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Weighted composition operators on extended analytic Lipschitz algebras

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Abstract

In this paper, we study weighted composition operators on extended analytic Lipschitz algebras $\mathrm{Lip}_A(X, K, \alpha)$ where X is a compact plane set, K is a closed subset of X with nonempty interior and $0 < \alpha \leq 1$. We first give necessary conditions and sufficient conditions on a function $u \in \mathbb{C}^X$ and self-map φ of X for which $T = uc_\varphi$ to be a weighted composition operator on $\mathrm{Lip}_A(X, K, \alpha)$. We next give the necessary conditions for these operators to be compact and provide some sufficient conditions for the compactness of such operators.

Keywords: Extended analytic Lipschitz algebra, Analytic uniform algebra, Banach function algebra, Compact operator, Composition operator, Weighted composition operator
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1 Introduction and preliminaries

Let X be a nonempty set, $\mathbb{C}^X$ denote the set of all complex-valued functions on X and A be a nonempty subset of $\mathbb{C}^X$. For each $u \in \mathbb{C}^X$ and every self-map φ of X, $f \to u \cdot (f \circ \varphi)$ defines a map from A to $\mathbb{C}^X$ that denotes by uC_φ. A map $T : A \to \mathbb{C}^X$ is called a *weighted composition operator* on A if there exists a function $u \in \mathbb{C}^X$ and a self-map φ of X such that $u \cdot (f \circ \varphi) \in A$ for all $f \in A$ and $T = uC_\varphi$ on A. In the case where $u = 1_X$, the constant function with value 1 on X, the weighted composition operator uC_φ on A reduces to the composition operator C_φ. Clearly, every weighted composition operator on A is linear if A is a linear subspace of $\mathbb{C}^X$.

Let X be a compact Hausdorff space. We denote by $C(X)$ the set of all complex-valued continuous functions on X. It is known that $C(X)$ is a unital commutative Banach algebra with the uniform norm $||\cdot||$ defined by

$$||f||_X = \sup\{|f(x)| : x \in X\} \qquad (f \in C(X)).$$

Let (X, d) be a metric space and $\alpha \in (0, 1]$. For each $f \in \mathbb{C}^X$ and every nonempty subset K of X, define

$$p_{(K,d^\alpha)}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d^\alpha(x, y)} : x, y \in X, x \neq y\right\}.$$

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We denote by $\mathrm{Lip}(X, d^{\alpha})$ the set of all bounded functions $f \in \mathbb{C}^X$ for which $p_{(X,d^{\alpha})} < \infty$. Then $\mathrm{Lip}(X, d^{\alpha})$ separates the points of X and $1_X \in \mathrm{Lip}(X, d^{\alpha})$. Furthermore, $\mathrm{Lip}(X, d^{\alpha})$ is a Banach algebra with the Lipschitz sum norm $\|\cdot\|_{\mathrm{Lip}(X,d^{\alpha})}$ defined by

$$\|f\|_{\mathrm{Lip}(X,d^{\alpha})} = \|f\|_X + p_{(K,d^{\alpha})}(f) \qquad (f \in \mathrm{Lip}(X, d^{\alpha})).$$

These algebras are called *Lipschitz algebras* and were introduced by Sherbet in [14, 15]. Jiménez-Vargas and Villegas-Vallecillos characterized the structure of compact composition operators between $\mathrm{Lip}(X, d)$-spaces in [10]. Weighted composition operators on $\mathrm{Lip}(X, d)$ studied in [1, 7, 9].

Let (X, d) be a compact metric space and let K be a nonempty closed subset of X. The set of all $f \in C(X)$ for which $f|_K \in \mathrm{Lip}(K, d^{\alpha})$ is denoted by $\mathrm{Lip}(X, K, d^{\alpha})$. It is clear that $\mathrm{Lip}(X, d^{\alpha})$ is a subset of $\mathrm{Lip}(X, K, d^{\alpha})$. In addition, $\mathrm{Lip}(X, K, d^{\alpha}) = \mathrm{Lip}(X, d^{\alpha})$ if $K = X$ and $\mathrm{Lip}(X, d^{\alpha}) = C(X)$ if K is finite. It is known that $\mathrm{Lip}(X, K, d^{\alpha})$ is a Banach function algebra on (X, d) with the extended Lipschitz sum norm $\|\cdot\|_{\mathrm{Lip}(X,K,d^{\alpha})}$ defined by

$$\|f\|_{\mathrm{Lip}(X,K,d^{\alpha})} = \|f\|_X + p_{(X,d^{\alpha})}(f) \qquad (f \in \mathrm{Lip}(X, K, d^{\alpha}))$$

These algebras are called *extended Lipschitz algebras* and were first introduced in [11]. Weighted composition operators on extended Lipschitz algebras studied in [6]. Some properties of extended Lipschitz algebras were investigated in [4, 12].

Let X be a compact plane set. For $\alpha \in (0, 1]$, we write $\mathrm{Lip}(X, \alpha)$ instead of $\mathrm{Lip}(X, d^{\alpha})$ where d is the Euclidean metric on X. The *analytic Lipschitz algebra* of order α on X is denoted by $\mathrm{Lip}_A(X, \alpha)$ and defined by $\mathrm{Lip}_A(X, \alpha) = \mathrm{Lip}(X, \alpha) \cap A(X)$, where $A(X)$ is the uniform function algebra of all continuous complex-valued functions on X which are analytic on int(X). It is known that $\mathrm{Lip}_A(X, \alpha)$ is a closed subalgebra of $\mathrm{Lip}(X, \alpha)$ and a Banach function algebra on X. Weighted composition operators on $\mathrm{Lip}_A(X, \alpha)$ studied by Amiri, Golbaharan and Mahyar in [5].

Let X be a compact plane set and K be a closed subset of X with nonempty interior. We denote by $A(X, K)$ the set of all $f \in C(X)$ for which f is analytic on int(K). It is known that $A(X, K)$ is a uniform function algebra on X. For $\alpha \in (0, 1]$, the *extended analytic Lipschitz algebra* on X of order α with respect to K is denoted by $\mathrm{Lip}(X, K, \alpha)$ and defined by

$$\mathrm{Lip}_A(X, K, \alpha) = \mathrm{Lip}(X, K, \alpha) \cap A(X, K).$$

In fact, $f \in \mathrm{Lip}_A(X, K, \alpha)$ if $f \in C(X)$, $f|_K \in \mathrm{Lip}(K, \alpha)$ and f is analytic on int(K). It is clear that $\mathrm{Lip}_A(X, K, \alpha) = \mathrm{Lip}_A(X, \alpha)$ if $K = X$. Note that $\mathrm{Lip}_A(X, K, \alpha)$ is a Banach function algebra on X with the extended Lipschitz sum norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$. Compact unital homomorphisms between extended analytic Lipschitz algebras studied in [2]. Power compact and quasicompact unital endomorphisms of extended analytic Lipschitz algebras were investigated in [3].

Let X be a compact plane set, K be a closed subset of X with nonempty interior and B be a subalgebra of $A(X, K)$ which is a Banach function algebra on X with an algebra norm. In section 2, we first give a necessary condition on a function $u \in \mathbb{C}^X$ and a self-map φ of X for which $T = uC_{\varphi} : B \to \mathbb{C}^X$ to be a weighted composition operator on B. In continue, we give some sufficient conditions on a function $u \in \mathbb{C}^X$ and a self-map φ of X for which $T = uC_{\varphi} : \mathrm{Lip}_A(X, K, \alpha) \to \mathbb{C}^X$ to be a weighted composition operator on $\mathrm{Lip}_A(X, K, \alpha)$ for $\alpha \in (0, 1]$. In section 3, we first give some necessary and sufficient conditions for a weighted composition operators $T = uC_{\varphi}$ on extended analytic Lipschitz algebras $\mathrm{Lip}_A(X, K, \alpha)$ to be compact, where $\alpha \in (0, 1]$. Next, we give some another necessary conditions for a weighted composition operator $T = uC_{\varphi}$ on $\mathrm{Lip}_A(X, K, 1)$ to be compact. Our results extend som results in [5, 2].

2 Weighted composition operators

Let X be a compact plane set, K be a closed subset of X with nonempty interior and B be a subalgebra of $A(X, K)$ which is a natural Banach function algebra on X under an algebra norm $\|\cdot\|$. It is interesting to know under which conditions on complex-valued function u on X and self-map φ of X, the map $T = uC_{\varphi} : B \to \mathbb{C}^X$ is a weighted composition operator on B. We first give a necessary condition on a complex-valued function u on X and a self-map $\varphi : X \to X$ for which $T = uC_{\varphi} : B \to \mathbb{C}^X$ be a weighted composition operator on B.

