8)	0.000970666666	0.001962666668	0.0001355079399
(0.9,0.9)	0.000039000000	0.001404000002	0.0000602666449

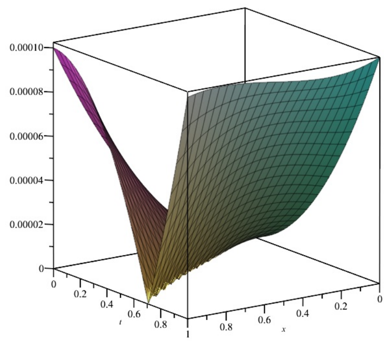

Figure 3: Absolut error for $m = 2$.

7 Conclusion

In this paper, Bernstein's polynomials and the operational matrices for integration, differentiations, products, and the dual were reviewed. Then, by using them and the features of Kronecker's product, the necessary operational matrices for two-dimensional Bernstein functions were calculated. Operational matrices obtained from two-dimensional Bernstein polynomials were used to change solving the hyperbolic differential equation to the solution of algebraic equations. The method presented in this paper can readily be generalized to other appropriate partial differential equations. The method is general, easy to implement, low cost, and very accurate. illustrative examples using the method developed in this article show that the new method produces accurate and acceptable results. Also, the suggested upper bound for error is proof of this claim. It convergence to the exact solution when m increases to infinity. It is worth noting that the new technique developed in the current paper can be extended to solve similar problems in higher dimensions.

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