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Generalizations of the Hilbert-Weierstrass theorem and Tonelli-Morrey theorem: The regularity of solutions of differential equations and optimal control problems

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Abstract

One of the basic problems in the "Calculus of Variations" is the minimization of the following functional:

$$F(x)=\int_a^b f(t,x(t),x'(t))dt,$$

over a class of functions x defined on the interval $[a,b]$. According to a regularity theorem, solutions to this fundamental problem are found in a smaller class of more regular functions. However, they were originally considered to belong to a larger class. In this context, two theorems attributed to "Hilbert-Weierstrass" and "Tonelli-Morrey" are two classical studies of the regularity of discussion for the solutions to this problem. As higher-order differential equations and higher-order optimal control problems become more prevalent in the literature, regularity issues for these problems should receive more attention. Therefore, a generalization of the above regularity theorems is presented here, namely the regularity of solutions to the following functional

$$F(x)=\int_a^b f(t,x(t),x'(t),\dots,x^{(n-1)}(t))dt$$

where $n\geq 2$. It is expected that this extension will be helpful in discussing the regularity of higher-order differential equations and optimal control problems.

Keywords: Boundary value problems, Classical solution, Regularity, Weak solution.
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1 Introduction

"Differential equations" play a key role in explaining how the physical world works. Systems of ordinary differential equations of the form

$$F(x,y,y',\dots,y^{(n-1)})=y^{(n)} \tag{1.1}$$

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are routinely used to model a wide variety of phenomena in fields as diverse as aeronautics, power generation, robotics, economic growth, and natural resources. To solve (1.1), one must find a $y \in C^n$ that satisfies the equation. In some exceptional cases, there are direct methods to obtain the exact solution (e.g., Bernoulli equations, etc.), which are usually treated in books on elementary differential equations (see, e.g., [16, 17]). In real applications, however, most differential equations have more complicated forms. Therefore, various numerical methods are used to approximate the solution of these problems. Alongside that, there are theoretical approaches that deal with the existence and the number of solutions, as well as with the analysis of the properties of the solutions of differential equations. These theoretical efforts pave the way for the numerical tasks.

An approach to theoretical work on the existence of solutions involves studying the solution in a space larger than C^n, namely $W^{n-1,2}$, which is the space of all $n-1$ weakly differentiable functions in L^2. In fact, the problem of finding a solution in the space C^n is replaced by the problem of finding a weak solution in the space $W^{n-1,2}$, a space that is reflexive and a much larger space than C^n. Subsequently, it is easier to study the problem of the existence of solutions from the point of view of the use of theorems of mathematical analysis. Having proved the existence of a weak solution in $W^{n-1,2}$, it remains only to prove that the weak solution belongs to C^n; this is called "the regularity of the weak solution". Also, proving that the solutions in C^{n-1} belong to C^n is called "the regularity of the classical solutions".

In the early days of differential equation theory, only differential equations of order 2 were the focus of interest. In other words, the mathematical interpretation of most applied problems in physics and engineering involved quadratic differential equations. Therefore, the regularity of the solutions of this type of differential equations was at the center of attention in the literature.

Among these efforts, [7, Theorems 7.1.13 and 7.1.14] are worth mentioning concerning the regularity of the classical solution and the regularity of the weak solution of a quadratic ordinary differential equations category. Theorem 7.1.13, a classical result of Hilbert and Weierstrass, dating back to around 1875, has appeared in countless books on the calculus of variations since then. It was initially documented in the lecture notes of Weierstrass circulating at that time. Another form of this theorem can be found, for example, in [4, Theorem 15.7]. There are also some explanations about it in Goldstein's 1980 book [9] (a history of the Calculus of Variations).

Theorem 7.1.14 is a version of what is known in the literature as the "Tonelli-Morrey" approach to regularity. It goes back to Tonelli's seminal 1921 book "Fondamenti del calcolo delle variazioni" [18] and Morrey's 1966 book [15]. A corresponding discussion can also be found in Chapter 16 of [4]: To recover Theorem 7.1.14, first use Theorem 16.13 to obtain Lipschitz regularity (see also the remark at the end of page 329), then Theorem 15.5 to obtain C^1, and then Theorem 15.7 for higher regularity. The approach in these theorems is to examine the *energy functional* corresponding to the differential equation under consideration and to apply the fact that the local extremum of the energy functional is a weak solution of the differential equation and vice versa. It is thus shown that if x is a local extremum of the energy functional, then $x \in C^2$.

Now, since the use of differential equations of order greater than 2 has increased due to their various interpretations in practical problems. (see, e.g., [10, 13, 14, 20]), theoretical discussions of this type of differential equations should be addressed more extensively. Therefore, here, we present the general form of ordinary differential equations of order $n \geq 2$ and prove the regularity of the weak solutions, taking into account the known existence of some solutions. In fact, we generalize the approach presented in [7, Theorem 7.1.13 and Theorem 7.1.14].

The following format can express the general form of the boundary value problems for the n-th order ordinary differential equations:

$$\begin{cases} G(t, x(t), x'(t), \ldots, x^{(n)}(t)) = 0, & t \in (a, b), \\ x^{(i)}(a) = u_i & \text{for} \quad i \in N, \\ x^{(j)}(b) = w_j & \text{for} \quad j \in N', \end{cases} \tag{1.2}$$

where $x^{(i)}$ is the i-th derivative of the function x, $u_i, w_j \in \mathbb{R}$ for $i \in N, j \in N'$, $n \geq 2$ and $N, N' \subseteq \{0, 1, \ldots, n-1\}$. We are interested in providing regularity results for solutions to problem (1.2). As mentioned earlier, the preoccupation with the existence of solutions to these equations has increased since applied interpretations for ODEs with degrees larger than 2 have come to the fore. Thus, in parallel with the existence results, some effort should be made to find regularity results for these equations. Therefore, this work aims to provide regularity results for the solution of problem (1.2). For this purpose, we consider the fact that every critical point of the energy functional F corresponds to a weak solution of (1.2), and vice versa:

$$F(x) = \int_a^b f(t, x(t), x'(t), \ldots, x^{(n-1)}(t))dt, \quad x \in \mathcal{N}_{N,N'} \tag{1.3}$$

where $\mathcal{N}_{N,N'} = \{u \in W^{n-1,2}(a,b) : u^{(i)}(a) = u_i, u^{(j)}(b) = w_j \text{ where } i \in N, j \in N'\}$ and f is a function defined on $[a,b] \times \mathbb{R}^n$ with continuous second partial derivatives with respect to all its variables.

The approach in this paper is to show that if u_0 is a local minimum of F, then $u_0 \in C^n[a,b]$. This work has already been done for $n = 2$ and $N, N' = \{0\}$ (see Theorems 7.1.13 and 7.1.14 from [7]). Here, we prove the general case; namely $n \geq 2$ and $N, N' \subseteq \{0, 1, \dots, n-1\}$.

The structure of the paper is as follows: Section 2 presents the "regularity of the classical solutions" for the functional F in (1.3) . In Section 3, it will be shown that if u is a local minimum of F in (1.3) with respect to $W^{n-1,2}$, then u is in C^n. Sections 4 illustrates the importance of the theorems in sections 2 and 3 through various examples.

