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Fuzzy q-Taylor theorem

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Abstract

The main purpose of this work is to introduce and investigate fuzzy quantum calculus. Our idea begins with a general definition of fuzzy q-derivative on arbitrary time scales using the generalized Hukuhara difference. It compiled some basic facts in the fields of the fuzzy q-derivative and the fuzzy q-integral and proved them in detail. Proceed with this work, specifying the particular concept of fuzzy q-Taylor's expansion, especially for continuous and fuzzy valued functions which are non-differentiable in the classical (usual) concept, as the best tool for approximating functions and solving the fuzzy initial value q-problems. Eventually, some numerical examples of fuzzy q-Taylor's expansion of special functions and functions with switching points, are solved for illustration.

Keywords: Generalized Hukuhara q-difference, q-Taylor's expansion, Fuzzy q-derivative, Fuzzy q-integral, Fuzzy q-Taylor
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1 Introduction

In any physical entity or some physical property, a quantum has been introduced as the smallest quantity which can take on only discrete values consisting of integer multiples of one quantum. Its nominal root comes from 'quantus' which is a Latin word and it means that 'how great' or 'how many'. In 1909, quantum calculus (for short q-calculus) was applied to show the relevance between mathematics and physics by the q-differential definition [13] and considered the beginning of q-calculus. For more applications see [6, 7, 9, 10, 11, 16, 17, 22].

In recent decades, fuzzy mathematics has attracted attention of researchers and is progressing continually. In this way, first section of our paper consists of some preliminaries and summary of the fuzzy details and q-calculus which will be used in the other sections. In the time to follow, most of researches in the area were mainly directed to the study of the fuzzy logic and the fuzzy calculus (see for ex. [1, 4, 20, 21, 24]). That is to say that at our best knowledge, the question of the fuzzy of the quantum calculus, had never been considered. It is possible that seems difficult, but we will develop this point in the main parts of the paper. After preliminaries, the basic definitions and fundamental results for fuzzy q-derivative and fuzzy q-integral based on the generalized Hukuhara difference will be considered in some details.

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Finding approximation of functions has always been one of the most interesting branches of mathematics. The most practical and important tool in this area is the Taylor's expansion representing of a function as an infinite plural of terms which is obtained from the derivatives of function at a specific grid. Morever, it is used for approximating functions by using a finite number of their terms [23]. It is well-known that real life phenomena such many problems in different sciences are characterized by functions which have been presented as non-differentiable functions.

With this interpretation, since the Taylor's expansion is introduced on the basis of the function's derivatives; finding Taylor's expansion non-differentiable continuous functions will be impossible by classic methods. One of these approaches is to take refuge to the quantum calculus. Thus, we apply the q-calculus to find the Taylor expansion of mentioned functions because of its abilities to find the derivative using a difference operator and without computing the limit of functions. As a result, q-Taylor theorem will also be important in the quantum calculus. The concept of a q-Taylor series was formulated due to F. H. Jackson [14] as follows:

Let $\Phi(z)$ be a function which it can written as a convergent power series and $q \neq 1$, then we can write

$$\Phi(z) = \sum_{\nu=0}^{\infty} \frac{(z-z_0)_q^{\nu}}{[\nu]!_q} D_q^{\nu} \Phi(z_0),$$

where

$$(z-z_0)_q^{\nu} = \prod_{i=0}^{\nu-1}(z-q^i z_0) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{\frac{k(k-1)}{2}} z^{\nu-k}(-z_0)^k.$$

In accordance with the above, it is not unexpected to confront the non-differentiable functions in the fuzzy quantum calculus. A new notation for q-Taylor's formula in 1995 with S. C. Jing et al. [15] and in 1999 via T. Ernst [8] appeared. After that, AL-Salam and Stanton in [12] presented the q-type interpolation series. In paper [18] authors introduced enough criteria to have convergence of Ismail-Stanton q-Taylor series by using the approach of [19]. In 2008, M. H. Annaby et al. [5] for Jackson q-difference operators proved q-Taylor series.

As consequently, we present fuzzy q-Taylor formula with integral remainder based on generalized Hukuhara difference for approximation of fuzzy functions, especially every continuous functions that are not differentiable or are differentiable except on a set of isolated points.

