Int. J. Nonlinear Anal. Appl. In Press, 1–8
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.33369.4967

# Stock portfolio selection with maximum predictability of Sharpe ratio based on hidden Markov model

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*(Communicated by Majid Eshaghi Gordji)*

**Abstract**

The Sharpe ratio is one of the performance evaluation criteria of the stock portfolio, which shows the return per unit of risk. This ratio is particularly important for risk-averse investors. In the current research, the hidden Markov model approach introduces a new stock portfolio model called the maximum predictability portfolio of the Sharpe ratio. The hidden Markov model used for each hidden state has a mixture of normals distribution output, which is used to calculate the return and standard deviation to calculate the Sharpe ratio in the investment horizon. The research portfolio calculates the weights of the portfolio in such a way that the Sharpe ratio is maximized in the horizon of the portfolio. The optimal research portfolio was optimized using the historical data of 10 indices from the Tehran Stock Exchange between 2018 and 2018 in a four-member mode space. The evaluation of the performance of the optimal portfolio in the Sharpe ratio criterion shows that the research model has a better performance than the mean-variance model and the equal-weight model.

Keywords: Sharpe ratio, Markov chain, hidden Markov model, mixture of normal
2020 MSC: 60J10, 62M05, 62P20

## 1 Introduction

The price or return of an asset or stock portfolio is inherently random at any given time. When we consider these random variables over a specific period, they collectively form a random process. The actual observed value of a portfolio over this timeframe represents a single realization, or path, of this random process. ARIMA, Wiener, and Levy process all fall under this framework [9].

Markov processes are a significant and widely used category of stochastic processes. In a Markov process, the future state depends solely on the current state. Past states provide no additional information for predicting the future. A discrete-time Markov chain is a specific type of Markov process where the set of possible states (known as the state space) is either finite or countably infinite. A Markov chain transitions between states over time, with these transitions governed by a probability distribution. The probability of transitioning between any two specific states is called the transition probability.

For example, a country's economic conditions can be modelled as a two-state Markov chain, with states representing prosperity and recession [5].

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*Received:* February 2024 *Accepted:* April 2024

Now, consider a Markov chain where the state of the chain influences another observable variable. For instance, a country's economic state (recession or prosperity) can impact the performance of a stock index, which is the observable variable. In addition to the transition matrix that describes state transitions, another matrix is needed. This emission matrix captures the probability of observing a specific value of the observable variable given each possible state. A hidden Markov model (HMM) is a type of Markov chain where the underlying states are hidden and unknown. HMMs are characterized by a transition matrix and an emission matrix that relates the distribution of the observable variable to the hidden states [13].

Markov chains and hidden Markov models have been extensively employed in financial research to predict asset or portfolio returns. This study utilizes the framework of hidden Markov models to address the problem of optimal stock portfolio selection. Optimizing stock portfolios is a dynamic field in financial research, encompassing both theoretical and practical aspects. Selecting an optimal and efficient portfolio arguably represents one of the primary objectives for investors in financial markets. Numerous stock portfolio models exist in the literature, each with its own assumptions and limitations. This research explores the novel application of hidden Markov models for portfolio optimization.

## 2 Theoretical foundations and research background

Optimizing the stock portfolio is the process of choosing the weight of the assets that make up the portfolio based on the limitations of the investment environment (such as a specific budget and the impossibility of borrowing) and the wishes of the investor [8].

Investor's demands are often modelled in the form of two parameters, return and risk. In this way, it is expected that a stock portfolio can estimate the investor's expected return at a minimal level of risk, and such a portfolio is called an efficient portfolio [3]. Therefore, in the research literature, there are various metrics to estimate the return. and risk has been used. As an example, the weighted average of individual asset returns or statistical models or artificial intelligence has been used to estimate the expected return. Also, various metrics have been used to measure risk, including standard deviation, half standard deviation, value at risk, expected loss, and maximum capital loss, etc. [11].