2 Regularity of the classical solution

To deduce that the solution of the notational map of (1.2) is not only of class C^{n-1} but also lies in C^n, we have to make some requirements. We start with the following lemma:

Lemma 2.1. Suppose that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a function whose partial derivatives exist and are continuous around the point $(x_0, x_1, x_2, \dots, x_n) \in \mathbb{R}^{n+1}$. Then

$$\lim_{r\to 0} \frac{f(x_0, x_1 + rm_1, x_2 + rm_2, \dots, x_n + rm_n) - f(x_0, x_1, x_2, \dots, x_n)}{r} = \sum_{i=1}^{n} m_i \frac{\partial f}{\partial x_i}(x_0, x_1, \dots, x_n).$$

Proof . First, note the following equality

$$\begin{aligned} &f(x_0, x_1 + rm_1, x_2 + rm_2, \dots, x_n + rm_n) - f(x_0, x_1, x_2, \dots, x_n) \\ =&f(x_0, x_1 + rm_1, x_2, \dots, x_n) - f(x_0, x_1, x_2, \dots, x_n) \\ +&\sum_{i=2}^{n} [f(x_0, x_1 + rm_1, \dots, x_i + rm_i, x_{i+1}, x_{i+2}, \dots, x_n) \\ &\quad - f(x_0, x_1 + rm_1, \dots, x_{i-1} + rm_{i-1}, x_i, x_{i+1}, \dots, x_n)]. \end{aligned}$$

Then, by this the proof comes from the following fact:

Fact: *Suppose that $j \in \{1, \dots, n\}$ and $m_i, t_i \in \mathbb{R}$; $1 \leq i \leq n$. If $m_i = t_i$ for $i \neq j$ and $m_j \neq t_j = 0$, then*

$$\lim_{r\to 0} \frac{f(x_0, x_1 + rm_1, \dots, x_n + rm_n) - f(x_0, x_1 + rt_1, \dots, x_n + rt_n)}{r} = m_j \frac{\partial f}{\partial x_j}(x_0, x_1, \dots, x_n).$$

Proof . Given that $m_i = t_i$ for $i \neq j$ and $m_j \neq t_j = 0$, we can simplify the expression within the limit. The terms involving m_i and t_i for $i \neq j$ cancel out, and we are left with

$$\begin{aligned} &\lim_{r\to 0} \frac{f(x_0, x_1 + rm_1, \dots, x_n + rm_n) - f(x_0, x_1 + rt_1, \dots, x_n + rt_n)}{r} \\ &= \lim_{r\to 0} \frac{f(x_0, x_1, \dots, x_j + rm_j, \dots, x_n) - f(x_0, x_1, \dots, x_j, \dots, x_n)}{r} \\ &= \lim_{r\to 0} \frac{f(x_0, x_1, \dots, x_j + rm_j, \dots, x_n) - f(x_0, x_1, \dots, x_j, \dots, x_n)}{rm_j} \cdot m_j. \end{aligned}$$

By the definition of the partial derivative, we know that

$$\frac{\partial f}{\partial x_j}(x_0, x_1, \dots, x_n) = \lim_{r\to 0} \frac{f(x_0, x_1, \dots, x_j + rm_j, \dots, x_n) - f(x_0, x_1, \dots, x_j, \dots, x_n)}{rm_j}.$$

Therefore, we have

$$\lim_{r\to 0} \frac{f(x_0, x_1 + rm_1, \dots, x_n + rm_n) - f(x_0, x_1 + rt_1, \dots, x_n + rt_n)}{r} = m_j \frac{\partial f}{\partial x_j}(x_0, x_1, \dots, x_n).$$

□

This concludes the proof. □

Moreover, the lemma presented below is derived as a straightforward application of the Implicit Function Theorem, as detailed in [12]. This lemma is essential for our further analysis:

Lemma 2.2. Assume that $\varphi := \varphi(t,s)$ is a function from $[a,b] \times \mathbb{R}$ to $\mathbb{R}$ such that

(i) $\varphi(t_0, s_0) = 0$;

(ii) $\frac{\partial \varphi}{\partial s}(t_0, s_0) \neq 0$;

(iii) $\varphi, \frac{\partial \varphi}{\partial s}$ are continuous in t_0.

Then,

$$\exists \delta_1, \hat{\delta} > 0 \ s.t.\ \forall t \in (t_0 - \delta_1, t_0 + \delta_1)\ \exists! z(t) \in (s_0 - \hat{\delta}, s_0 + \hat{\delta}),\ \varphi(t, z(t)) = 0.$$

Moreover, the function $t \longrightarrow z(t)$ is continuous on $(t_0 - \delta_1, t_0 + \delta_1)$. Note that the symbol "$\exists! x,\ P(x)$" stands for "there exists a unique x satisfying P(x)", or "there is exactly one x for which P(x) holds".

We recall the Fundamental Lemma in the Calculus of Variations, as established by du Bois-Reymond [8] (also see [7, Lemma 7.1.9]):

Lemma 2.3. Let $\mathcal{I}$ be an open interval and f be a function in $L^1_{loc}(\mathcal{I})$. If for any function φ in $C_0^\infty(\mathcal{I})$, the following condition holds:

$$\int_{\mathcal{I}} f(x)\varphi'(x)\, dx = 0,$$

then it can be concluded that f is almost everywhere constant within $\mathcal{I}$.

Here, $C_0^\infty(\mathcal{I})$ denotes the set of all infinitely differentiable functions on $\mathcal{I}$ that vanish, along with their derivatives, at the endpoints of $\mathcal{I}$. The statement "almost everywhere constant" implies that f is constant throughout $\mathcal{I}$, except possibly on a set of measure zero.

We introduce a generalization of the Fundamental Lemma in the Calculus of Variations, crucial for establishing Theorem 2.5. This lemma asserts that under specific integral conditions, a function f behaves almost like a polynomial within a given interval.

Lemma 2.4. Let $\mathcal{M} = \{u \in C^n[a,b] : u^{(i)}(a) = u_i, u^{(i)}(b) = w_i;\ 0 \leq i \leq n\}$ and $f^{(n-1)} \in L^1_{loc}(a,b)$ for $n \geq 2$. If for every $V \in \mathcal{M}$,

$$\int_a^b f(t)V^{(n)}(t)\, dt = c,$$

then f is almost everywhere a polynomial of degree $n-1$ in $[a,b]$; i.e., there exist constants $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$ such that:

$$f(t) = c_{n-1}t^{n-1} + \cdots + c_1 t + c_0 \quad \text{almost everywhere in } [a,b].$$

Proof . We begin by defining an auxiliary set $\mathcal{M}' = \{u \in C^n[a,b] : u^{(i)}(a) = u^{(i)}(b) = 0;\ 0 \leq i \leq n\}$ and functions $V_0(t)$ and $G(t; A, B)$ as follows:

$$V_0(t) = \frac{w_0 - u_0}{b-a}(t-a) + u_0,$$
$$G(t; A, B) = \frac{B-A}{b-a}(t-a) + A.$$