Most of the issues in nature are led to solve of differential equations. For this reason, solving differential equations always has been one of the concerns of various sciences. The Taylor's expansion is the simple, introductory and original method for solving differential equations especially initial value problems. Therefore, we can consider the fuzzy q-Taylor's expansion method to solve fuzzy initial value q-problems as another one of its most important applications. The final of this paper, some numerical relevant examples have been indicated. Conclusions and future perspectives would close the paper. For this aim, we will focus on the following topics **:**

- Introducing the fuzzy q-derivative, fuzzy q-integral and some of their features
- Expressing and proving fuzzy q-Taylor theorem.
- Calculating the fuzzy q-Taylor's expansion of special functions.

2 Basic Concepts

The aim of this section is to present some details and definitions of fuzzy and quantum calculus that will be used throughout this paper and can be found in [2, 3, 16, 6], etc.

$\triangleleft$ fuzzy calculus:

Set $\mathbb{R}_{\mathcal{F}} = \{\psi : \mathbb{R}^n \to [0,1]\}$ is the fuzzy numbers set if

I. ψ is normal: $\exists z_0 \in \mathbb{R}^n$ s.t. $\psi(z_0) = 1$,

II. ψ is convex: for $0 \le \lambda \le 1$, $\psi(\lambda_{z_1} + (1-\lambda_{z_2})) \ge \min\{\psi(z_1), \psi(z_2)\}$,

III. ψ is upper semi-continuous: $\forall z_0 \in \mathbb{R}^n$, it holds that $\psi(z_0) \ge \lim_{z \to z_0^{\pm}} \psi(z)$,

IV. $[\psi]^0 = \overline{supp(\psi)} = cl\{z \in \mathbb{R}^n \mid \psi(z) > 0\}$ is a compact subset.

The r-level set is $[\psi]_r = \{z \in \mathbb{R}^n \mid \psi(z) \ge r,\ 0 < r \le 1\}$. A triangular fuzzy number is defined as $\psi = (a_1, a_2, a_3) \in \mathbb{R}^3$ with $a_1 \le a_2 \le a_3$ such that r-level set it is $\forall r \in [0,1] \Rightarrow [\underline{\psi}]_r = [\overline{\psi}(r), \underline{\psi}(r)] = [a_1 + (a_2 - a_1)r, a_3 - (a_3 - a_2)r]$.

For arbitrary $\psi_1,\ \psi_2 \in \mathbb{R}_{\mathcal{F}}$ and scalar $k \in \mathbb{R}$, we have

$\diamondsuit$ addition : $[\psi_1 \oplus \psi_2]_r = [\underline{\psi_1}(r) + \underline{\psi_2}(r), \overline{\psi_1}(r) + \overline{\psi_2}(r)]$,

$\diamondsuit$ scalar multiplication : $\begin{array}{l} k \geq 0 \Rightarrow [k \odot \psi_1]_r = [k\underline{\psi_1}(r), k\overline{\psi_1}(r)]. \\ k < 0 \Rightarrow [k \odot \psi_1]_r = [k\overline{\psi_1}(r), k\underline{\psi_1}(r)]. \end{array}$

The Hukuhara difference (H-difference for short) of $\psi_1, \psi_2 \in \mathbb{R}_{\mathcal{F}}$ is $\psi_3 = \psi_1 \ominus \psi_2$ if there exists $\psi_3 \in \mathbb{R}_{\mathcal{F}}$ such that $\psi_1 = \psi_2 \oplus \psi_3$ and the generalized Hukuhara difference (gH-difference for short) of these fuzzy numbers is defined as follows

$$\psi_1 \ominus_{gH} \psi_2 = \psi_3 \Leftrightarrow \begin{cases} (i) \quad \psi_1 = \psi_2 \oplus \psi_3, \\ \\ or(ii) \quad \psi_2 = \psi_1 \oplus (-1)\psi_3, \end{cases}$$

where (i) and (ii) are both valid iff $\psi_3 \in \mathbb{R}$. Let $[\psi_1]_r = [\underline{\psi_1}(r), \overline{\psi_1}(r)]$ and $[\psi_2]_r = [\underline{\psi_2}(r), \overline{\psi_2}(r)]$ be two fuzzy numbers, the Hausdorff distance $d_{\mathcal{H}} : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+ \cup \{0\}$ can be defined as follows

$$d_{\mathcal{H}}(\psi_1, \psi_2) = \sup_{0 \leq r \leq 1} \max\{|\underline{\psi_1}(r) - \underline{\psi_2}(r)|, |\overline{\psi_1}(r) - \overline{\psi_2}(r)|\},$$

that $\forall \psi_1,\ \psi_2,\ \psi_3,\ \psi_4\ \in \mathbb{R}_{\mathcal{F}},\ \lambda \in \mathbb{R}$ the following properties are valid:

I. $d_{\mathcal{H}}(\psi_1 \oplus \psi_3, \psi_2 \oplus \psi_3) = d_{\mathcal{H}}(\psi_1, \psi_2)$,

II. $d_{\mathcal{H}}(\lambda\psi_1, \lambda\psi_2) = \mid \lambda \mid d_{\mathcal{H}}(\psi_1, \psi_2)$.

III. $d_{\mathcal{H}}(\psi_1 \oplus \psi_2, \psi_3 \oplus \psi_4) \leq d_{\mathcal{H}}(\psi_1, \psi_3) + d_{\mathcal{H}}(\psi_2, \psi_4)$,

IV. $d_{\mathcal{H}}(\psi_1 \ominus \psi_2, \psi_3 \ominus \psi_4) \leq d_{\mathcal{H}}(\psi_1, \psi_3) + d_{\mathcal{H}}(\psi_2, \psi_4)$, as long as $\psi_1 \ominus \psi_2$ and $\psi_3 \ominus \psi_4$ exist.

Function ϕ is a fuzzy valued function for $\phi : \theta \to \mathbb{R}_{\mathcal{F}},\ \theta \subseteq \mathbb{R}$ and a r-cut of this function can be defined as $[\phi(z)]_r = [\underline{\phi}(z; r), \bar{\phi}(z; r)]$. This function is integrable on $[a, b]$, if be continuous by the metric $d_{\mathcal{H}}$, its definite Riemann-integral exists and belongs to $\mathbb{R}_{\mathcal{F}}$. Furthermore it holds

$$\left[\int_a^b \phi(z)dz\right]_r = \left[\int_a^b \underline{\phi}(z; r)dz, \int_a^b \bar{\phi}(z; r)dz\right].$$

$\triangleleft$ quantum calculus:

In this work, there is no loss of generality in assuming that $q \in (0, 1)$. Let $\mu \in \mathbb{R}$, the time-scale $\mathbb{T}_\mu$, for $0 < q < 1$ define $\Rightarrow \mathbb{T}_\mu = \{z : z = \mu q^n,\ n \in \mathbb{Z}^+\} \cup \{0\}$. The q-analogue of non-negative integer number n is the polynomial

$$[\nu]_q := \frac{1 - q^\nu}{1 - q} = \sum_{i=0}^{\nu-1} q^i, \quad [0]_q := 0.$$

The q-differential of an arbitrary function $\phi : \mathbb{T}_\mu \to \mathbb{R}$ is $d_q\phi(z) = \phi(z) - \phi(qz)$ and the q-derivative it is

$$D_q\phi(z) = \frac{d_q\phi(z)}{d_q z} = \frac{\phi(z) - \phi(qz)}{(1 - q)z}, \quad t \in \mathbb{T}_\mu - \{0\}.$$

Further, the different order q-derivatives of ϕ as follows

$$D_q^{(0)}\phi = \phi, \quad D_q^{(1)}\phi = D_q\phi, \quad D_q^{(\nu)}\phi = D_q(D_q^{(\nu-1)}\phi), \quad (\nu = 1, 2, 3, ...).$$

The operator q-antiderivative ${}_qI_a$ or Jackson integral of ϕ is following form

$$\int_0^z \phi(s)d_q s = (1 - q)z \sum_{i=0}^{\infty} q^i \phi(zq^i), \quad 0 < q < 1.$$

This definition implies that

$$\int_a^b \phi(s) d_q s = b(1-q)\sum_{i=0}^{\infty} q^i \phi(bq^i) - a(1-q)\sum_{i=0}^{\infty} q^i \phi(aq^i).$$

$$ {}_qI_a^0\phi(z) = \phi(z), \quad {}_qI_a^{(\nu)}\phi(z) = {}_qI_a({}_qI_a^{(\nu-1)}\phi(z)), \ (\nu = 1,\ 2,\ 3, ...).$$

For $\nu \in \mathbb{N}$, the following relations valid:

$$D_q^{(\nu)}\ {}_qI_a^{(\nu)}\phi(z) = \phi(z), \quad {}_qI_a^{(\nu)}\ D_q^{(\nu)}\phi(z) = \phi(z) - \sum_{k=0}^{\nu-1} \frac{(z-a)_q^k}{[k]_q!} D_q^{(k)}\phi(a).$$