Various types of optimal portfolio models have been developed in the research literature. The Markowitz mean-variance portfolio model is the first quantification of an optimal stock portfolio model, in which the return of the stock portfolio is calculated through the average returns of the portfolio assets and the risk is modelled with the help of the standard deviation and different versions of the mean-variance model have been developed. The risk parity portfolio model is another category of stock portfolio models that deal with risk allocation instead of asset allocation and aims to make the risk share of assets equal to the total risk of the stock portfolio as much as possible. Kelly's growth benchmark portfolio also has its own philosophy and its objective function is the logarithmic return. The maximum predictability portfolio is also a type of stock portfolio model that allocates the stock portfolio weights in such a way that its yield is optimized in criteria such as the determined adjustment coefficient or mean squared error. In this way, one can hope with greater certainty about the realization of the expected return of the stock portfolio. In the current research, the maximum predictability approach is used in the Sharpe ratio.

The Sharpe ratio is one of the most well-known risk-reward ratios in finance, widely used by professionals. This ratio is the average return earned over the risk-free rate per unit of volatility or total risk. Volatility is a measure of price fluctuations of an asset or portfolio [1]. The model that will be used to predict the Sharpe ratio in the optimal research portfolio is the hidden Markov model. In the following, this model is introduced. A sequence of like $\{X_t\}$ random variables is called a random process. A discrete-time chain is a random process whose state space, i.e. the set of values that the random variables that make up the process can take, is a countable or finite set and in which the time index is $t \in \{0, 1, 2, \cdots\}$ discrete set. In the following, it is always assumed that time is discrete. Formally, the $\{X_t\}$ chain is a Markov chain if relation (2.1) holds:

$$P\{X_{n+1} = j\,|X_0 = i_0, \cdots, X_{n-1} = i_{n-1}, X_n = i\} = P\{X_{n+1} = j\,|X_n = i\} \tag{2.1}$$

This means that the future values of the process depend only on its current value. Suppose $P_{ij}$ is the probability of changing state from $i$ to $j$. Usually, the numbers $P_{ij}$ are represented as a matrix as shown below.

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0n} \\ p_{10} & p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n0} & p_{n1} & \cdots & p_{nn} \end{bmatrix} \tag{2.2}$$

$P = [p_{ij}]$ matrix is called state change matrix or process transition probability matrix. A Markov chain is completely known by specifying the transition probability matrix and the probability distribution $X_0$ (the starting state of the chain). In the normal Markov model, the state is directly observable by the observer, and therefore the probability of transitions between states is the only available parameter. In a hidden Markov model, the state is not directly observable, but the output that depends on the state is. Each state has a probability distribution on the output; So the sequence of outputs produced by a hidden Markov model gives information about the sequence of states. Therefore, the hidden attribute in a hidden Markov model refers to the sequence of states that the model passes through, not to the parameters of the model, even if the parameters of the model are precisely known, the model is still hidden. Hidden Markov models are mostly known for their application in pattern recognition, such as voice and handwriting recognition, gesture and movement recognition, labelling of speech components, bioinformatics, etc.

Figure 1 shows the general architecture of a hidden Markov model. Each ellipse is a random variable where Xt represents the state at time $t$ and $y(t)$ represents the output or observable variable at that time. Arrows mean conditional dependencies. It is clear from the figure that the conditional probability distribution of the hidden variable $x(t)$ depends only on the value of the hidden variable $x(t-1)$ and the values of the times before that have no effect. As mentioned, this characteristic is called the Markov property. Similarly, the value of the observation variable $y(t)$ depends only on the value of the state variable $x(t)$. In the standard case in the hidden Markov model, the state space of the hidden variables is discrete, while the observed variables can be discrete or continuous (of normal distribution).

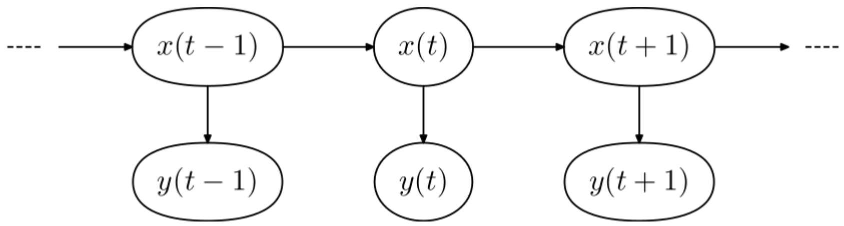


Figure 1: Hidden Markov model

In the Markov model, there are two types of parameters: the probability of transitions between states and the probability of outputs (observations) under the condition of hidden states. The transition probability controls the transition from state t-1 to state t. The output probability specifies the probability of occurrence of each member of the observation set for each possible hidden state that can follow a probability distribution. The number of members of the observation set depends on the nature of the observation variable [4]. The task of learning in a Markov model is to find the best probabilities of transitions and outputs based on a sequence or sequences of outputs. Usually, these parameters are estimated by the maximum likelihood method based on the given output. In order for the hidden Markov model to be expressed mathematically and computationally, the assumptions of the Markov sequence of states, the assumption of stationarity of the transfer probability and the assumption of independence of the output are considered [6].