Theorem 2.1. Let X be a compact plane set, K be a closed subset of X with nonempty interior and B be a subalgebra of $A(X, K)$ which is a Banach function algebra on X under an algebra norm. Let u be a complex-valued function on

X, φ be a self-map on X and $T = uC_\varphi : B \to \mathbb{C}^X$ be a weighted composition operator on B. Then the following assertions hold:

(i) $u = T(1_X)$ *and so* $u \in B$.

(ii) T *is a bounded linear operator.*

(iii) *If the cordinate function* Z_X *belongs to* B*, then* $u\varphi = T(Z_X)$*,* $u\varphi \in B$*,* φ *is continuous on* $\mathrm{coz(u)}$ *and analytic on* $\mathrm{int(K)} \cap \mathrm{coz(u)}$.

(iv) *If* $B = \mathrm{Lip}_A(X, K, \alpha)$ *for* $\alpha \in (0, 1]$*, then*

$$\sup\{|u(z)|\frac{|\varphi(z) - \varphi(w)|}{|z - w|^\alpha} : z, w \in K, z \neq w\} \leq M,$$

where $M = \|T\|_{op}(\mathrm{diam(X)})^{1-\alpha}((\mathrm{diam(X)})^\alpha + 1)$.

Proof .

(i) Since $1_X \in B$ and $T = uC_\varphi$ is a weighted composition operator on B, (i) holds.

(ii) Clearly, T is a linear operator on B. Assume that $\|\cdot\|$ is the given norm on B. Let $\{f_n\}_{n=1}^\infty$ be a sequence in B with

$$\lim_{n\to\infty} f_n = 0_X \qquad (\text{in } (B, \|.\|)), \tag{2.1}$$

and $g \in B$ with

$$\lim_{n\to\infty} T(f_n) = g \qquad (\text{in } (B, \|.\|)). \tag{2.2}$$

We show that $g = 0_X$. Since $\|h\|_X \leq \|h\|$ for all $h \in B$, according to (2.2) we deduce that $T(f_n)$ converges uniformly to g on X. This implies that

$$\lim_{n\to\infty} u(z)f_n(\varphi(z)) = g(z) \tag{2.3}$$

for all $z \in X$. According to (2.1), we deduce of that

$$\lim_{n\to\infty} f_n(\varphi(z)) = 0 \tag{2.4}$$

for all $z \in X$. By (i), $u \in B$. This implies that u is a bounded function on X. Therefore,

$$\lim_{n\to\infty} u(z)f_n(\varphi(z)) = 0 \tag{2.5}$$

for all $z \in X$ since (2.4) holds for all $z \in X$. According to (2.4) and (2.5) hold for all $z \in X$, we deduce that $g = 0$. Therefore, T is continuous by the closed graph theorem. Hence, (ii) holds.

(iii) Let $Z_X \in B$. It is clear that $u\varphi = T(Z_X)$ and so $u\varphi \in B$. Since $u, u\varphi \in B$, we deduce that $u, u\varphi \in C(X)$. This implies that φ is continuous on $\mathrm{coz(u)}$. According to $u, u\varphi \in B$ and $B \subseteq A(X, K)$, we deduce that $u, u\varphi$ are analytic on $\mathrm{int(K)}$. Therefore, φ is analytic on $\mathrm{coz(u)} \cap \mathrm{int(K)}$.

(iv) Let $B = \mathrm{Lip}_A(X, K, \alpha)$ for $\alpha \in (0, 1]$. Then B is a subalgebra of $A(X, K)$ and a Banach function algebra on X under the algebra norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$. Moreover, $Z_X \in B$. Take

$$M = \|T\|_{op}(\mathrm{diam(X)})^{1-\alpha}((\mathrm{diam(X)})^\alpha + 1).$$

For each $w \in X$, we define the function $f_w : X \to \mathbb{C}$ by

$$f_w = Z_X - \varphi(w)1_X.$$

Clearly, $f_w \in B$, $\|f_w\|_X \leq \mathrm{diam}(X)$ and $p_{K,\alpha}(f_w) \leq (\mathrm{diam}(K))^{1-\alpha}$. Therefore, for each $z, w \in K$ with $z \neq w$ we get

$$|u(z)|\frac{|\varphi(z) - \varphi(w)|}{|z - w|^\alpha} \leq \|T\|_{op}(\mathrm{diam(X)})^{1-\alpha}((\mathrm{diam(X)})^\alpha + 1) = M.$$

Since the above inequality holds for all $z, w \in K$ with $z \neq w$, we deduce that

$$\sup\{|u(z)|\frac{|\varphi(z) - \varphi(w)|}{|z - w|^\alpha} : z, w \in K, z \neq w\} \leq M.$$

Hence, (iv) holds and the proof is complete. □

Here we give some sufficient conditions on complex-valued functions u on X and self-maps φ of X, that the map $T = uC_\varphi : B \to \mathbb{C}^X$ is a weighted composition operator on B.

Theorem 2.2. Let X be a compact plane set, K be a closed subset of X with nonempty interior and B be a subalgebra of $A(X, K)$ which is a natural Banach function algebra on X under an algebra norm. Let $u \in B$ and $\varphi \in B$ with $\varphi(X) \subseteq \text{int(K)}$. Then $T = uC_\varphi : B \to \mathbb{C}^X$ is a weighted composition operator on B and $u = T(1_X)$. In addition, if $Z_X \in B$ then $uC_\varphi = T(Z_X)$ and $u\varphi \in B$.

Proof . By [2, Proposition 2.1], $f \circ \varphi \in B$ for all $f \in B$. This implies that $u \cdot (f \circ \varphi) \in B$ since $u \in B$. This implies that $T = uC_\varphi$ is a weighted composition operator on B. Therefore, the proof completes by parts (i) and (iii) of Theorem 2.1. □

We now give another sufficient condition on functions $u : X \to \mathbb{C}$ and self maps φ of X that the map $T = uC_\varphi : B \to \mathbb{C}^X$ is a weighted composition operator on B, where $B = \text{Lip}_A(X, K, \alpha)$ for $\alpha \in (0, 1]$.

Theorem 2.3. Let X be a compact plane set, K be a closed subset of X with nonempty interior and $B = \text{Lip}_A(X, K, \alpha)$ for $\alpha \in (0, 1]$. Let $u \in B$ and $\varphi \in \text{Lip}_A(X, K, 1)$ with $\varphi(X) \subseteq X$ and $\varphi(\text{int(K)}) \subseteq \text{int(K)}$. Then $T = uC_\varphi : B \to \mathbb{C}^X$ is a weighted composition operator on B.

Proof . Let $f \in B$. Then $f \in C(X)$ and f is analytic on int(K). Since $\varphi \in \text{Lip}_A(X, K, 1)$, we have $\varphi \in C(X)$ and φ is analytic on int(K). According to $\varphi(X) \subseteq X$ and $\varphi(\text{int(K)}) \subseteq \text{int(K)}$, we deduce that $f \circ \varphi$ is continuous on X and analytic on int(K). Thus $\varphi \in A(X, K)$.