Iteratively defining $V_n(t) = V_{n-1}(t) + G_n(t)(V_0(t) - u_0)^n(V_0(t) - w_0)^n$, where $G_n(t) = G(t; A_n, B_n)$ and

$$A_n = (-1)^n \frac{(b-a)^n}{n!(w_0 - u_0)^{2n}}(u_n - V_{n-1}^{(n)}(a)),$$
$$B_n = \frac{(b-a)^n}{n!(w_0 - u_0)^{2n}}(w_n - V_{n-1}^{(n)}(b)),$$

we conclude that $V_n + \mathcal{M}' = \mathcal{M}$. Thus, for all $V \in \mathcal{M}'$,

$$\int_a^b f(t)[V^{(n)}(t) + V_n^{(n)}(t)]\, dt = c.$$

Therefore, for all $V \in \mathcal{M}'$,
$$\int_a^b f(t)V^{(n)}(t)\,dt = c - \int_a^b f(t)V_n^{(n)}(t)\,dt := c'.$$
Since $\alpha\mathcal{M}' = \mathcal{M}'$ for $\alpha \neq 0$, we deduce for all $V \in \mathcal{M}'$,
$$\int_a^b f(t)V^{(n)}(t)\,dt = \frac{c'}{\alpha}.$$
Hence, $\frac{c'}{\alpha} = c'$, implying $c' = 0$. By integrating by parts iteratively for all $V \in \mathcal{M}'$, we obtain
$$\int_a^b f^{(n-1)}(t)V'(t)\,dt = 0.$$
Consequently, as $C_0^\infty(a,b) \subseteq \mathcal{M}'$, by Lemma 2.3, we deduce that $f^{(n-1)}$ is a constant almost everywhere in $[a,b]$. Thus, f is a polynomial of degree $n-1$ almost everywhere in $[a,b]$. □

The theorem we present next deals with the regularity of functional extrema, which forms the basis for our analysis. This theorem deepens our understanding of the intricate interplay between the functional properties and regularity of extreme points. It shows the nuanced relationship between local extremes of functionals and the smoothness of the functions that reach these extremes.

Theorem 2.5. Suppose $n \geq 1$, $N, N' \subseteq \{0, 1, 2, \ldots, n\}$, and
$$\mathcal{M}_{N,N'} = \{u \in C^n[a,b] : u^{(i)}(a) = u_i, u^{(j)}(b) = w_j \textit{ where } i \in N, j \in N'\}.$$
Define the functional F on $\mathcal{M}_{N,N'}$ by
$$F(u) = \int_a^b f(t, u(t), u'(t), \ldots, u^{(n)}(t))dt$$
where $f = f(x_1, \ldots, x_{n+2})$ is a function defined on $[a,b] \times \mathbb{R}^{n+1}$ with continuous second partial derivatives with respect to all its variables. Let $u_0 \in \mathcal{M}_{N,N'}$ be a local extremum of F with respect to $\mathcal{M}_{N,N'}$, and let $t_0 \in (a,b)$ be such that
$$\frac{\partial^2 f}{\partial x_{n+2}^2}(t_0, u_0(t_0), u_0'(t_0), \ldots, u_0^{(n)}(t_0)) \neq 0.$$
Then there exists $\delta > 0$ such that $u_0 \in C^{n+1}(t_0 - \delta, t_0 + \delta)$.

Proof . Let $V \in \mathcal{M}_{N,N'}$. Then by Lemma 2.1,
$$\begin{aligned}
\delta F(u_0; V) &= \lim_{r \to 0} \frac{F(u_0 + rV) - F(u_0)}{r} \\
&= \int_a^b \lim_{r \to 0} \frac{f(t, u_0(t) + rV(t), u_0'(t) + rV'(t), \ldots, u_0^{(n)}(t) + rV^{(n)}(t)) - f(t, u_0(t), \ldots, u_0^{(n)}(t))}{r} dt \\
&= \int_a^b \sum_{i=2}^{n+2} \frac{\partial f}{\partial x_i}(t, u_0(t), \ldots, u_0^{(n)}(t))V^{(i-2)}(t)dt.
\end{aligned}$$
Therefore, by Euler Necessary Condition,
$$\delta F(u_0; V) = 0 \quad \text{for} \quad V \in \mathcal{M}_{N,N'}.$$
Consequently,
$$\sum_{i=2}^{n+2} \int_a^b \frac{\partial f}{\partial x_i}(t, u_0(t), \ldots, u_0^{(n)}(t))V^{(i-2)}(t)dt = 0 \quad \text{for} \quad V \in \mathcal{M}_{N,N'}.$$

Define $h_{0,j}(t) := \frac{\partial f}{\partial x_j}(t, u_0(t), \ldots, u_0^{(n)}(t))$ for $2 \leq j \leq n+2$ and $h_{k,j}$ iteratively as follows:

$$h_{k,j}(t) := \int_a^t h_{k-1,j}(\xi) d\xi \quad \text{for} \quad k \geq 1,\ 2 \leq j \leq n+2.$$

Integrating by parts iteratively implies that

$$\int_a^b \frac{\partial f}{\partial x_{n-m+2}}(t, u_0(t), \ldots, u_0^{(n)}(t)) V^{(n-m)}(t) dt = \sum_{j=1}^{m} (-1)^{j+1} h_{j,n-m+2}(b) V^{(j+n-m-1)}(b) + (-1)^m \int_a^b h_{m,n-m+2}(t) V^{(n)}(t) dt \quad \text{for} \quad 1 \leq m \leq n.$$

Let's delve deeper into the process of integrating by parts iteratively, a key step in our proof. We aim to transform the integral

$$\int_a^b \frac{\partial f}{\partial x_{n-m+2}}(t, u_0(t), \ldots, u_0^{(n)}(t)) V^{(n-m)}(t)\, dt$$

into a more useful form.

1. **Initial Setup**: We recognize that $h_{0,n-m+2}(t) = \frac{\partial f}{\partial x_{n-m+2}}(t, u_0(t), \ldots, u_0^{(n)}(t))$ and integrate it with $V^{(n-m)}(t)$.
2. **First Iteration**:
   - Apply integration by parts: $\int u\, dv = uv - \int v\, du$.
   - Set $u = V^{(n-m)}(t)$ and $dv = h_{0,n-m+2}(t)\, dt$.
   - Then $du = V^{(n-m+1)}(t)\, dt$ and $v = \int_a^t h_{0,n-m+2}(\xi)\, d\xi = h_{1,n-m+2}(t)$.
   - This yields $\int_a^b h_{0,n-m+2}(t) V^{(n-m)}(t)\, dt = h_{1,n-m+2}(t) V^{(n-m)}(t) \Big|_a^b - \int_a^b h_{1,n-m+2}(t) V^{(n-m+1)}(t)\, dt$.
3. **Subsequent Iterations**:
   - Continue this process. In each step, the derivative order of V increases by 1, and the index of h increases by 1.
   - After m iterations, the integral transforms into a sum of boundary terms and a final integral.
4. **Final Formulation**:
   - The boundary terms from each step contribute to the sum $\sum_{j=1}^m (-1)^{j+1} h_{j,n-m+2}(b) V^{(j+n-m-1)}(b)$.
   - The last integral term is $(-1)^m \int_a^b h_{m,n-m+2}(t) V^{(n)}(t)\, dt$.