In the field of hidden Markov models, there are three questions:

1. Having the observed data set and the hidden Markov model, how to calculate the probability of the observed vector?
2. With the observed data set and the hidden Markov model, how can the vector of hidden states corresponding to the observations be calculated?
3. Having the observed data set, how to calculate the parameters of the hidden Markov model in such a way that the probability of observing the output is maximized?

Forward algorithm or back entry is used to answer the first question. Viterbi algorithm is used to answer the second question. Although the answer to the first question is unique, but the answer to the current question is not unique. The Viterbi algorithm gives a sequence of states which is called the best fit. In response to the third question, no algorithm has been presented to find the maximum point. The Baum-Welch algorithm calculates the relative maximum point in an iterative process. This algorithm uses the maximum likelihood estimation method.

What has been said so far is the introduction of the hidden Markov model tool. In the following, the main idea of the research is expressed in the optimization of the stock portfolio. For this purpose, in the hidden Markov model, for each unobservable state, a mixed normal distribution is attributed, which represents the distribution of the returns under the condition of that state. The mixed normal distribution helps to fit and model the stock portfolio distribution well if it deviates from the normal distribution. A stock portfolio can be considered as a composite asset and its return can be calculated based on the historical data of its component assets. In this way, each selection of stock portfolio weights results in a time series of historical returns of the portfolio. If the hidden Markov model is used for the series of historical returns of the stock portfolio, the normal distribution of its mixture can be calculated for the investment horizon of the portfolio. By predicting the distribution of the stock portfolio in the investment horizon, its average and standard deviation can be extracted, which in turn causes the Sharpe ratio to be extracted. With the above statement, the weights of the stock portfolio can be optimized in such a way that the Sharpe ratio is maximized. In this way, the maximum predictability stock portfolio for the Sharpe ratio is formed with the hidden Markov model approach. The process of changing the weights until reaching the optimal weights can also be controlled with the help of a particle aggregation algorithm.

In the following, we refer to a number of researches conducted with the tool of hidden Markov model or Markov chains in forecasting and optimizing the stock portfolio. Park presented a method for predicting stock returns using Principal Component Analysis and Hidden Markov Model and tested stock trading results based on this approach. The results show that with appropriate hyperparameters, the resulting model has a higher annual Sharpe ratio than the buy-and-hold strategy [9] Jiang et al [4]. A deep learning framework based on the combination of hidden Markov model and short-term memory model- provided long term. Experiments on 6 real-world datasets show that the designed model achieves an average of 10% improvement on all datasets [4]. Majumder & Neerchal [7] fit a hidden Markov model to relative stock price changes for S&P 500 stocks from 2011 to 2016 based on weekly closing values. The state processes are correlated using a Gaussian copula so that the states of the component chains at any given time point are correlated. Su and Yi [13] extended the discrete hidden Markov model to the continuous hidden Markov model and then presented the upward and downward trend forecasting model based on the continuous model. The first-order continuous model is extended to the second-order continuous model, and the stock price is predicted by combining the fluctuation range prediction method. The evaluation results in six months of the Hong Kong stock market index show that the predicted value of the proposed model is very close to the actual value and it performs better than the three criteria in terms of the adjusted coefficient of determination. Lu et al [6] combined quantum computing and hidden Markov in the financial industry and proposed a new stock forecasting model based on the quantum hidden Markov algorithm for the first time. Coulombe & Goebel [2] developed a collaborative machine-learning algorithm that optimizes portfolio weights so that the resulting security is maximally predictable. For this purpose, a random forest is combined on one side of the equation, and a restricted ridge regression on the other side.