Now, we show that $(f \circ \varphi)|_K \in \text{Lip}(K, \alpha)$. Since $\varphi \in \text{Lip}_A(X, K, 1)$, we deduce that$\varphi|_K \in \text{Lip}(K, 1)$. If $z, w \in K$ with $\varphi(z) \neq \varphi(w)$, then

$$\begin{aligned}\frac{|(f \circ \varphi)(z) - (f \circ \varphi)(w)|}{|z - w|^\alpha} &= \frac{|(f \circ \varphi)(z) - (f \circ \varphi)(w)|}{|\varphi(z) - \varphi(w)|^\alpha} \frac{|\varphi(z) - \varphi(w)|^\alpha}{|z - w|^\alpha} \\ &\leq p_{K,\alpha}(f)(p_{K,\alpha}(\varphi))^\alpha.\end{aligned}$$

If $z, w \in K$ with $z \neq w$ and $\varphi(z) = \varphi(w)$, then

$$\frac{|(f \circ \varphi)(z) - (f \circ \varphi)(w)|}{|\varphi(z) - \varphi(w)|^\alpha} = 0 \leq p_{K,\alpha}(f)(p_{K,\alpha}(\varphi))^\alpha.$$

Therefore,

$$\frac{|(f \circ \varphi)(z) - (f \circ \varphi)(w)|}{|z - w|^\alpha} \leq p_{K,\alpha}(f)(p_{K,\alpha}(\varphi))^\alpha.$$

for all $z, w \in K$ with $z \neq w$. This implies that $(f \circ \varphi)|_K \in \text{Lip}(K, \alpha)$. Hence, $(f \circ \varphi) \in B$ and so $u \cdot (f \circ \varphi) \in B$ since $u \in B$. Therefore, $T = uC_\varphi$ is a weighted composition operator on B. □

Theorem 2.4. Let X be a compact plane set, K be a closed subset of X with nonempty interior and $B = \text{Lip}_A(X, K, \alpha)$ for $\alpha \in (0, 1]$. Suppose that $u \in B$, $\varphi \in A(X, K)$ with $\varphi(X) \subseteq X$, $\varphi(\text{int(K)}) \subseteq \text{int(K)}$ and

$$\sup \{|u(z)| \frac{|\varphi(z) - \varphi(w)|^\alpha}{|z - w|^\alpha} : z, w \in K, z \neq w\} < \infty.$$

Then $T = uC_\varphi : B \to \mathbb{C}^X$ is a weighted composition operator on B. In particular, if H is a nonempty compact subset of $K \cap \text{coz(u)}$ then $\varphi|_H \in \text{Lip}(H, \alpha)$.

Proof . Take

$$C = \sup \{|u(z)| \frac{|\varphi(z) - \varphi(w)|^\alpha}{|z - w|^\alpha} : z, w \in K, z \neq w\}.$$

Then $C < \infty$. Let $f \in B$. Then $f \in C(X)$, $f|_K \in A(K)$ and $f|_K \in \text{Lip}(K, \alpha)$. Since $\varphi \in A(X, K)$, we have $\varphi \in C(X)$ and $\varphi|_K \in A(K)$. According to $f, \varphi \in C(X)$ and $\varphi(X) \subseteq X$, we deduce that $f \circ \varphi \in C(X)$. Therefore,

$u \cdot (f \circ \varphi) \in C(X)$ since $u \in B \subseteq C(X)$. Since $\varphi|_K \in A(K)$, $\varphi(\text{int(K)}) \subseteq \text{int(K)}$ and $f|_K \in A(K)$, we deduce that $(f \circ \varphi)|_K \in A(K)$. Note that $u|_K \in A(K)$ since $u \in B$. Therefore, $T(f)|_K \in A(K)$.

We now show that $T(f)|_K \in \text{Lip}(K, \alpha)$. If $z, w \in K$ with $\varphi(z) \neq \varphi(w)$, we have

$$\begin{aligned}\frac{|T(f)(z) - T(f)(w)|}{|z-w|^\alpha} &= \frac{|u(z)f(\varphi(z)) - u(w)f(\varphi(w))|}{|z-w|^\alpha}\\ &\leq \frac{|u(z)||\varphi(z)-\varphi(w)|^\alpha}{|z-w|^\alpha}\frac{|f(\varphi(z)) - f(\varphi(w))|}{|\varphi(z)-\varphi(w)|^\alpha} + \frac{|u(z)-u(w)|}{|z-w|^\alpha}|f(\varphi(z))|\\ &\leq Cp_{K,\alpha}(f) + p_{K,\alpha}(u)\|f\|_X.\end{aligned}$$

If $z, w \in K$ with $z \neq w$ and $\varphi(z) = \varphi(w)$, then

$$\begin{aligned}\frac{|T(f)(z) - T(f)(w)|}{|z-w|^\alpha} &= \frac{|u(z)f(\varphi(z)) - u(w)f(\varphi(w))|}{|z-w|^\alpha}\\ &= \frac{|u(z)-u(w)|}{|z-w|^\alpha}|f(\varphi(z))|\\ &\leq p_{K,\alpha}(u)\|f\|_X\\ &\leq Cp_{K,\alpha}(f) + p_{K,\alpha}(u)\|f\|_X.\end{aligned}$$

Therefore,

$$\frac{|T(f)(z) - T(f)(w)|}{|z-w|^\alpha} \leq Cp_{K,\alpha}(f) + p_{K,\alpha}(u)\|f\|_X$$

for all $z, w \in K$ with $z \neq w$. This implies that $T(f)|_K \in \text{Lip}(K, \alpha)$. According to $T(f) \in A(X, K)$ and $T(f)|_K \in \text{Lip}(K, \alpha)$, we get $T(f) \in B$. Therefore, $T = uC_\varphi$ is a weighted composition operator on B.

By part (iv) of Theorem 2.1, we have

$$\sup\{|u(z)|\frac{|\varphi(z)-\varphi(w)|^\alpha}{|z-w|^\alpha} : z, w \in K, z \neq w\} \leq M,$$

where $M = \|T\|_{op}(\text{diam(X)})^{1-\alpha}((\text{diam(X)})^\alpha + 1)$. Let H be a nonempty compact subset of $K \cap \text{coz(u)}$. The continuity of u on H implies that there exists $z_0 \in H$ such that $|u(z_0)| \leq |u(z)|$ for all $z \in H$. Note that $|u(z_0)| > 0$ since H is a subset of coz(u). Let $z, w \in H$ with $z \neq w$. According to $H \subseteq K$, we have

$$\frac{|\varphi(z)-\varphi(w)|}{|z-w|^\alpha} \leq \frac{1}{|u(z_0)|}|u(z)|\frac{|\varphi(z)-\varphi(w)|}{|z-w|^\alpha} \leq \frac{M}{|u(z_0)|}.$$

Since the above inequality holds for all $z, w \in H$ with $z \neq w$, we deduce that $\varphi|_H \in \text{Lip}(H, \alpha)$. Hence, the proof is complete. □

The following example shows that the conditions $\varphi \in \text{Lip}_A(X, K, 1)$ in Theorem 2.3 and $\varphi \in A(X, K)$ in Theorem 2.4 do not necessary in general.

Example 2.5. Let $X = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $K = \overline{\mathbb{D}_r} = \{z \in \mathbb{C} : |z| \leq r\}$ where $0 < r \leq 1$. Assume that $\alpha \in (0, 1]$. Define the self-map φ of X by

$$\varphi(z) = \begin{cases} \frac{1}{2} & z = 0,\\ z & 0 < |z| \leq r,\\ \frac{rz}{|z|} & r < |z| \leq 1.\end{cases}$$

Then $\varphi(X) \subseteq X$ and $\varphi(\text{int(k)}) \subseteq \text{int(k)}$. In addition, φ is not continuous on X. Clearly, $u \in \text{Lip}_A(X, K, \alpha)$. Therefore, $\varphi \notin A(X, K)$ and so $\varphi \notin \text{Lip}_A(X, K, 1)$. Define the function $u : X \to \mathbb{C}$ by $u(z) = z, z \in X$. Clearly, $u \in \text{Lip}_A(X, K, \alpha)$. We show that

$$\sup\left\{|u(z)|\frac{|\varphi(z)-\varphi(w)|^\alpha}{|z-w|^\alpha} : z, w \in K, z \neq w\right\} \leq (\frac{3}{2})^\alpha. \tag{2.6}$$

If $z, w \in K\backslash\{0\}$ with $z \neq w$, then

$$|u(z)|\frac{|\varphi(z)-\varphi(w)|^{\alpha}}{|z-w|^{\alpha}} = |z|\frac{|z-w|^{\alpha}}{|z-w|^{\alpha}} = |z| \leq r \leq 1 \leq (\frac{3}{2})^{\alpha}.$$

If $z = 0$ and $w \in K\backslash\{0\}$, then

$$|u(z)|\frac{|\varphi(z)-\varphi(w)|}{|z-w|^{\alpha}} = 0 \leq (\frac{3}{2})^{\alpha}.$$

If $z \in K\backslash\{0\}$ and $w = 0$, then

$$|u(z)|\frac{|\varphi(z)-\varphi(w)|^{\alpha}}{|z-w|^{\alpha}} = |z|\frac{|z-\frac{1}{2}|^{\alpha}}{|z-0|^{\alpha}} \leq |z|^{1-\alpha}(|z|+\frac{1}{2})^{\alpha} \leq (\frac{3}{2})^{\alpha}.$$

Hence, (2.6) holds. Now, we show that $T = uC_{\varphi} : \mathrm{Lip}(X, K, \alpha) \to \mathbb{C}^X$ is a weighted composition operator on $\mathrm{Lip}_A(X, K, \alpha)$. Let $f \in \mathrm{Lip}_A(X, K, \alpha)$. It is easy to see that

$$T(f)(z) = \begin{cases} zf(z) & z \in \mathbb{C}, |z| \leq 1, \\ zf(\frac{rz}{|z|}) & z \in \mathbb{C}, r \leq |z| \leq 1. \end{cases}$$

This implies that $T(f) \in C(X)$, $T(f)|_K \in \mathrm{Lip}(K, \alpha)$ and $T(f)$ is analytic on $\mathrm{int(K)} = \{z \in \mathbb{C} : |z| < r\}$. Therefore, $T(f) \in \mathrm{Lip}_A(X, K, \alpha)$. It follows that $T = uC_{\varphi}$ is a weighted composition operator on $\mathrm{Lip}(X, K, \alpha)$.