For $z \neq 0$, we can define the q-fractional function as

$$(z-s)_q^{\nu} = \begin{cases} \prod_{i=0}^{\nu-1}(z - q^i s), & \nu \in \mathbb{N}, \\ z^{\nu}\prod_{i=0}^{\infty} \dfrac{1-\frac{s}{z}q^i}{1-\frac{s}{z}q^{i+\nu}}, & o.\ w. \end{cases}$$

Also, we can assert that the following relations are satisfied for the q-fractional function

I. $(z-s)_q^{\nu+m} = (z-s)_q^{\nu}(z-q^{\nu}s)_q^m$,

II. $(az-as)_q^{\nu} = a^{\nu}(z-s)_q^{\nu}$.

III. $(z-s)_q^{\nu} \neq (-1)^{\nu}(s-z)_q^{\nu}$.

IV. for all $\nu \in \mathbb{Z} \rightsquigarrow \int_a^z (z-qs)_q^{\nu-1} d_q s = \dfrac{\Gamma_q(\nu)}{\Gamma_q(\nu+1)}(z-a)_q^{\nu}$.

V. for all $\nu \in \mathbb{Z} \rightsquigarrow D_q(z-s)_q^{\nu} = \begin{cases} [\nu]_q(z-s)_q^{\nu-1}, & w.\ r.\ t.\ (z), \\ -[\nu]_q(z-qs)_q^{\nu-1}, & w.\ r.\ t.\ (s). \end{cases}$

The q-gamma function denoted by $\Gamma_q(\cdot)$ and can be found using

$$\Gamma_q(\nu) = \frac{(1-q)_q^{\nu-1}}{(1-q)^{\nu-1}}, \quad \nu \in \mathbb{R}/\{0\} \cup \mathbb{Z}_-, \quad 0 < q < 1,$$

which for non-negative integer number ν we get

$$\Gamma_q(\nu+1) = [\nu]_q\Gamma_q(\nu) = \frac{1-q^{\nu}}{1-q}\Gamma_q(\nu) = [\nu]_q!,$$
$$\Gamma_q(1) = 1, \quad [\nu]_q! = [\nu]_q[\nu-1]_q...[2]_q[1]_q, \quad [0]_q! = 1.$$

Now the q-gamma function can be explicitly written as

$$\Gamma_q(\nu) = \frac{\Gamma(\nu)\Gamma(\frac{1}{1-q}+1)}{(1-q)^{\nu}\Gamma(\frac{1}{1-q}+\nu+1)}, \ 0 < q < 1.$$

For $0 \leq k \leq \nu$, q-binomial coefficient is given by

$$\binom{\nu}{\nu-k}_q = \binom{\nu}{k}_q = \frac{[\nu]_q!}{[k]_q![\nu-k]_q!}.$$

Note that when $q \to 1$, the q-binomial coefficient is reduced to the binomial coefficient of the standard calculus. The q-sin(z) and the q-cos(z) functions are defined as $\cos_q(z) = \frac{e_q^{iz}+e_q^{-iz}}{2}$. Then $D_q\cos_q(z) = -\sin_q(z)$, $\sin_q(z) = \frac{e_q^{iz}-e_q^{-iz}}{2i}$. This implies that $D_q\sin_q(z) = \cos_q(z)$. Then

$$e_q^{iz} = \cos_q(z) + i\sin_q(z) = \sum_{\nu=0}^{\infty}\frac{(iz)^{\nu}}{[\nu]_q!},$$

for all $z \in \mathbb{T}_{\mu}$, where e_q^{iz} is q-exponential function.

3 Fuzzy Quantum Calculus

In what follows, fuzzy q-derivative based on concept of generalized Hukuhara q-differentiability has been introduced. Consequently, obtained results and properties of fuzzy q-differentiability are useful.

Definition 3.1. Let $\phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ is arbitrary fuzzy-valued function and $\phi(z) \ominus_{gH} \phi(qz)$ exists, q-differential of ϕ by gH-difference is

$$^{gH}d_q\phi(z) = \phi(z) \ominus_{gH} \phi(qz).$$

On the other hand, we have

$$[\phi(z) \ominus_{gH} \phi(qz)]_r = \Big[\min\{\underline{\phi}(z;r) - \underline{\phi}(qz;r), \overline{\phi}(z;r) - \overline{\phi}(qz;r)\}, \max\{\underline{\phi}(z;r) - \underline{\phi}(qz;r), \overline{\phi}(z;r) - \overline{\phi}(qz;r)\}\Big],$$