By this detailed iterative process, we successfully express the original integral in terms of the function $h_{m,n-m+2}(t)$ and the derivatives of V, thus facilitating further analysis in our proof. Consequently, since

$$\begin{aligned}\sum_{m=1}^{n} \int_a^b \frac{\partial f}{\partial x_{n-m+2}}(t, u_0(t), \ldots, u_0^{(n)}(t)) V^{(n-m)}(t) dt &= -\int_a^b \frac{\partial f}{\partial x_{n+2}}(t, u_0(t), \ldots, u_0^{(n)}(t)) V^{(n)}(t) dt \\ &= -\int_a^b h_{0,n+2}(t) V^{(n)}(t) dt,\end{aligned}$$

for any $V \in \mathcal{M}_{N,N'}$

$$\int_a^b h_{0,n+2}(t) V^{(n)}(t) dt + \sum_{m=1}^{n} \left(\sum_{j=1}^{m} (-1)^{j+1} h_{j,n-m+2}(b) V^{(j+n-m-1)}(b) + (-1)^m \int_a^b h_{m,n-m+2}(t) V^{(n)}(t) dt \right) = 0,$$

or

$$\sum_{m=1}^{n} \left(\sum_{j=1}^{m} (-1)^{j+1} h_{j,n-m+2}(b) V^{(j+n-m-1)}(b) \right) + \sum_{m=0}^{n} \left((-1)^m \int_a^b h_{m,n-m+2}(t) V^{(n)}(t) dt \right) = 0.$$

Since $\mathcal{M} := \mathcal{M}_{\{0,1,\ldots,n\},\{0,1,\ldots,n\}} \subseteq \mathcal{M}_{N,N'}$, we also have the above equality for each $V \in \mathcal{M}$. Let us define

$$c := -\sum_{m=1}^{n}\left(\sum_{j=1}^{m}(-1)^{j+1}h_{j,n-m+2}(b)V^{(j+n-m-1)}(b)\right).$$

The constant c represents the summation of boundary terms arising from the integration by parts. Consequently, we obtain

$$\forall V \in \mathcal{M}, \int_a^b \left(\sum_{m=0}^{n}(-1)^m h_{m,n-m+2}(t)\right) V^{(n)}(t)dt = c.$$

By Lemma 2.4, we conclude that there exist constants $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$ such that

$$\sum_{m=0}^{n}(-1)^m h_{m,n-m+2}(t) = c_{n-1}t^{n-1} + \cdots + c_1 t + c_0 \quad \text{a.e. in} \quad [a,b].$$

This is an important result showing that the variational derivative leads to a polynomial expression. Since $u_0 \in C^n[a,b]$, the function $u_0^{(n)}$ is continuous. Therefore, we have

$$\frac{\partial f}{\partial x_{n+2}}(t, u_0(t), \ldots, u_0^{(n)}(t)) + \sum_{m=1}^{n}(-1)^m h_{m,n-m+2}(t) = c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$$

for all $t \in [a,b]$. Define the function φ by

$$\varphi(t,s) = \frac{\partial f}{\partial x_{n+2}}(t, u_0(t), \ldots, u_0^{(n-1)}(t), s) + \sum_{m=1}^{n}(-1)^m h_{m,n-m+2}(t) - c_{n-1}t^{n-1} - \cdots - c_1 t - c_0.$$

The function φ satisfies the following properties:

(i) $\varphi(t_0, u_0^{(n)}(t_0)) = 0$.

(ii) $\frac{\partial \varphi}{\partial s}$ and $\frac{\partial \varphi}{\partial t}$ exist and are continuous.

(iii) $\frac{\partial \varphi}{\partial s}(t_0, u_0^{(n)}(t_0)) = \frac{\partial^2 f}{\partial x_{n+2}^2}(t_0, u_0(t_0), \ldots, u_0^{(n-1)}(t_0), u_0^{(n)}(t_0)) \neq 0$.

Therefore, Lemma 2.2 implies that there exist $\delta_1, \delta_2 > 0$ such that for all $t \in (t_0 - \delta_1, t_0 + \delta_1)$ there exists a unique $z(t) \in (u_0^{(n)}(t_0) - \delta_2, u_0^{(n)}(t_0) + \delta_2)$, such that

$$\varphi(t, z(t)) = 0, \qquad z \in C^1(t_0 - \delta_1, t_0 + \delta_1), \quad \text{and} \quad z(t_0) = u_0^{(n)}(t_0).$$

On the other hand, the continuity of $u_0^{(n)}$ implies that there is a $\delta_0 > 0$ such that

$$\forall t \in (t_0 - \delta_0, t_0 + \delta_0);\ u_0^{(n)}(t) \in (u_0^{(n)}(t_0) - \delta_2, u_0^{(n)}(t_0) + \delta_2).$$

Therefore, if $\delta := \min\{\delta_0, \delta_1\}$, for every $t \in (t_0 - \delta, t_0 + \delta)$, we have

$$u_0^{(n)}(t) \in (u_0^{(n)}(t_0) - \delta_2, u_0^{(n)}(t_0) + \delta_2)$$

and also

$$\exists! z(t) \in (u_0^{(n)}(t_0) - \delta_2, u_0^{(n)}(t_0) + \delta_2),\ \varphi(t, z(t)) = 0.$$

Since $\varphi(t, u_0^{(n)}(t)) = 0$ for $t \in (a,b)$, we have

$$u_0^{(n)}(t) = z(t) \quad \text{for} \quad t \in (t_0 - \delta, t_0 + \delta)$$

and consequently, since $z \in C^1(t_0 - \delta_1, t_0 + \delta_1)$, we conclude that

$$u_0^{(n)} \in C^1(t_0 - \delta, t_0 + \delta),$$

and therefore

$$u_0 \in C^{n+1}(t_0 - \delta, t_0 + \delta).$$

□

3 Regularity of the weak solution

The following lemmas are necessary to prove Theorem 3.4, the main theorem in this section.

Lemma 3.1. Let Ω be an open set in $\mathbb{R}$. Suppose $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ has the following properties:

(i) for all $(y_1, \dots, y_n) \in \mathbb{R}^n$, the function $x \mapsto f(x, y_1, \dots, y_n)$ is measurable on Ω;
(ii) for a.a. (almost all in the sense of the Lebesgue measure) $x \in \Omega$, the function $(y_1, \dots, y_n) \mapsto f(x, y_1, \dots, y_n)$ is continuous on $\mathbb{R}^n$.

If $\varphi_i : \Omega \to \mathbb{R}$ for $i = 1, \dots, n$ are (Lebesgue) measurable on Ω, then

$$x \longmapsto f(x, \varphi_1(x), \dots, \varphi_n(x))$$

is a measurable function on Ω.