In the following example, we give compact plane sets X, a closed subset K of with nonempty interior, a self-map φ of X and a function $u \in \mathrm{Lip}_A(X, K, \alpha)$ for $\alpha \in (0, 1]$ such that the map $T = uC_{\varphi}$ from $\mathrm{Lip}_A(X, K, \alpha)$ into $\mathbb{C}^X$ is a weighted composition operator on the algebra $\mathrm{Lip}_A(X, K, \alpha)$ but φ is not a Lipschitz mapping of order $\alpha \in (\frac{1}{2}, 1]$ on $\mathrm{coz(u)} \cap \mathrm{int(K)}$.

Example 2.6. Let $X = \overline{\mathbb{D}}$, $K = \overline{\mathbb{D}_{\delta}}$ where $0 < \delta \leq 1$ and $\alpha \in (0, 1]$. Note that for each $z \in \overline{\mathbb{D}_{\delta}}\backslash\{-\delta\}$ we have $|\frac{\delta+z}{2}| \leq 1$ and $0 \leq \mathrm{Re}(\frac{\delta+z}{2}) \leq 1$ which implies that $\mathrm{Arg}(\frac{\delta+z}{2}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Define the map $\psi : \overline{\mathbb{D}_{\delta}} \to \mathbb{C}$ as follows. Let $\psi(z)$ be the principle value of $(\frac{\delta+z}{2})^{\frac{1}{2}}$ for $z \in \overline{\mathbb{D}_{\delta}}\backslash\{-\delta\}$ and let $\psi(-\delta) = 0$. Then ψ is continuous on $K = \overline{\mathbb{D}_{\delta}}$ and analytic on $\mathrm{int(K)} = \mathbb{D}_{\delta} = \{z \in \mathbb{C} : |z| < \delta\}$. By Tietze's extension theorem [13, Theorem 20.4], there exists a complex-valued continuous function φ on $X = \overline{\mathbb{D}}$ with $\varphi|_{\overline{\mathbb{D}_{\delta}}} = \psi$ on $\overline{\mathbb{D}_{\delta}} = K$ and $\|\varphi\|_{\overline{\mathbb{D}}} = \|\psi\|_{\overline{\mathbb{D}_{\delta}}}$.Therefore, φ is a self-map of $X = \overline{\mathbb{D}}$ which is continuous on $X = \overline{\mathbb{D}}$ and analytic on $\mathrm{int(K)} = \mathbb{D}_{\delta}$. Define the function $u : X \to \mathbb{C}$ by $u(z) = \delta + z$, $z \in X$. Clearly, $u \in \mathrm{Lip}_A(X, K, \alpha)$. We now show that

$$\sup\{|u(z)|\frac{|\varphi(z)-\varphi(w)|}{|z-w|^{\alpha}} : z, w \in K, z \neq w\} \leq 2^{1-\alpha}, \tag{2.7}$$

for all $z, w \in K$ with $z \neq w$. To this aim, pick $z, w \in K$ with $z \neq w$. Let us distinguish following cases.

Case1. $z, w \in K\backslash\{-\delta\}$ and $z \neq w$. Assume that $r = |\frac{\delta+z}{2}|$ and $\rho = |\frac{\delta+w}{2}|$. Then $0 < r \leq 1$, $0 < \rho \leq 1$ and there exists $\theta, \eta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that

$$\frac{\delta+z}{2} = re^{i\theta}, \qquad \frac{\delta+w}{2} = \rho e^{i\eta}.$$

Therefore,

$$\begin{aligned} |(\frac{\delta+z}{2})^{\frac{1}{2}} + (\frac{\delta+w}{2})^{\frac{1}{2}}|^2 &= |\sqrt{r}e^{i\theta} + \sqrt{\rho}e^{i\eta}|^2 \\ &= (\sqrt{r}\cos\frac{\theta}{2} + \sqrt{\rho}\cos\frac{\eta}{2})^2 + (\sqrt{r}\sin\frac{\theta}{2} + \sqrt{\rho}\sin\frac{\eta}{2})^2 \\ &= r + \rho + 2\sqrt{r\rho}\cos(\frac{\theta-\eta}{2}) \\ &\geq r. \end{aligned}$$

This implies that

$$\begin{aligned}
|u(z)|\frac{|\varphi(z)-\varphi(w)|^{\alpha}}{|z-w|^{\alpha}} &= |\delta+z|\frac{|(\frac{\delta+z}{2})^{\frac{1}{2}}-(\frac{\delta+w}{2})^{\frac{1}{2}}|^{\alpha}}{|z-w|^{\alpha}}\\
&= |\delta+z|\frac{|(\frac{\delta+z}{2})^{\frac{1}{2}}-(\frac{\delta+w}{2})^{\frac{1}{2}}|^{\alpha}}{|(\delta+z)-(\delta+w)|^{\alpha}}\\
&= 2^{-\alpha}\frac{|\delta+z|}{|(\frac{\delta+z}{2})^{\frac{1}{2}}+(\frac{\delta+w}{2})^{\frac{1}{2}}|^{\alpha}}\\
&\leq \frac{2^{-\alpha}|\delta+z|}{r^{\frac{\alpha}{2}}}\\
&= \frac{2^{-\alpha}\cdot 2r}{r^{\frac{\alpha}{2}}}\\
&= 2^{1-\alpha}r^{\frac{2-\alpha}{2}}\\
&\leq 2^{1-\alpha}.
\end{aligned}$$

Case 2. $z=-\delta$ and $w\in K\backslash\{-\delta\}$. Then

$$|u(z)|\frac{|\varphi(z)-\varphi(w)|^{\alpha}}{|z-w|^{\alpha}}=0\leq 2^{1-\alpha}.$$

Case 3. $z\in K\backslash\{-\delta\}$ and $w=-\delta$. Then

$$\begin{aligned}
|u(z)|\frac{|\varphi(z)-\varphi(w)|^{\alpha}}{|z-w|^{\alpha}} &= |\delta+z|\frac{|(\frac{\delta+z}{2})^{\frac{1}{2}}-(\frac{-\delta+\delta}{2})^{\frac{1}{2}}|^{\alpha}}{|\delta+z|^{\alpha}}\\
&= \frac{|\delta+z|^{1+\frac{\alpha}{2}}}{2^{\frac{\alpha}{2}}|\delta+z|^{\alpha}}\\
&= 2^{1-\alpha}|\frac{\delta+z}{2}|^{1-\frac{\alpha}{2}}\\
&\leq 2^{1-\alpha}.
\end{aligned}$$

Summarising, we have proved (2.7) holds for all $z,w\in K$ with $z\neq w$ and so

$$\sup\{|u(z)|\frac{|\varphi(z)-\varphi(w)|}{|z-w|^{\alpha}}:z,w\in K,z\neq w\}\leq 2^{1-\alpha}<\infty.$$

Therefore, $T=uC_{\varphi}:\mathrm{Lip}_A(X,K,\alpha)\to\mathbb{C}^X$ is a weighted composition operator on $\mathrm{Lip}_A(X,K,\alpha)$ by Theorem 2.4. We now show that φ is not a Lipschitz mapping of order $\alpha\in(\frac{1}{2},1]$ on $\mathrm{coz(u)}\cap\mathrm{int(K)}$. Note that

$$\mathrm{coz(u)}\cap\mathrm{int(K)}=\{-\delta\}\cap\{z\in\mathbb{C}:|z|<\delta\}=\{z\in\mathbb{C}:|z|<\delta\}.$$

Take $\alpha\in(\frac{1}{2},1]$. Then $-\delta+\frac{8}{n^2},-\delta+\frac{2}{n^2}\in\mathrm{coz(u)}\cap\mathrm{int(K)}$ and

$$\frac{|\varphi(-\delta+\frac{8}{n^2})-\varphi(-\delta+\frac{2}{n^2})|}{|(-\delta+\frac{8}{n^2})-(-\delta+\frac{2}{n^2})|^{\alpha}}=\frac{|(\frac{8}{2n^2})^{\frac{1}{2}}-(\frac{2}{2n^2})^{\frac{1}{2}}|}{|\frac{6}{n^2}|^{\alpha}}=\frac{n^{2\alpha-1}}{6^{\alpha}}$$

for all $n\in\mathbb{N}$. Therefore, φ is not a Lipschitz mapping of order α on $\mathrm{coz(u)}\cap\mathrm{int(K)}$.