that existence conditions of $^{gH}d_q\phi(z) = \phi(z) \ominus_{gH} \phi(qz) \in \mathbb{R}_\mathcal{F}$ are

$$\text{case}(i) \begin{cases} d_q\underline{\phi}(z;r) = \underline{\phi}(z;r) - \underline{\phi}(qz;r), \text{ is increasing,} \\ d_q\overline{\phi}(z;r) = \overline{\phi}(z;r) - \overline{\phi}(qz;r), \text{ is decreasing,} \\ d_q\underline{\phi}(z;r) \le d_q\overline{\phi}(z;r), \end{cases}$$

$$\text{case}(ii) \begin{cases} d_q\underline{\phi}(z;r) = \overline{\phi}(z;r) - \overline{\phi}(qz;r), \text{ is increasing,} \\ d_q\overline{\phi}(z;r) = \underline{\phi}(z;r) - \underline{\phi}(qz;r), \text{ is decreasing,} \\ d_q\underline{\phi}(z;r) \le d_q\overline{\phi}(z;r). \end{cases}$$

So, the fuzzy generalized Hukuhara q-derivative (for short $^F_q[gH]$-derivative) of ϕ can be presented as follows

$$^F_qD\phi(z) = \frac{^{gH}d_q\phi(z)}{d_qz} = \frac{\phi(z) \ominus_{gH} \phi(qz)}{(1-q)z}, \quad z \in \mathbb{T}_\mu - \{0\}.$$

Definition 3.2. Let $\phi : \mathbb{T}_q \to \mathbb{R}_\mathcal{F}$ and $^{gH}d_q\phi$ exists. we say that ϕ is $^F_q[i.gH]$-differentiable(for short $^F_q[i.gH]$-D) if

$$[^F_qD\phi_{i.gH}(z)]_r = [D_q\underline{\phi}(z;r),\ D_q\overline{\phi}(z;r)],\ 0 \le r \le 1,$$

and ϕ is $^F_q[ii.gH]$-differentiable(for short $^F_q[ii.gH]$-D) if

$$[^F_qD\phi_{ii.gH}(z)]_r = [D_q\overline{\phi}(z;r),\ D_q\underline{\phi}(z;r)],\ 0 \le r \le 1,$$

where

$$D_q\underline{\phi}(z;r) = \frac{\underline{\phi}(z;r) - \underline{\phi}(qz;r)}{(1-q)z}, \quad D_q\overline{\phi}(z;r) = \frac{\overline{\phi}(z;r) - \overline{\phi}(qz;r)}{(1-q)z}.$$

Definition 3.3. For fuzzy valued function ϕ on $\mathbb{T}_\mu$ a point $\xi_0 \in \mathbb{T}_\mu - \{0\}$ is said to be a switching point (S.P) for the $^F_q[gH]$-D of ϕ, if in any neighborhood V of ξ_0 there exist points $z_1 < \xi_0 < z_2$ such that

(type I) ϕ is $^F_q[i.gH]$-D at z_1 while ϕ is not $^F_q[ii.gH]$-D at z_1, and ϕ is $^F_q[ii.gH]$-D at z_2 while ϕ is not $^F_q[i.gH]$-D at z_2, or

(type II) ϕ is $^F_q[ii.gH]$-D at z_1 while ϕ is not $^F_q[i.gH]$-D at z_1, and ϕ is $^F_q[i.gH]$-D at z_2 while ϕ is not $^F_q[ii.gH]$-D at z_2.

Definition 3.4. Function $\phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ is $^F_q[gH]$-D of the ν^{th}-order at z_0 whenever the function ϕ is $^F_q[gH]$-D of the order m $(m = 1, 2, ..., \nu - 1)$ at z_0, moreover there is not any S.P on $\mathbb{T}_\mu$, then there exist $^F_qD^\nu\phi(z_0) \in \mathbb{R}_\mathcal{F}$ such that

$$^F_qD^\nu\phi(z_0) = \frac{^F_qD^{\nu-1}\phi(z_0) \ominus_{gH} {}^F_qD^{\nu-1}\phi(qz_0)}{(1-q)z_0}.$$