Proof . Since each φ_i is a measurable function on Ω, there exists a sequence of step functions $\{s_{i,m}\}_{m=1}^{\infty}$ such that $s_{i,m} \to \varphi_i$ almost everywhere in Ω for each $i = 1, \dots, n$. A step function s_i can be written as:

$$s_i(x) = \sum_{j=1}^{k_i} \alpha_{i,j} \chi_{\Omega_{i,j}}(x),$$

where $\Omega_{i,j}$ are pairwise disjoint measurable subsets of Ω, and $\chi_{\Omega_{i,j}}$ is the characteristic function of $\Omega_{i,j}$. For step functions $s_1, \dots, s_n$, the function $f(x, s_1(x), \dots, s_n(x))$ can be written as:

$$f(x, s_1(x), \dots, s_n(x)) = f\left(x, \sum_{j=1}^{k_1} \alpha_{1,j} \chi_{\Omega_{1,j}}(x), \dots, \sum_{j=1}^{k_n} \alpha_{n,j} \chi_{\Omega_{n,j}}(x)\right).$$

By property (i), $f(x, \alpha_{1,j}, \dots, \alpha_{n,j})$ is measurable in x for each fixed $(\alpha_{1,j}, \dots, \alpha_{n,j})$. Therefore,

$$f(x, s_1(x), \dots, s_n(x)) = \sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} f(x, \alpha_{1,j_1}, \dots, \alpha_{n,j_n}) \chi_{\Omega_{1,j_1}}(x) \cdots \chi_{\Omega_{n,j_n}}(x)$$

is a measurable function of x. Since $s_{i,m} \to \varphi_i$ almost everywhere, by property (ii), we have:

$$\lim_{m \to \infty} f(x, s_{1,m}(x), \dots, s_{n,m}(x)) = f(x, \varphi_1(x), \dots, \varphi_n(x))$$

for almost all $x \in \Omega$. Since the pointwise limit of measurable functions is measurable, it follows that:

$$x \mapsto f(x, \varphi_1(x), \dots, \varphi_n(x))$$

is measurable on Ω. □

Lemma 3.2. Assume that $g := g(t, s)$ is a function from $[a, b] \times \mathbb{R}$ to $\mathbb{R}$ such that

(i) $\forall t \in [a, b] \exists! s(t) \in \mathbb{R}$, $g(t, s(t)) = 0$;

(ii) $\frac{\partial g}{\partial s} > 0$ on $[a, b] \times \mathbb{R}$;

(iii) g, $\frac{\partial g}{\partial s}$ are continuous on $[a, b] \times \mathbb{R}$.

Then, the function $t \mapsto s(t)$ is continuous on $[a, b]$.

Proof . To prove the continuity of $t \mapsto s(t)$ on (a, b), we will apply the Implicit Function Theorem. The Implicit Function Theorem states that if $g(t, s)$ is continuously differentiable and $\frac{\partial g}{\partial s}(t, s) \neq 0$ at a point (t_0, s_0) where $g(t_0, s_0) = 0$, then there exists an open interval I containing t_0 and an open interval J containing s_0 such that for $t \in I$, there is a unique $s \in J$ for which $g(t, s) = 0$, and s is a continuously differentiable function of t.

We are given that $\frac{\partial g}{\partial s} > 0$ on $[a, b] \times \mathbb{R}$, ensuring that $\frac{\partial g}{\partial s} \neq 0$. By the Implicit Function Theorem, for each $t \in (a, b)$, there exists an interval around t where $s(t)$ is defined and continuous. Since $[a, b]$ is a compact interval, we can cover

$[a,b]$ with a finite number of such intervals by compactness, ensuring that $t \mapsto s(t)$ is continuous on the whole interval (a,b). Next, we prove that $t \mapsto s(t)$ is continuous at the endpoints a and b. Consider the endpoint b. Suppose $\{t_n\}$ is a sequence in (a,b) that converges to b. Since $g(t_n, s(t_n)) = 0$, we have

$$\lim_{n\to\infty} g(t_n, s(t_n)) = 0.$$

On the other hand, based on the assumptions, $g(a, s(b)) = 0$ and s is unique. Since s is continuous on (a,b) and g, $\frac{\partial g}{\partial s}$ are continuous and also $\frac{\partial g}{\partial s} > 0$, we conclude that $\lim_{n\to\infty} s(t_n) = s(b)$. Similarly, we can prove the continuity of s in the endpoint a. If we combine these results, we conclude that $t \mapsto s(t)$ is continuous on the entire interval $[a,b]$. Please also refer to [7, Exercise 7.1.21]. □

Lemma 3.3. Suppose F is a functional from $W^{n,2}(a,b)$ to $\mathbb{R}$ and let $u_0 \in C^n[a,b]$ be a local extremum of F. Then, u_0 is a local extremum of $F|_{C^n[a,b]}$.

Proof . For $u \in C^n[a,b]$,

$$\|u^{(i)}\|_2^2 = \int_a^b |u^{(i)}(x)|^2 dx \leq (b-a)\|u^{(i)}\|_\infty^2,$$

so

$$\|u^{(i)}\|_2 \leq \sqrt{b-a}\|u^{(i)}\|_\infty$$

and

$$\sum_{i=0}^n \|u^{(i)}\|_2 \leq \sqrt{b-a}\sum_{i=0}^n \|u^{(i)}\|_\infty.$$

Therefore,

$$\|u\|_{W^{n,2}(a,b)} \leq \sqrt{b-a}\|u\|_{C^n[a,b]}.$$

Now, by the following fact, the proof would be complete.

Fact: *Suppose that X, Y are normed spaces such that $Y \subseteq X$ and there exists $M > 0$ such that for all $u \in Y$, $\|u\|_X < M\|u\|_Y$. Let F be a functional from X to $\mathbb{R}$ and $u_0 \in Y$ be a local extremum of F. Then, u_0 would be a local extremum of $F|_Y$.*

Proof: Since u_0 is a local extremum of F in X, there exists $\epsilon_X > 0$ such that for all $u \in X$ with $\|u - u_0\|_X < \epsilon_X$, $F(u) \leq F(u_0)$ (if u_0 is a local minimum) or $F(u) \geq F(u_0)$ (if u_0 is a local maximum). Given the norm inequality $\|u\|_X \leq M\|u\|_Y$ for all $u \in Y$, we can express $\|u - u_0\|_X \leq M\|u - u_0\|_Y$ for all $u \in Y$. Let $\epsilon_Y = \frac{\epsilon_X}{M}$. Then, if $u \in Y$ satisfies $\|u - u_0\|_Y < \epsilon_Y$, we have

$$\|u - u_0\|_X \leq M\|u - u_0\|_Y < M \cdot \frac{\epsilon_X}{M} = \epsilon_X.$$

Since $\|u - u_0\|_X < \epsilon_X$, by the extremum property of u_0 in X,

$$F(u) \leq F(u_0) \quad \text{or} \quad F(u) \geq F(u_0),$$

depending on whether u_0 is a local minimum or maximum of F in X. Therefore, for all $u \in Y$ with $\|u - u_0\|_Y < \epsilon_Y$, $F(u) \leq F(u_0)$ (or $F(u) \geq F(u_0)$). This implies u_0 is a local extremum of $F|_Y$ in Y. Thus, we have shown that if u_0 is a local extremum of F in X, it is also a local extremum of the restriction $F|_Y$ in Y.

This completes the proof. □

The following regularity theorem is our major goal in this section.