3 Compact weighted composition operators

We first give a necessary condition for which a weighted composition operator on extended analytic Lipschitz algebras to be compact.

Theorem 3.1. Let X be a compact plane set, K be a closed subset of X with nonempty interior, u be a complex-valued function on X, φ be a self-map of X, $\alpha\in(0,1]$ and $T=uC_{\varphi}:\mathrm{Lip}_A(X,K,\alpha)\to\mathbb{C}^X$ be a weighted composition operator on $\mathrm{Lip}_A(X,K,\alpha)$. If T is compact, then $\{T(f_n)\}_{n=1}^{\infty}$ converges to 0_X in $(\mathrm{Lip}_A(X,K,\alpha),\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)})$ for each bounded sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathrm{Lip}_A(X,K,\alpha)$ which converges uniformly to 0_X on X.

Proof . Let $T = uC_\varphi$ be compact. Assume that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $(\mathrm{Lip}_A(X,K,\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,\alpha)})$ which converges uniformly to 0_X on X. We show that

$$\lim_{n\to\infty} T(f_n) = 0_X \qquad (\text{in } (\mathrm{Lip}_A(X,K,\alpha), \|\cdot\|_{\mathrm{Lip}(X,K,\alpha)})). \tag{3.1}$$

Suppose that (3.1) does not hold. Then there exists $\varepsilon > 0$ and a strictly increasing function $q : \mathbb{N} \to \mathbb{N}$ such that

$$\|T(f_{q(j)})\|_{\mathrm{Lip}(X,K,\alpha)} \geq \varepsilon \tag{3.2}$$

for all $j \in \mathbb{N}$. Define the sequence $\{g_j\}_{j=1}^{\infty}$ in $\mathrm{Lip}_A(X,K,\alpha)$ by

$$g_j = f_{q(j)} \qquad (j \in \mathbb{N}).$$

Since $\{f_n\}_{n=1}^{\infty}$ converges uniformly to 0_X on X, we deduce that $\{g_j\}_{j=1}^{\infty}$ converges uniformly to 0_X on X. This implies that $\{g_j \circ \varphi\}_{j=1}^{\infty}$ converges uniformly to 0_X on X because φ is a self-map of X. Since $T = uC_\varphi$ is a weighted composition operator on $\mathrm{Lip}_A(X,K,\alpha)$, by part (i) of Theorem 2.1 we deduce that $u = T(1_X) \in \mathrm{Lip}_A(X,K,\alpha)$ and so u is a bounded complex-valued function on X. Therefore, $\{u \cdot (g_j \circ \varphi)\}_{j=1}^{\infty}$ converges uniformly to 0_X on X. Thus

$$\lim_{j\to\infty} \|T(g_j) - 0_X\|_X = 0. \tag{3.3}$$

According to the boundedness of $\{f_n\}_{n=1}^{\infty}$ in $\mathrm{Lip}_A(X,K,\alpha)$ with the norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$, we deduce that $\{g_j\}_{j=1}^{\infty}$ is a bounded sequence in $\mathrm{Lip}_A(X,K,\alpha)$ with the norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$. The compactness of $T : \mathrm{Lip}_A(X,K,\alpha) \to \mathrm{Lip}_A(X,K,\alpha)$ implies that there exists a strictly increasing function $r : \mathbb{N} \to \mathbb{N}$ and a function $g \in \mathrm{Lip}_A(X,K,\alpha)$ such that

$$\lim_{j\to\infty} \|T(g_{r(j)}) - g\|_{\mathrm{Lip}(X,K,\alpha)} = 0. \tag{3.4}$$

According to $\|h\|_X \leq \|h\|_{\mathrm{Lip}(X,K,\alpha)}$ for all $h \in \mathrm{Lip}_A(X,K,\alpha)$, by (3.4) we deduce that

$$\lim_{j\to\infty} \|T(g_{r(j)}) - g\|_X = 0. \tag{3.5}$$

Since $\{g_{r(j)}\}_{j=1}^{\infty}$ is a subsequence of $\{g_j\}_{j=1}^{\infty}$, by (3.3) we have

$$\lim_{j\to\infty} \|T(g_{r(j)}) - 0_X\|_X = 0. \tag{3.6}$$

According to (3.5) and (3.6), we get $g = 0_X$. Therefore, by (3.4) we have

$$\lim_{j\to\infty} \|T(g_{r(j)}) - 0_X\|_{\mathrm{Lip}(X,K,\alpha)} = 0$$

which implies that there exists $N \in \mathbb{N}$ such that

$$\|T(g_{r(N)})\|_{\mathrm{Lip}(X,K,\alpha)} < \varepsilon. \tag{3.7}$$

Since $r(N) \in \mathbb{N}$, $g_{r(N)} = f_{q(r(N))}$ and the inequality (3.2) holds for all $j \in \mathbb{N}$, we deduce that

$$\|T(g_{r(N)})\|_{\mathrm{Lip}(X,K,\alpha)} = \|T(f_{q(r(N))})\|_{\mathrm{Lip}(X,K,\alpha)} \geq \varepsilon$$

which contradicts to (3.7). Therefore, (3.1) holds and so the proof is complete. □

Note that Theorem 3.1 is a generalization of the necessity part of [5, Corollary 2.1]. We now give a sufficient condition for which a weighted composition operator on extended analytic Lipschitz algebras to be compact.

Theorem 3.2. Let X be a compact plane set, K be a closed subset of X with nonempty interior, u be a complex-valued function on X, φ be a self-map of X with $\varphi(X) \subseteq K$ and $T = uC_\varphi : \mathrm{Lip}_A(X,K,\alpha) \to \mathbb{C}^X$ be a weighted composition operator on $\mathrm{Lip}_A(X,K,\alpha)$. Then T is compact if $\varphi(\mathrm{coz(u)}) \subseteq \mathrm{int(K)}$ and

$$\lim_{\substack{z,w\in K\\ z\neq w\\ |u(z)|\to 0}} |u(z)| \frac{|\varphi(z) - \varphi(w)|^\alpha}{|z-w|^\alpha} = 0. \tag{3.8}$$

Proof . Let $\varphi(\text{coz(u)}) \subseteq \text{int(K)}$ and (3.8) to be hold. To prove the compactness of $T = uC_\varphi$, let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\text{Lip}_A(X, K, \alpha)$ with $\|f_n\|_{\text{Lip}(X,K,\alpha)} \leq 1$. Then $\|f_n\|_X \leq 1$ and $p_{K,\alpha}(f_n) \leq 1$ for all $n \in \mathbb{N}$. According to $K \subseteq X$ and $\|f_n\|_X \leq 1$, we deduce that $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded sequence on K. We claim that $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on metric space (K, d). Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon^{\frac{1}{\alpha}}$. Then $\delta > 0$ and $\delta^\alpha = \varepsilon$. Assume that $z, w \in K$ with $|z - w| < \delta$. According to $p_{K,\alpha}(f_n) \leq 1$ for all $n \in \mathbb{N}$, we have

$$|f_n(z) - f_n(w)| \leq p_{K,\alpha}(f_n)|z - w|^\alpha \leq |z - w|^\alpha \leq \delta^\alpha = \varepsilon$$