Harris [3] investigated the predictability of 10G currencies concerning lagged currency returns from the perspective of an American investor, using the maximum predictable portfolio approach. The proposed model has a higher Sharpe ratio, a higher cumulative return and a lower maximum withdrawal than a portfolio with an equal weight of currencies and an equally weighted portfolio of momentum trading strategies. Ta et al [14] used the maximum predictability portfolio optimization model to create an investment portfolio. This model can capture the predictability of the market and avoid forecasting errors, which is a serious obstacle for the mean-variance framework. The obtained results show that the maximum predictability portfolio optimization model performs better than the traditional mean-variance Markowitz model, thus partially showing its potential for further development in the progress of portfolio optimization. Pasricha et al [10] presented a semi-Markov process-based approach for the optimal selection of a portfolio of bonds. The credit portfolio optimization criteria are based on infinite soft, and the proposed optimization model is transformed into a linear programming problem assuming that the bond credit rating follows a semi-Markov process. In the end, the portfolio is optimized on 10 bonds with a maturity of 9 years. Burkett et al [1] presented an innovative new approach to portfolio design that applies a discrete, state-based method to define market states and asset allocation decisions with respect to current and future state membership. The transition dynamics of derived states are modelled as a Markov process. Asset weighting and portfolio allocation decisions are made through an approach based on heuristic optimization. The research portfolio is optimized on a 7-share portfolio between 2004 and 2019. Ryou et al [12] proposed an investment strategy that uses a hidden Markov model to select stocks in bullish mode. By identifying the stock position, it buys stocks in a rising position and rebalances after the holding period. This study shows that investment strategy using the hidden Markov model is useful in the Korean stock market. Ruiz et al [11] proposed a trading strategy for a stock portfolio, the proposed strategy is mainly based on the key-means clustering algorithm to determine and learn internal hidden patterns in the time series of stock market prices, the predictive algorithm based on a simple Markov chain and a system Fuzzy inference is for making decisions in transactions. The performance of the trading algorithm is confirmed through simulation using the real prices of the Mexican stock exchange. Petković et al [8] optimized the stock portfolio in the Belgrade Stock Exchange and for this purpose, they used the Markov chain method as a simple and non-parametric method. In the end, considering the three cases, stocks will be present in the portfolio that will maximize the probability of being in the range of expected returns.

## 3 Research model

In this section, we introduce the optimal portfolio selection model of the research.
**Assumptions:**

- The portfolio consists of $N$assets $(1, 2, \cdots, N)$.
- $rit$ denotes the historical return series of asset $i$ for time periods $(1, 2, \cdots, T)$.
- $w1, w2, \cdots, wN$ are the portfolio weights, such that $\sum_{i=1}^{N} wi = 1$.

**Historical portfolio return:**
The historical return of the portfolio is calculated using Equation (3.1):

$$r_t = w_1 r_{1t} + w_2 r_{2t} + \cdots + w_N r_{Nt}, \qquad t \in \{1, 2, \cdots, T\} \tag{3.1}$$

For the resulting time series $\{r_t\}_{t=1}^{T}$, a hidden Markov model is trained. The output of the model for each hidden state in the hidden Markov model is a one-dimensional mixed normal random variable that represents the distribution of the return of the stock portfolio under the condition of that state. The reason for using the mixed normal distribution is the proper fit of the distribution in case of yield data deviation from the normal distribution. By training the hidden Markov model, the mean and standard deviation of the portfolio return distribution can be extracted as criteria for profitability and risk for the future time step (stock portfolio horizon). By predicting these two values, the Sharpe ratio is calculated by dividing the yield by the standard deviation. The maximum predictability portfolio of the Sharpe ratio in the framework of the hidden Markov model by changing the stock portfolio weights vector, seeks to find the optimal weights that maximize the predicted Sharpe ratio. Based on what was stated, the optimal stock portfolio model of the research is as (3.2).