Theorem 3.4. Suppose that $n \geq 1$, $N, N' \subseteq \{0, 1, \ldots, n\}$, and

$$\mathcal{N}_{N,N'} = \{u \in W^{n,2}(a,b) : u^{(i)}(a) = u_i,\ u^{(j)}(b) = w_j \text{ where } i \in N,\ j \in N'\}.$$

Define the functional F on $\mathcal{N}_{N,N'}$ by

$$F(u) = \int_a^b f(t, u(t), u'(t), \ldots, u^{(n)}(t))\, dt, \tag{3.1}$$

where $f = f(x_1, \dots, x_{n+2})$ is a function defined on $[a,b]\times\mathbb{R}^{n+1}$ with continuous second partial derivatives with respect to all its variables. Let $h \in L_2(a,b)$, $c_1 \geq 0$ be such that for a.a. $x_1 \in [a,b]$ and for all $(x_2, \dots, x_{n+2}) \in \mathbb{R}^{n+1}$,

$$|f(x_1, x_2, \dots, x_{n+2})| \leq h(x_1) + c_1(x_2^2 + \cdots + x_{n+2}^2), \tag{3.2}$$

$$\left|\frac{\partial f}{\partial x_i}(x_1, x_2, \dots, x_{n+2})\right| \leq h(x_1) + c_1(|x_2| + \cdots + |x_{n+2}|) \text{ for } i \in \{2, \dots, n+2\}. \tag{3.3}$$

Let $u_0 \in \mathcal{N}_{N,N'}$ be a local extremum of F with respect to $\mathcal{N}_{N,N'}$. For $t \in [a,b]$ and $s \in \mathbb{R}$ set

$$\psi(t,s) = \frac{\partial f}{\partial x_{n+2}}(t, u_0(t), u_0'(t), \dots, u_0^{(n-1)}(t), s).$$

Assume that $\frac{\partial \psi}{\partial s} > 0$ on $[a,b]\times\mathbb{R}$ and that for every fixed $t \in [a,b]$ the function $s \longrightarrow \psi(t,s)$ maps $\mathbb{R}$ onto $\mathbb{R}$. Then $u_0 \in C^{n+1}[a,b]$.

Proof . We begin by noting that for any $u \in \mathcal{N}_{N,N'}$, the function defined by $t \mapsto f(t, u(t), \dots, u^{(n)}(t))$ is measurable on (a,b) according to Lemma 3.1. Moreover, employing condition (3.2) and considering the fact that h, u, $u', \dots, u^{(n)}$ are in $L^2(a,b)$, we can deduce that:

$$\begin{aligned}\left|\int_a^b f(t, u(t), u'(t), \dots, u^{(n)}(t))dt\right| &\leq \int_a^b |f(t, u(t), u'(t), \dots, u^{(n)}(t))|dt \\ &\leq \int_a^b h(t)dt + c_1\left(\sum_{i=0}^{n}\int_a^b u^{(i)}(t)dt\right) \\ &<\infty.\end{aligned}$$

Thus, for every $u \in \mathcal{N}_{N,N'}$, $F(u)$ is finite, implying that F is a well-defined functional. Continuing, Lemma 3.1 also assures that for each i, $2 \leq i \leq n+2$, the function $t \mapsto \frac{\partial f}{\partial x_i}(t, u_0(t), \dots, u_0^{(n)}(t))$ is measurable on (a,b). Utilizing condition (3.3) alongside Hölder's inequality, for any $v \in \mathcal{N}_{N,N'}$, we establish that:

$$\int_a^b \sum_{i=2}^{n+2} \frac{\partial f}{\partial x_i}(t, u_0(t), \dots, u_0^{(n)}(t)) V^{(i-2)}(t)\, dt < \infty. \tag{3.4}$$

Thus, following the procedure used in the proof of Theorem 2.5, we reach the following equality which holds for almost all $t \in [a,b]$:

$$\frac{\partial f}{\partial x_{n+2}}(t, u_0(t), \dots, u_0^{(n)}(t)) + \sum_{m=1}^{n}(-1)^m h_{m,n-m+2}(t) - c_{n-1}t^{n-1} - \cdots - c_1 t - c_0 = 0.$$

We now define a function g by:

$$g(t,s) = \psi(t,s) + \sum_{m=1}^{n}(-1)^m h_{m,n-m+2}(t) - c_{n-1}t^{n-1} - \cdots - c_1 t - c_0.$$

For $\varphi_t(s) := \psi(t,s)$, we observe that:

$$\varphi_t'(s) = \frac{\partial \psi}{\partial s}(t,s) = \frac{\partial^2 f}{\partial x_{n+2}^2}(t, u_0(t), \dots, u_0^{(n-1)}(t), s) > 0,$$

implying that φ_t is a one-to-one function. On the other hand, by the assumptions of the theorem, φ_t is surjective. Hence

$$\forall t \in [a,b]\ \exists! s(t) \in \mathbb{R},\ \varphi_t(s(t)) = c_{n-1}t^{n-1} + \cdots + c_1 t + c_0 - \sum_{m=1}^{n}(-1)^m h_{m,n-m+2}(t),$$

or

$$\forall t \in [a,b]\ \exists! s(t) \in \mathbb{R},\ \psi(t,s(t)) + \sum_{m=1}^{n} (-1)^m h_{m,n-m+2}(t) - c_{n-1}t^{n-1} - \cdots - c_1 t - c_0 = 0.$$

Consequently, for all $t \in [a,b]$, there exists a unique $s(t) \in \mathbb{R}$ such that

$$g(t,s(t)) = 0.$$

By Lemma 3.2 the function $t \longrightarrow s(t)$ is continuous on $[a,b]$. On the other hand, we have for every $t \in [a,b]$ that $g(t,u_0^{(n)}(t)) = 0$. Therefore, for all $t \in [a,b]$,

$$u_0^{(n)}(t) = s(t),$$

implying the continuity of $u_0^{(n)}$. Thus, $u_0 \in C^n[a,b]$, and by Lemma 3.3, it is a local extremum of $F|_{C^n[a,b]}$. The assertion now follows from Theorem 2.5.

□

Remark 3.5. The growth conditions (3.2) and (3.3) were included in the assumptions of Theorem 3.4, mainly to ensure the integrability of expressions (3.1) and (3.4). These conditions ensure that the integrals involved in the definition of the functional F and its variations remain finite and well-defined. However, if integrability can be proved for a given problem by other means, then Theorem 3.4 can be applied without the need to strictly adhere to the growth conditions (3.2) and (3.3). This observation suggests that the theorem has a broader scope of application, contingent upon the satisfaction of the integrability criteria, regardless of whether this is achieved through conditions (3.2) and (3.3) or through other properties inherent to the problem.