for all $n \in \mathbb{N}$. Hence, our claim is justified. By Arzela-Ascoli theorem, $\{f_n\}_{n=1}^{\infty}$ has a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that $\{f_{n_j}\}_{j=1}^{\infty}$ converges uniformly on K. According to $\varphi(X) \subseteq K$, we deduce that $\{f_{n_j} \circ \varphi\}_{j=1}^{\infty}$ converges uniformly on X. This implies that $\{f_{n_j} \circ \varphi\}_{j=1}^{\infty}$ is a Cauchy sequence in $(C(X), \|\cdot\|_X)$. According to $T = uC_\varphi$ is a weighted composition operator on $\text{Lip}_A(X, K, \alpha)$, by part (i) of Theorem 2.1 we deduce that $u = T(1_X) \in \text{Lip}_A(X, K, \alpha)$ and so u is bounded complex-valued function on X. Therefore, $\{T(f_{n_j})\}_{j=1}^{\infty}$ is a Cauchy sequence in $(C(X), \|\cdot\|_X)$. We claim that $\{T(f_{n_j})\}_{j=1}^{\infty}$ is a Cauchy sequence in $(\text{Lip}_A(X, K, \alpha), \|\cdot\|_{\text{Lip}(X,K,\alpha)})$. Let $\varepsilon > 0$ be given. According to $\{T(f_{n_j})\}_{j=1}^{\infty}$ is a Cauchy sequence in $(C(X), \|\cdot\|_X)$, there exists $N_1 \in \mathbb{N}$ such that

$$\|u \cdot (f_{n_j} - f_{n_k}) \circ \varphi)\|_X < \frac{\varepsilon}{2}, \tag{3.9}$$

for all $j, k \in \mathbb{N}$ with $j \geq N_1$ and $k \geq N_1$. By the definition of limit (3.8), there exists a $\delta > 0$ such that

$$|u(z)| \frac{|\varphi(z) - \varphi(w)|^\alpha}{|z - w|^\alpha} < \frac{\varepsilon}{4}, \tag{3.10}$$

wherever $z, w \in K$ with $w \neq z$ and $|u(z)| < \delta$. Let $F_\delta = \{z \in K : |u(z)| \geq \delta\}$. By the continuity of u, we deduce that F_δ is a compact subset of coz(u). Since $T = uC_\varphi$ is a weighted composition operator on $\text{Lip}_A(X, K, \alpha)$, the part (iii) of Theorem 2.1 implies that φ is continuous on coz(u). Therefore, $\varphi(F_\delta)$ is a compact plane set. Since $\varphi(\text{coz(u)}) \subseteq \text{int(K)}$ and $\varphi(F_\delta) \subseteq \varphi(\text{coz(u)})$, we get $\varphi(F_\delta) \subseteq \text{int(K)}$. Take

$$C = \sup\{|u(z)| \frac{|\varphi(z) - \varphi(w)|^\alpha}{|z - w|^\alpha} : z, w \in K, z \neq w\}.$$

According to T is a weighted composition operator on $\text{Lip}_A(X, K, \alpha)$, part (iv) of Theorem 2.1 implies that $C < \infty$. Since $\{f_{n_j}\}_{n=1}^{\infty}$ is uniformly convergent on K and f_{n_j} is analytic on int(K) for all $j \in \mathbb{N}$, by Montel's theorem the sequences $\{f_{n_j}\}_{n=1}^{\infty}$ and $\{f'_{n_j}\}_{n=1}^{\infty}$ are uniformly convergent on the compact subsets of int(K). According to the compactness of K and $\varphi(K)$ in the complex plane $\mathbb{C}$, by using [8, Lemma 1.5] we deduce that there exist a finite union of uniformly regular sets in int(K) containing $\varphi(K)$, namely Y, and a positive constant C_0 such that for every analytic complex-valued function f on int(K) and any $z, w \in \varphi(K)$,

$$|f(z) - f(w)| \leq C_0|z - w|(\|f\|_Y + \|f'\|_Y). \tag{3.11}$$

Since $\{f_{n_j}\}_{j=1}^{\infty}$ and $\{f'_{n_j}\}_{j=1}^{\infty}$ are uniformly convergent on Y and $\{f_{n_j}\}_{j=1}^{\infty}$ is uniformly convergent on K, there exist $N_2, N_3, N_4 \in \mathbb{N}$ such that

$$\|f_{n_j} - f_{n_k}\|_Y < \frac{\varepsilon}{8CC_0} \tag{3.12}$$

for all $j, k \in \mathbb{N}$ with $j \geq N_2$ and $k \geq N_2$,

$$\|f'_{n_j} - f'_{n_k}\|_Y < \frac{\varepsilon}{8CC_0} \tag{3.13}$$

for all $j, k \in \mathbb{N}$ with $j \geq N_3$ and $k \geq N_3$,

$$\|f_{n_j} - f_{n_k}\|_K < \frac{\varepsilon}{4p_{K,\alpha}(u) + 1} \tag{3.14}$$

for all $j, k \in \mathbb{N}$ with $j \geq N_4$ and $k \geq N_4$. Put $N = \max\{N_1, N_2, N_3, N_4\}$. Let $j, k \in \mathbb{N}$ with $j \geq N$ and $k \geq N$. Then (3.11), (3.12), (3.13) and (3.14) hold.

If $z, w \in K$ with $\varphi(z) \neq \varphi(w)$, then

$$\begin{aligned}
\frac{|T(f_{n_j} - f_{n_k})(z) - T(f_{n_j} - f_{n_k})(w)|}{|z-w|^\alpha} =& \frac{|u(z)(f_{n_j}(\varphi(z)) - f_{n_k}(\varphi(z))) - u(w)(f_{n_j}(\varphi(w)) - f_{n_k}(\varphi(w)))|}{|z-w|^\alpha} \\
\leq& \frac{|u(z)||(f_{n_j} - f_{n_k})(\varphi(z)) - (f_{n_j} - f_{n_k})(\varphi(w))|}{|z-w|^\alpha} \\
&+ \frac{|u(z) - u(w)|}{|z-w|^\alpha}|(f_{n_j} - f_{n_k})(\varphi(w))| \\
=& |u(z)| \frac{|\varphi(z) - \varphi(w)|}{|z-w|^\alpha} \frac{|(f_{n_j} - f_{n_k})(\varphi(z)) - (f_{n_j} - f_{n_k})(\varphi(w))|}{|\varphi(z) - \varphi(w)|} \\
&+ \frac{|u(z) - u(w)|}{|z-w|^\alpha}|(f_{n_j} - f_{n_k})(\varphi(w))| \\
\leq& CC_0(\|f_{n_j} - f_{n_k}\|_Y + \|f'_{n_j} - f'_{n_k}\|_Y) + p_{K,\alpha}(u)\|f_{n_j} - f_{n_k}\|_K \\
\leq& CC_0(\frac{\varepsilon}{8CC_0} + \frac{\varepsilon}{8CC_0}) + p_{K,\alpha}(u)\frac{\varepsilon}{4p_{K,\alpha}(u)+1} \\
\leq& \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} \\
=& \frac{\varepsilon}{2}.
\end{aligned}$$

If $z, w \in K$ with $z \neq w$ and $\varphi(z) = \varphi(w)$, then

$$\frac{|T(f_{n_j} - f_{n_k})(z) - T(f_{n_j} - f_{n_k})(w)|}{|z-w|^\alpha} = 0 < \frac{\varepsilon}{2}.$$

Therefore,

$$\frac{|T(f_{n_j} - f_{n_k})(z) - T(f_{n_j} - f_{n_k})(w)|}{|z-w|^\alpha} < \frac{\varepsilon}{2}$$

for all $z, w \in K$ with $z \neq w$. This implies that

$$p_{K,\alpha}(T(f_{n_j} - f_{n_k})) \leq \frac{\varepsilon}{2}. \tag{3.15}$$

According to $j \geq N_1$ and $k \geq N_1$, by (3.9) we have

$$\|T(f_{n_j} - f_{n_k})\|_X < \frac{\varepsilon}{2}. \tag{3.16}$$

By (3.15) and (3.16), we get

$$\|T(f_{n_j} - f_{n_k})\|_{\mathrm{Lip}(X,K,\alpha)} < \varepsilon. \tag{3.17}$$

According to (3.17) and the linearity of T, we deduce that

$$\|T(f_{n_j}) - T(f_{n_k})\|_{\mathrm{Lip}(X,K,\alpha)} < \varepsilon.$$

Therefore, our claim is justified . Since $(\mathrm{Lip}(X, K, \alpha), \|\cdot\|_{\mathrm{Lip}(X,K,\alpha)})$ is a Banach space, we deduce that $\{T(f_{n_j})\}_{j=1}^\infty$ converges in $(\mathrm{Lip}(X, K, \alpha)$ with the norm $\|\cdot\|_{\mathrm{Lip}(X,K,\alpha)}$. Hence, T is compact and so the proof is complete. □

Note that Theorem 3.2 is a generalization of [5, Theorem 2.2]. We now give some necessary condition on a function $u \in \mathbb{C}^X$ and a self-map φ of X by omiting the condition $\varphi(X) \subseteq K$ that the weighted composition operator $T = uC_\varphi$ on $\mathrm{Lip}(X, K, 1)$ to be compact. We will need some preliminaries including angular derivatives.