$$\begin{aligned}
\max_{w_1, w_2, \cdots, w_N} \quad & \text{Sharpe ratio}_{T+1} = \frac{\mu_{T+1}(\{r_t\}_{t=1}^{T})}{\sigma_{T+1}(\{r_t\}_{t=1}^{T})} \\
& \text{s.t :} \\
& r_t = w_1 r_{1t} + w_2 r_{2t} + \cdots + w_N r_{Nt}, \quad t \in \{1, 2, \cdots, T\} \\
& \mu_{T+1}(\{r_t\}_{t=1}^{T}) \text{ is the mean of mixture of normals distribution} \\
& \text{for time } T+1 \text{ predicted by HMM} \\
& \sigma_{T+1}(\{r_t\}_{t=1}^{T}) \text{ is the} s\tan \text{ dard deviation of mixture of normals distribution} \\
& \text{for time } T+1 \text{ predicted by HMM} \\
& \sum_{i=1}^{N} w_i = 1 \\
& \forall i \ w_i \geq 0
\end{aligned} \tag{3.2}$$

In model (3.2), $\mu_{T+1}$ is the average of the mixed normal distribution predicted by the hidden Markov model and $\sigma_{T+1}$ is its standard deviation for the investment horizon. Particle swarm optimization algorithm is used to optimize model (3.2). This algorithm consists of a mass of particles. Each particle is settled in a region of the search space. The value of the objective function for each particle shows the degree of fitness of that particle's location. Particles in the search area move at a certain speed. The speed (direction and speed value) of each particle is under two factors. One is the best experience that the particle has had so far (the best value of fitness it has had so far) and the other factor is the best experience that neighbouring particles have had so far, and finally, the movement of particles will converge towards the optimal point. In the optimization of model (3.2), each measurement represents the weights of the stock portfolio. In this way, the cumulative particle algorithm calculates the optimal stock portfolio by changing the weights of the stock portfolio over several generations.

## 4 Research findings

In this section, the details of the implementation of the stock portfolio selection model are discussed in the study of the Tehran Stock Exchange. The stock portfolio of the research consists of 10 assets according to table 1, where each asset is an index of the Tehran Stock Exchange. Choosing an index as an asset means a diverse stock portfolio of stocks under that index. As an example, the index of metal minerals is a diverse portfolio of stocks under that index. In this way, the study portfolio is a diverse portfolio.

Table 1: List of research portfolio assets

| **Asset number** | **Index title** | **Asset number** | **Index title** |
|---|---|---|---|
| 1 | Metallic minerals | 6 | Chemical |
| 2 | Non-metallic minerals | 7 | medicine |
| 3 | oil products | 8 | machinery |
| 4 | car | 9 | Foodstuffs |
| 5 | Cement | 10 | technical |

The historical data of 10 assets were extracted from the beginning of 2018 to the end of 2011, which includes 3442 daily data. The time horizon of the stock portfolio is weekly, where each week is equivalent to 5 working days. Based on this, the number of historical weekly returns is 688 data, of which 588 data were used as training data to train the hidden Markov model and 100 data were used to test the profitability of the stock portfolio. The hidden Markov model used was designed according to the parameters of table 2 in MATLAB software and HMM software package.

Table 2: Hidden Markov model parameters

| **parameter** | **amount** |
|---|---|
| Output type | The mixed normal distribution consists of the convex combination of two normal distributions |
| Number of training data | 588 |
| Type and number of mode space | Discrete and includes 4 modes |
| Initial distribution of states | $A = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$ |

After extracting the data and setting the hidden Markov model, optimization of the stock portfolio selection model (3.2) was done with the help of particle cumulative algorithm in MATLAB software with 1000 particles and 1000 repetitions, and constraints were added as penalties to the objective function. As seen, the output of the normal model is mixed with two Gaussian distributions. Table 3 shows the mean of the normal distribution with two mixed components and four states.

Table 3: The average distribution of output observations according to the hidden state in two mixed normal components

| Status / component | Status 1 | Status 2 | Status 3 | Status 4 |
|---|---|---|---|---|
| The first component | 0.00902159 | -0.0249263 | -0.0060606 | -0.0023457 |
| The second component | 0.03719011 | 0.00034018 | 0.0000534 | 0.06348524 |

Also, the variance of the normal distribution of the output mixture in two components and four states is presented in table 4.