Remark 3.6. The differentiability condition of f in Theorem 3.4 can be relaxed in certain categories of differential equations. For instance, consider a scenario where it is proved that the following differential equation has a weak solution for any continuous function f:

$$x''(t) = f(t,x(t)), \quad t \in (0,1). \tag{3.5}$$

Even though f is merely continuous and not differentiable, it can be shown, with the aid of Theorem 3.4 and under the consideration that $\overline{C^2(0,1)} = C(0,1)$, that the weak solution lies in C^2. Suppose $x_\circ$ is a weak solution of the differential equation (3.5). For an arbitrary n, let f_n be a function satisfying the differentiability conditions of Theorem 3.4 such that

$$\|f_n - f\|_\infty < \frac{1}{n}.$$

Since f_n's are continuous, the equations

$$x''(t) = f_n(t,x(t)), \quad n \in \mathbb{N},$$

have weak solutions. Moreover, as f_n's meet the conditions of Theorem 3.4, these solutions are in C^2, i.e., there exist $x_n \in C^2(0,1)$ such that

$$x_n''(t) = f_n(t,x_n(t)).$$

Now, we have

$$\begin{aligned}\|x_n'' - x_m''\|_\infty &= \sup_{t\in(0,1)} |x_n''(t) - x_m''(t)| \\ &= \sup_{t\in(0,1)} |f_n(t,x_n(t)) - f_m(t,x_m(t))| \\ &\leqslant \|f_n - f_m\|_\infty.\end{aligned}$$

Thus, $\{x_n''\}_{n=1}^\infty$ is Cauchy in $C(0,1)$. Therefore, there exists $z \in C(0,1)$ such that

$$x_n'' \to z \ \text{ uniformly as } \ n \to \infty,$$

and so there exists $z \in C(0,1)$ such that

$$x_n' \to \int_0^t z(s)\,ds \text{ uniformly as } n \to \infty.$$

Consequently,

$$\forall y \in C_0^\infty(0,1) \ \int_0^1 \left(x_n'(t) - \int_0^t z(s)ds \right) y'(t)dt \to 0 \ \text{ as } \ n \to \infty. \tag{3.6}$$

On the other hand, since $x_\circ$ is a weak solution of the following equation

$$x''(t) = f(t, x(t)); \quad t \in (0,1),$$

we have

$$-\int_0^1 x_\circ'(t)y'(t)dt = \int_0^1 f(t, x(t))y(t)dt. \tag{3.7}$$

for all $y \in C_0^\infty(0,1)$. Moreover, x_n for every $n \in \mathbb{N}$ is a weak solution of the following equation:

$$x''(t) = f_n(t, x(t)); \quad t \in (0,1),$$

so

$$-\int_0^1 x_n'(t)y'(t)dt = \int_0^1 f_n(t, x(t))y(t)dt. \tag{3.8}$$

for all $y \in C_0^\infty(0,1)$. Now, by (3.7) and (3.8), for every $y \in C_0^\infty(0,1)$,

$$\begin{aligned} \left| \int_0^1 [x_\circ'(t) - x_n'(t)]y'(t)dt \right| &= \left| \int_0^1 [f(t, x_n(t)) - f_n(t, x_\circ(t))]y(t)dt \right| \\ &\leqslant \|f_n - f\|_\infty \int_0^1 |y(t)|dt. \end{aligned}$$

Consequently,

$$\int_0^1 \left(x_\circ'(t) - x_n'(t) \right) y'(t)dt \to 0 \ \text{ as } \ n \to \infty. \tag{3.9}$$

for all $y \in C_0^\infty(0,1)$. Then, by (3.6) and (3.9), it is concluded that

$$\int_0^1 \left(x_\circ'(t) - \int_0^t z(s)ds \right) y'(t)dt = 0.$$

for all $y \in C_0^\infty(0,1)$. So, by Lemma 2.3, the following is resulted:

$$x_\circ'(t) = \int_0^t z(s)ds + c; \quad z \in C(0,1),$$

hence,

$$x_\circ \in C^2(0,1).$$

Remark 3.7. The method for proving the existence of a classical solution for differential equations is not based exclusively on theories that find weak solutions. See, for example, these articles [21, 22, 23, 24], in which another theory directly proves the existence of solutions to some differential equations in engineering and physics without the need to find weak solutions.

4 Examples in Differential Equations, Optimal Control Problems, Imaging Sienece and Structural Analysis

In this section, we illustrate the application of the generalized regularity theorems discussed in the previous sections with some examples. The following example demonstrates the practical implications of our theoretical results by applying them to specific boundary value problems with higher-order differential equations:

Example 4.1. A Dirichlet Boundary Value Problem: We illustrate the application of Theorem 3.4 to the following Dirichlet boundary value problem:

$$\begin{cases} x^{(2n)}(t) + x''(t) + x^3(t) = f(t, x(t)), \quad t \in (0,1), \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $n \in \mathbb{N}$ and f is a continuous function on $[0,1] \times \mathbb{R}$. Let

$$H := \{u \in W^{2n-1,2}(0,1) : u^{(0)}(0) = 0,\ u^{(0)}(1) = 0\}.$$

The functional

$$\psi(x) := \int_0^1 \int_0^{x(t)} f(t,s)\,ds\,dt$$

defined on H is of the class $C^1(H, \mathbb{R})$ and

$$\psi'(x)(h) = \int_0^1 f(t, x(t))h(t)\,dt, \quad x, h \in H.$$

Then the functional

$$F(x) = \int_0^1 \left[\frac{(-1)^n}{2} |x^{(n)}(t)|^2 - \frac{1}{2}|x'(t)|^2 + \frac{1}{4}|x(t)|^4 - \int_0^{x(t)} f(t,s)\,ds \right] dt$$

is of the class $C^1(H, \mathbb{R})$ and its critical points correspond to weak solutions of (4.1). The regularity argument in Theorem 3.4 applied to (4.1) implies that every weak solution is a classical solution in the sense that

$$x \in C_0^{2n}[0,1] := \{x \in C^{2n}[0,1] : x(0) = x(1) = 0\},$$

and the equation in (4.1) holds at every point t. Note that, in this example, the differentiability condition of f has been omitted based on Remark 3.6.

The techniques discussed in this article can be applied to a variety of practical problems in the applied sciences, including the following example of image denoising:

Example 4.2. Image Denoising: Many practical problems in applied sciences, such as image denoising, can be expressed as the following minimization problem (see references [1, 3, 5, 6, 11]):

$$\|x - x_\circ\|^2_{L_2(I)} + \lambda_1 \|x\|_{Y_1} + \cdots + \lambda_n \|x\|_{Y_n}, \tag{4.2}$$

where

$$\|x - x_\circ\|_{L_2(I)} := \left(\int_I |x(t) - x_\circ(t)|^2\,dt \right)^{\frac{1}{2}}$$

represents the root-mean-square error (or more generally, the difference) between x and $x_\circ$, with x being the variable and $x_\circ$ a given reference image. Moreover, $\|x\|_{Y_i}$ for $i = 1, \dots, n$ are the norms of different smoothness spaces Y_i respectively. The parameters λ_i for $i = 1, \dots, n$ influence the smoothness of the solution; a large λ_i enforces a smaller $\|x\|_{Y_i}$ at the minimum, implying that x must be smoother, while a small λ_i allows x to be rougher. In cases where $\|x\|_{Y_i}$ for $i = 1, \dots, n$ take the form

$$\|x\|_{Y_i} = \int_I f_i(t, x(t), x'(t), \dots, x^{(n)}(t))\,dt,$$

with $f_i = f_i(x_1, \dots, x_{n+2})$ being a function defined on $I \times \mathbb{R}^n$ with continuous second partial derivatives with respect to all its variables, (4.2) would be a problem of the type (3.1). Consequently, all papers addressing discussions around solutions of (3.1), including this note, could potentially be important in investigating (4.2).