Definition 3.3. Let$z_0 \in \mathbb{C}$, $r > 0$ and $\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

(a) A *sector* in $\mathbb{D}(z_0, r)$ at a point $\omega \in \partial\mathbb{D}(z_0, r)$ is the region between two straight lines in $\mathbb{D}(z_0, r)$ that meet at ω and are symmetric about the radius to ω.

(b) If f is a complex-valued function on $\mathbb{D}(z_0, r)$ and $\omega \in \partial\mathbb{D}(z_0, r)$, then $\angle\lim_{z\to\omega} f(z) = \mathrm{L}$ means $f(z) \to \mathrm{L}$ as $z \to \omega$ through any sector at ω. When this happens, we say that L is *angular* (or *non-tangential*) *limit* of f at ω.

(c) An analytic function $\psi : \mathbb{D}(z_0, r) \to \mathbb{D}_\rho$ has an *angular derivation* at a point $\omega \in \mathbb{D}(z_0, r)$ if for some $\eta \in \partial\mathbb{D}_\rho$

$$\angle \lim_{z\to\omega} \frac{\eta - \psi(z)}{\omega - z} \tag{3.18}$$

exists (finitely). We call this limit the *angular derivation* of ψ at ω and denote it by $\angle\psi'(\omega)$.

Proposition 3.4 ([2, Proposition 2.10]). *Let $z_0 \in \mathbb{C}$, $r > 0$, $K = \overline{\mathbb{D}(z_0, r)}$ and X be compact plane set with $K \subseteq X$. Suppose that $\omega \in \partial\mathbb{D}(z_0, r)$ and $\varphi \in \mathrm{Lip}_A(X, K, \alpha)$ is a nonconstant function with $|\varphi(c)| = \|\varphi\|_{\overline{\mathbb{D}(z_0,r)}}$. Then the angular derivative of φ at ω exists and is nonzero.*

Definition 3.5. Let X be a plane set with int(X) $\neq \varnothing$ and $\partial X \neq \varnothing$.

(a) We say that X at $\omega \in \partial X$ has an *internal circular tangent* if there exists a disc D in the complex plane $\mathbb{C}$ such that $\omega \in \partial X$ and $\overline{D}\backslash\{\omega\} \subseteq$ int(X).

(b) X is a called *strongly accessible from the interior* if it has an internal circular tangent at each point of its boundary. Such sets include the closed unit disc $\overline{\mathbb{D}}$ and $\overline{\mathbb{D}(z_0, r)}\backslash \cup_{k=1}^{n} \mathbb{D}(z_k, r_k)$, where closed discs $\mathbb{D}(z_k, r_k)$ are mutually disjoint in $\mathbb{D}(z_0, r)$.

(c) We say that X has a *peak boundary* with respect a family B of complex-valued bounded functions on X if for each $\zeta \in \partial X$ there exists a nonconstant function $h \in B$ such that $\|h\|_X = h(\zeta) = 1$.

Theorem 3.6. Suppose that X is a nonempty compact plane set such that int(X) is a connected set, X is the closure of int(X) and X has a peak boundary with respect to $\mathrm{Lip}_A(X, 1)$. Let u be a complex-valued function on X and K be a closed subset of X such that int(K) $\cap$ coz(u) is a nonempty connected set in $\mathbb{C}$, int(K $\cap \overline{\text{coz(u)}}$) = int(K) $\cap$ coz(u) and $K \cap \overline{\text{coz(u)}}$ is strongly accessible from the interior. Let φ ba a continuous self-map of X and $T = uC_\varphi : \mathrm{Lip}_A(X, K, 1) \to \mathbb{C}^X$ be a weighted composition operator on $\mathrm{Lip}_A(X, K, 1)$. If T is compact, then φ is constant on $K \cap$ coz(u) or $\varphi(K \cap$ coz(u)) $\subseteq$ int(X).

Proof . Let $T = uC_\varphi$ be compact. Suppose that φ is not constant on $K \cap$ coz(u). By part (iii) of Theorem 2.1, φ analytic on int(K) $\cap$ coz(u). According to the open mapping theorem in complex analysis, we deduce that φ(int(K) $\cap$ coz(u)) is an open set in $\mathbb{C}$. This implies that

$$\varphi(\text{int(K)} \cap \text{coz(u)}) \subseteq \text{int(X)}, \tag{3.19}$$

since $\varphi(X) \subseteq X$. We prove that

$$\varphi(K \cap \text{coz(u)}) \subseteq \text{int(X)}. \tag{3.20}$$

Suppose that (3.20) does not hold. Then there exists $\zeta \in K \cap$ coz(u) such that $\varphi(\zeta) \notin$ int(X). According to (3.19), we get $\zeta \notin$ int(X) $\cap$ coz(u). Since int(K $\cap \overline{\text{coz(u)}}$) = int(K) $\cap$ coz(u), we deduce that $\zeta \notin$ int(K $\cap \overline{\text{coz(u)}}$). Therefore, $\zeta \in \partial(K \cap \overline{\text{coz(u)}})$ because $K \cap \overline{\text{coz(u)}}$ is a closed set in $\mathbb{C}$.

Since X is the closure of int(X) in $\mathbb{C}$, $\varphi(X) \subseteq X$ and $\varphi(\zeta) \notin$ int(K), we deduce that $\varphi(\zeta) \in \partial(X)$. Therefore, there exists a nonconstant function $h \in \mathrm{Lip}_A(X, 1)$ with $\|h\|_X = h(\varphi(\zeta)) = 1$ because X has a peak boundary with respect to $\mathrm{Lip}_A(X, 1)$. Since $K \cap \overline{\text{coz(u)}}$ is strongly accessible from the interior and $\zeta \in \partial(K \cap \overline{\text{coz(u)}})$, there exists an open disc $D = \mathbb{D}(z_0, r)$ in $\mathbb{C}$ such that $\zeta \in \partial D$ and $\overline{D}\backslash\{\zeta\}$ is a subset of int(K $\cap \overline{\text{coz(u)}}$). Therefore, φ is analytic on D since $D \subseteq \overline{D}\backslash\{\zeta\}$ and $\overline{D}\backslash\{\zeta\} \subseteq$ int(k $\cap \overline{\text{coz(u)}}$) = int(K) $\cap$ coz(u). According to $\overline{D}\backslash\{\zeta\} \subseteq K \cap$ coz(u), $\zeta \in K \cap$ coz(u) and the compactness of $\overline{D}$, we deduce that $\overline{D}$ is a compact subset of $K \cap$ coz(u). Therefore, φ is a Lipschitz mapping of order 1 on $\overline{D}$. This implies that $h \circ \varphi$ is a Lipschitz mapping on $\overline{D}$. On the other hand, $h \circ \varphi \in C(X)$since $\varphi : X \to \mathbb{C}$ is continuous on X, $\varphi(K) \subseteq X$ and $h \in C(X)$. Therefore, $h \circ \varphi \in \mathrm{Lip}_A(X, \overline{D}, 1)$. We claim that $h \circ \varphi$ is constant on D. Otherwise, by Proposition 3.4 we deduce that $\angle(h \circ \varphi)'(\zeta)$ exists and

$$\angle(h \circ \varphi)'(\zeta) \neq 0. \tag{3.21}$$

Let $n \in \mathbb{N}$. Define the function $f_n : X \to \mathbb{C}$ by

$$f_n(z) = \frac{h^n(z)}{n} \qquad (z \in X). \tag{3.22}$$

Then $f_n \in \mathrm{Lip}_A(X,K,1)$ and

$$\|f_n\|_X = \|\frac{1}{n}h^n\|_X = \frac{1}{n}(\|h\|_X)^n = \frac{1}{n}. \tag{3.23}$$

Moreover, for each $z, w \in K$ with $z \neq w$ we have

$$\begin{aligned}\frac{|f_n(z) - f_n)(w)|}{|z-w|} &= \frac{|(h(z))^n - (h(w))^n|}{n|z-w|}\\ &\leq \frac{|h(z)-h(w)|}{n|z-w|}n(\|h\|_X)^n\\ &= \frac{|h(z)-h(w)|}{|z-w|}\\ &= p_{K,1}(h)\\ &\leq p_{X,1}(h).\end{aligned}$$

Therefore,

$$p_{K,1}(f_n) \leq p_{X,1}(h). \tag{3.24}$$

According to (3.23) and (3.24), we get

$$\|f_n\|_{\mathrm{Lip}(X,K,1)} \leq \frac{1}{n} + p_{X,1}(h) \leq 1 + p_{X,1}(h). \tag{3.25}$$