Table 4: output variance according to the hidden state in the first component of the Gaussian mixture

| Status / component | Status 1 | Status 2 | Status 3 | Status 4 |
|---|---|---|---|---|
| first component | 0.0000128 | 0.0000055 | 0.00000832 | 0.0000644 |
| second component | 0.0000186 | 0.0002541 | 0.00238921 | 0.0000688 |

The combination coefficients (convex coefficients that combine the two components of the mixed normal distribution) are presented in table 5:

Finally, the optimal transfer matrix of four states in the designed hidden Markov model is also presented in Table 6.

After calculating the optimal stock portfolio, the weekly returns obtained from the optimal stock portfolio on 100 test data are presented in graph 2.

Table 5: normal mixture coefficients

| Status \ component | first component | second component |
|---|---|---|
| Status 1 | 0.81396745 | 0.18603255 |
| Status 2 | 0.60547567 | 0.39452433 |
| Status 3 | 0.54695714 | 0.45304286 |
| Status 4 | 0.06364043 | 0.93635957 |

Table 6: state transfer matrix

| Status | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0.292270864 | 0.19702078 | 0.372172309 | 0.13853605 |
| 2 | 0.234671121 | 0.42547715 | 0.061834724 | 0.278017 |
| 3 | 0.276933712 | 0.11385168 | 0.451352968 | 0.15786164 |
| 4 | 0.126402608 | 0.05194813 | 0.155911635 | 0.66573762 |

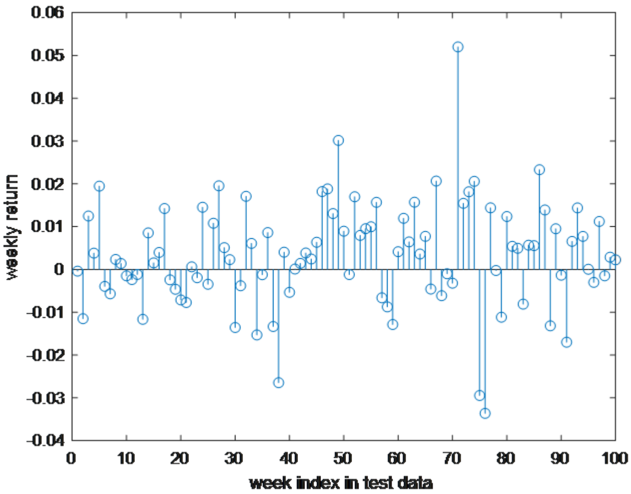


Figure 2: Weekly returns of the optimal research portfolio on test data

Based on the returns obtained on the test data, the profitability performance was calculated in the average criteria of weekly return, standard deviation and Sharpe ratio in table 7. In order to validate the model, two basic stock portfolio models of equal weight and normal mean variance were used. In the equal weight model, the weight of all assets is assumed to be the same, which means that this portfolio does not use any information.

Table 7: Profitability performance of the research portfolio

| portfolio \ Criterion | Research portfolio | Equal weight basket | Mean-variance basket |
|---|---|---|---|
| Average weekly return | 0.0035 | 0.0033 | 0.0031 |
| standard deviation | 0.00901 | 0.0123 | 0.0124 |
| Sharpe ratio | 0.3846 | 0.2687 | 0.2500 |

Based on the results obtained in table 7, the Sharpe ratio of the equal weight and mean-variance model is almost the same. This is even though the research model in the sample portfolio has a higher Sharpe ratio than the last two models.

## 5 Conclusions and suggestions

The goal of an investor is to be able to form a suitable investment portfolio by analyzing the investment environment in order to achieve his goals in achieving a desired return or final wealth. For this purpose, various models have been developed in the stock portfolio optimization literature. The major group of investors are risk averse. This does not mean that they do not want to take risks, but rather that they want to receive a return commensurate with the risk they bear. In fact, they seek to take as much risk as possible. Sharpe ratio is a measure of risk-adjusted return and is especially important for risk-averse investors. In the present research, this criterion was proposed as an objective function, and it was predicted with the help of the hidden Markov model tool as a non-linear model. Thus, in the present research, the optimal stock portfolio with the maximum predictability of the Sharpe ratio was introduced with the hidden Markov model approach. The results of the profitability evaluation of the research model in the sample portfolio of the research show that this portfolio can increase the Sharpe ratio by at least 40% compared to the normal average-variance and equal-weight portfolios. Therefore, the use of this portfolio model is recommended for risk-averse investors.

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