The next example concerns energy systems. We study the management of energy distribution in a smart grid and use higher-order differential equations to dynamically balance energy production, storage and consumption to improve the stability and efficiency of the grid:

Example 4.3. Optimal Control in Smart Grid Energy Management: Consider the problem of managing energy distribution in a smart grid system to optimize both the grid's stability and the cost of energy production and distribution. The smart grid integrates various renewable energy sources and manages demand-response strategies to enhance efficiency and reliability (see references [25, 26, 27, 28]). The objective is to minimize the operational costs and maintain energy supply-demand balance over time, formulated as:

$$J(u) = \int_0^T \left(c(t)u(t)^2 + q(t)(s(t) - d(t))^2 \right) dt,$$

where $u(t)$ represents the control actions such as the amount of energy generated or stored, $c(t)$ is the cost of generating or storing energy, $q(t)$ is a penalty term for deviations from the demand $d(t)$, and $s(t)$ is the total energy supplied to the grid. The dynamics of the energy supply in the grid are described by:

$$\frac{d^2 s}{dt^2} + \alpha \frac{ds}{dt} + \beta s = \gamma u(t),$$

where α, β, and γ are parameters that represent the responsiveness of the grid to control actions and the natural decay rates of energy. This second-order dynamic model captures both the inertia and damping effects in the energy supply system. Boundary conditions specify the initial and final states of energy storage and generation capacities:

$$s(0) = s_0, \quad \frac{ds}{dt}(0) = v_0, \quad s(T) = s_T, \quad \frac{ds}{dt}(T) = v_T,$$

where s_0 and s_T are the initial and final energy states, and v_0 and v_T are the initial and final rates of change of energy. Our regularity theorems show that the solutions to this control problem are smooth enough for implementation. This ensures that the smart grid operates efficiently and adapts to energy demand and supply fluctuations while minimizing operational costs.

In the following example, we consider the problem of beam deflection under uniform load in structural engineering, as described by the Euler-Bernoulli beam theory:

Example 4.4. Bending of an Elastic Beam under Uniform Load: In structural engineering, beam bending is a critical problem. The Euler-Bernoulli beam theory provides a simplified model for bending slender beams under a load. For a beam subjected to a uniformly distributed load, the deflection is determined by a fourth-order linear differential equation (see references [29, 30, 31]).

Consider an elastic beam of length L, fixed at both ends, subjected to a uniform load q (force per unit length). According to the Euler-Bernoulli beam theory, the deflection $w(x)$ of the beam satisfies the following boundary value problem:

$$EIw^{(4)}(x) = q, \quad x \in [0, L],$$

where E is the modulus of elasticity of the beam material, and I is the moment of inertia of the cross-section about the bending axis. The boundary conditions, assuming the beam is clamped at both ends, are:

$$w(0) = 0, \quad w(L) = 0, \quad w'(0) = 0, \quad w'(L) = 0.$$

The general solution to the homogeneous part of the differential equation is:

$$w_h(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3,$$

and a particular solution to the non-homogeneous equation can be:

$$w_p(x) = \frac{qx^4}{24EI}.$$

Combining these, the total solution becomes:

$$w(x) = \frac{qx^4}{24EI} + c_1 + c_2 x + c_3 x^2 + c_4 x^3.$$

Applying the boundary conditions, the coefficients c_1, c_2, c_3, and c_4 are determined, resulting in a specific expression for $w(x)$ that describes the beam's deflection. This example demonstrates the application of higher-order differential equations in structural engineering. By solving the fourth-order differential equation, we can determine the deflection of the beam under a uniform load, which is crucial for ensuring the structural integrity and safe design of beams in engineering projects. Understanding and solving such differential equations is essential for engineers to predict the behaviour of structural elements under various loading conditions, thereby ensuring safety and reliability in construction and design.

In order to illustrate the application of the regularity theorems to practical engineering problems, we also consider the dynamic behaviour of a cantilever beam subjected to harmonic excitation, a common scenario in the design and analysis of mechanical structures:

Example 4.5. Structural Dynamics of a Vibrating Beam: Consider the analysis of the dynamic behaviour of a cantilever beam subject to harmonic excitation. This problem is crucial in the design of mechanical structures to ensure they can withstand dynamic loads without experiencing resonant conditions that could lead to failure (see references [32, 33, 34]).

The equation governing the transverse vibrations of the beam is given by the Euler-Bernoulli beam theory:

$$\begin{cases} EI\frac{d^4y(x,t)}{dx^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} = F_0\cos(\omega t), & 0 < x < L, \\ y(0,t) = 0, \quad \frac{\partial y(0,t)}{\partial x} = 0, & \text{(clamped end)}, \\ EI\frac{\partial^2 y(L,t)}{\partial x^2} = 0, \quad EI\frac{\partial^3 y(L,t)}{\partial x^3} = 0, & \text{(free end)}, \end{cases}$$

where $y(x,t)$ represents the transverse displacement of the beam at position x and time t, E is the modulus of elasticity, I is the moment of inertia of the beam's cross-section, ρ is the density, A is the cross-sectional area, and $F_0\cos(\omega t)$ is the harmonic external force applied to the beam with amplitude F_0 and frequency ω.

The objective is to determine the displacement $y(x,t)$ and ensure that the beam's vibrations are within safe limits. The corresponding energy functional for this system is:

$$F(y) = \int_0^L \int_0^T \left[\frac{EI}{2}\left(\frac{\partial^2 y(x,t)}{\partial x^2}\right)^2 + \frac{\rho A}{2}\left(\frac{\partial y(x,t)}{\partial t}\right)^2 - F_0\cos(\omega t)y(x,t)\right] dx\, dt.$$

The regularity theorems from our paper show that the solution $y(x,t)$ is smooth and continuous. This regularity is crucial for accurately predicting the dynamic response of the beam and ensures that the design can withstand dynamic loads without catastrophic failure. This example highlights the critical application of higher-order differential equations in structural dynamics, demonstrating how theoretical insights can lead to safer and more reliable mechanical designs.

5 Conclusion

After three centuries, the study of the problem

$$F(x) = \int_a^b f(t, x(t), x'(t))\, dt$$

and its variants continues to attract considerable attention. This problem finds numerous applications in diverse fields such as geometry and differential equations, mechanics and physics, and extends to areas as varied as engineering, medicine, economics, and renewable resources.

In this paper, we have discussed a generalization of this classical problem, with a particular focus on the regularity of its solutions. Our aim is to contribute to the understanding of the regularity properties of solutions arising in these diverse disciplines. As noted, dealing with weak solutions—a generalization of classical solutions—is particularly beneficial in differential equations, as many nonlinear analysis methods are more aptly suited to obtaining weak solutions. However, upon finding a weak solution, one is naturally led to inquire about its finer properties, such as the continuity of its first and second derivatives. To address these concerns, we have generalized two theorems in regularity theory pertinent to differential equations. Furthermore, the area of optimal control, especially problems involving higher orders, is receiving increasing attention each year (refer to [19] for examples). Thus, discussing the regularity properties of solutions in this domain is becoming increasingly important. Given that the Hilbert-Weierstrass theorem and the Tonelli-Morrey theorem are instrumental in proving the regularity properties of optimal control problems (as elaborated in Chapter 23 of [4]), this article aims to inspire further research in proving the regularity of solutions for higher-order problems in this realm as well. It is also noteworthy that certain series of optimal control problems are equivalent to higher-order variational problems (see [2] for instances). In conclusion, this article is expected to be of interest to mathematicians who are passionate about Nonlinear Analysis and its rich history.

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