This implies that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathrm{Lip}_A(X,K,1)$ with the norm $\|\cdot\|_{\mathrm{Lip}(X,K,1)}$. Since (3.23) hold for all $n \in \mathbb{N}$, we deduce that

$$\lim_{n\to\infty} \|f_n\|_X = 0 \tag{3.26}$$

and so $\{f_n\}_{n=1}^{\infty}$ converge uniformly to 0_X on X. Since $T = uC_\varphi$ is compact, by Theorem 3.1 we deduce that

$$\lim_{n\to\infty} T(f_n) = 0 \qquad (\text{in } (\mathrm{Lip}_A(X,K,1), \|\cdot\|_{\mathrm{Lip}(X,K,1)})).$$

This implies that

$$\lim_{n\to\infty} p_{K,1}(T(f_n)) = 0. \tag{3.27}$$

Let $n \in \mathbb{N}$. Assume that $z, w \in K$ with $z \neq w$. Then

$$\begin{aligned}|u(z)|\frac{|f_n(\varphi(z)) - f_n)(\varphi(w))|}{|z-w|} &= \frac{|u(z)f_n(\varphi(z)) - u(z)f_n)(\varphi(w))|}{|z-w|}\\ &\leq \frac{|u(z)f_n(\varphi(z)) - u(z)f_n)(\varphi(w))|}{|z-w|} + \frac{|u(z)-u(w)|}{|z-w|}|f_n(\varphi(w))|\\ &\leq p_{K,1}(T(f_n)) + p_{K,1}(u)\|f_n\|_X.\end{aligned}$$

Let $z \in K$ with $z \neq \zeta$. Since $\zeta \in K$, by the argument above, we have

$$|u(z)|\frac{f_n(\varphi(z)) - f_n(\varphi(\zeta))|}{z-\zeta} \leq p_{K,1}(T(f_n)) + p_{K,1}(u)\|f_n\|_X.$$

This implies that

$$\sup_{\substack{z\in K\\ z\neq\zeta}} |u(z)|\frac{|f_n(\varphi(z)) - f_n(\varphi(\zeta))|}{|z-\zeta|} \leq p_{K,1}(T(f_n)) + p_{K,1}(u)\|f_n\|_X. \tag{3.28}$$

Since (3.28) holds for all $n \in \mathbb{N}$, according to (3.27) and (3.26) we get

$$\lim_{n\to\infty} \sup\{|u(z)|\frac{|f_n(\varphi(z)) - f_n(\varphi(\zeta))|}{|z-\zeta|} : z \in K, z \neq \zeta\} = 0. \tag{3.29}$$

Let $\varepsilon > 0$ be given. According to $\zeta \in \text{coz(u)}$ and (3.29), there exists a natural number N such that

$$\sup\left\{|u(z)|\frac{|f_N(\varphi(z)) - f_N(\varphi(\zeta))|}{|z-\zeta|} : z \in K, z \neq \zeta\right\} < \frac{\varepsilon}{2}|u(\zeta).|.$$

By the definition of f_N, we get

$$\sup\left\{|u(z)|\frac{|((h\circ\varphi)(z))^N - ((h\circ\varphi)(\zeta))^N|}{N|z-\zeta|} : z \in K, z \neq \zeta\right\} < \frac{\varepsilon}{2}|u(\zeta)|. \tag{3.30}$$

Let Γ be a sector in D at $\zeta \in \partial D$. According to (3.30), we have

$$\sup\left\{|u(z)|\frac{|((h\circ\varphi)(z))^N - ((h\circ\varphi)(\zeta))^N|}{N|z-\zeta|} : z \in \Gamma, z \neq \zeta\right\} < \frac{\varepsilon}{2}|u(\zeta)|. \tag{3.31}$$

Thus

$$\sup\left\{|u(z)|\frac{|((h\circ\varphi)(z))^N - ((h\circ\varphi)(\zeta))^N|}{N|z-\zeta|} : z \in \Gamma, z \neq \zeta\right\} < \frac{\varepsilon}{2}|u(\zeta)|.$$

According to the continuity of u and $h\circ\varphi$ at ζ, $(h\circ\varphi)(\zeta) = 1$ and the existence of $\angle(h\circ\varphi)^{'}(\zeta)$, we deduce that

$$\lim_{\substack{z\to\zeta\\ z\in\Gamma}}|u(z)|\frac{|((h\circ\varphi)(z))^N - ((h\circ\varphi)(\zeta))^N|}{N|z-\zeta|} = N|u(\zeta)||\angle(h\circ\varphi)^{'}(\zeta)|. \tag{3.32}$$

By (3.31) and (3.32), we get

$$|\angle(h\circ\varphi)^{'}(\zeta)| \leq \frac{\varepsilon}{2} < \varepsilon. \tag{3.33}$$

Since (3.33) holds for each $\varepsilon > 0$, we conclude that $|\angle(h\circ\varphi)^{'}(\zeta)| = 0$ which implies that $\angle(h\circ\varphi)^{'}(\zeta) = 0$. This contradicts to (3.21). Hence, our claim is justified. Therefore, h is constant on $\varphi(D)$. Since φ is a nonconstant analytic function on the connected open set D, by the open mapping theorem in complex analysis we deduce that $\varphi(D)$ is a connected open set in $\mathbb{C}$.

According to $\varphi(D) \subseteq \text{int(K)}$ and int(K) is a connected open set in $\mathbb{C}$, we conclude that h is constant on int(K). Therefore, h is constant on K since h is continuous on K and K is the closure of int(K). This contradicts to h is nonconstant on K. Therefore, (3.20) holds and so the proof is complete. □

Note that Theorem 3.6 is a generalization of [2, Theorem 2.14].

References

[1] D. Alimohammadi and S. Daneshmand, *Weighted composition operators between Lipschitz algebras of complex-valued bounded functions*, Caspian J. Math. Sci. **9** (2020), no. 1, 100–123.

[2] D. Alimohammadi and M. Mayghani, *Unital compact homomorphisms between extended analytic Lipschitz algebras*, Abstract Appl. Anal. **2011** (2011), Article ID: 146758, 15 pages.

[3] D. Alimohammadi and M. Mayghani, *Unital power compact and quasicompact endomorphisms of extended analytic Lipschitz algebras*, Int. J. Math. Anal. **6** (2012), no. 42, 2067–2082.

[4] D. Alimohammadi and S. Moradi, *Sufficient conditions for density in extended Lipschitz algebras*, Caspian J. Math. Sci. **3** (2014), no. 1, 141–151.

[5] S. Amiri, A. Golbaharan, and H. Mahyar, *Weighted composition operators on Analytic Lipschitz spaces*, Results Math. **73** (2022), no. 1, Article 46.

[6] R. Bagheri and D. Alimohammadi, *Weighted composition operators between extended Lipschitz algebras on compact metric spaces*, Sahand Commu. Math. Anal. **17** (2020), no.3, 33–70.

[7] Sh. Behrouzi, A. Golbaharan, and H. Mahyar, *Weighted composition operators between pointed Lipschitz spaces*, Results Math. **77** (2022), no.4, Article 157.

[8] F. Behrouzi and H. Mahyar, *Compact endomorphisms of certain analytic Lipschitz algebras*, Bull. Belg. Math. Soc. Simon Stevin **12** (2005), 301–312.

[9] A. Golbaharan and H. Mahyar, *Weighted composition operators on Lipschitz spaces*, Houst. J. Math. **42** (2016), no. 3, 905–917.

[10] A. Jiménez-Vargas and M. Villegas-Vallecillos, *Compact composition operators on noncompact Lipschitz spaces*, J. Math. Anal. Appl. **398** (2013), 221–229.

[11] T.G. Honary and S. Moradi, *On the maximal ideal space of extended analytic Lipschitz algebras*, Q. Math. **30** (2007), 349–353.

[12] M. Mayghani and D. Alimohammadi, *The structure of ideals, point derivations, amenability and weak amenability of extended Lipschitz algebras*, Int, J. Nonlinear Anal. Appl. **8** (2017), no. 1, 389–404.

[13] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, Third Edition, Singapore, 1987.

[14] D.R. Sherbert, *Banach algebras of Lipschitz functions*, Pacific. J. Math. **13** (1963), 1387–1399.

[15] D.R. Sherbert, *The structure of ideals and point derivations of Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. **111** (1964), 240–272.