Theorem 3.5. [21] Suppose that $[\phi(z)]_r = [\underline{\phi}(z;r), \bar{\phi}(z;r)]$ and $\phi(z) \ominus_{gH} \phi(qz)$ exists, the function ϕ is ${}_q^F[gH]$-D if and only if for all $r \in [0,1]$, $\underline{\phi}(z;r)$and $\bar{\phi}(z;r)$ are q-differentiable with respect to z and

$$[{}_q^F D\phi(z)]_r = [\min\{D_q\underline{\phi}(z;r),\ D_q\bar{\phi}(z;r)\}, \max\{D_q\underline{\phi}(z;r),\ D_q\bar{\phi}(z;r)\}].$$

Now, on account of the process described in [14] for q-antiderivative, we have following definition for the fuzzy valued function:

Definition 3.6. Let $\phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$, by Definition 3.1 of ${}_q^F[gH]$-derivative and the operator $\hat{M}_q$ defined by $\hat{M}_q(\Phi(z)) = \Phi(qz)$, we have

$$\frac{1}{(1-q)z}(1-\hat{M}_q)\Phi(z) = \frac{\Phi(z) \ominus_{gH} \Phi(qz)}{(1-q)z} = \phi(z),$$

then, we can find the fuzzy q-antiderivative as

$$\Phi(z) = \frac{1}{1-\hat{M}_q}((1-q)z\phi(z)) = (1-q) \odot \sum_{i=0}^{\infty} \hat{M}_q^i(z\phi(z)),$$

putting $a = 0$ and applying the geometric series, the fuzzy Jackson integral can be obtained as

$$\begin{aligned}
[\Phi(z)]_r &= \left[\int_0^z \phi(s)d_qs\right]_r = \left[\int_0^z \underline{\phi}(s;r)d_qs, \int_0^z \overline{\phi}(s;r)d_qs\right] \\
&= \left[(1-q)z\sum_{i=0}^{\infty} q^i\underline{\phi}(q^iz;r), (1-q)z\sum_{i=0}^{\infty} q^i\overline{\phi}(q^iz;r)\right] \\
&= (1-q)z\sum_{i=0}^{\infty} q^i[\phi(q^iz)]_r.
\end{aligned}$$

Then

$$\Phi(z) = (1-q)z \odot \sum_{i=0}^{\infty} q^i \odot \phi(q^iz).$$

In the remainder of this study, $\mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$ is applied to show the fuzzy valued continuous functions which is defined on $\mathbb{T}_\mu$ and $\mathcal{C}_f^\nu(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$, $n \in \mathbb{N}$ for the space of fuzzy-valued functions f on $\mathbb{T}_\mu$ such that itself and its first n, q-derivatives are all in $\mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$.

Lemma 3.7. Suppose that $(\mathbb{R}_\mathcal{F}, d_\mathcal{H})$ is a Banach space and $\phi \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$ then for all $a, z \in \mathbb{T}_\mu$, $\int_a^z \phi(s)d_qs$ exists and belongs to $\mathbb{R}_\mathcal{F}$(i.e. ϕ is q-integrable).

Proof . See the proof in [6]. This reasoning is similar to those in the Remark 4.3 and 4.5 appears in [6, Ch.4]. □

Lemma 3.8. Let $\phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ be a fuzzy continuous function then $\int_a^z \phi(s)d_qs$ is a continuous function for $z \in \mathbb{T}_\mu$.

Proof . By the definition of q-integral (Definition 3.6) and under the assumption of theorem, functions $\phi(aq^i)$ and $\phi(zq^i)$ are continuous for $a, z \in \mathbb{T}_\mu$. On the other hand, $a(1-q)$ and $z(1-q)$ are non-negative and continuous on $\mathbb{T}_\mu$. So, for all $z \in \mathbb{T}_\mu$, $\int_a^z \phi(s)d_qs$ is a continuous function. □

Lemma 3.9. Let $\phi \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$, $m \in \mathbb{N}$ and $z_{m-1}, z_{m-2}, ..., z \geq a$, all belong to $\mathbb{R}$, then the fuzzy q-integrals $\int_a^{z_{m-1}} \phi(z_m)d_qz_m$, $\int_a^{z_{m-2}}(\int_a^{z_{m-1}} \phi(z_m)d_qz_m)d_qz_{m-1}, ..., \int_a^z(\int_a^{z_1} ... \int_a^{z_{m-2}}(\int_a^{z_{m-1}} \phi(z_m)d_qz_m)d_qz_{m-1}...)d_qz_1$ are continuous functions in $z_{m-1}, z_{m-2}, ..., z$, respectively.

Proof . The proof is by induction on $m \in \mathbb{N}$:

Assume that the lemma holds for (m)-times operator q-integrating from function f, we will prove it for $(m+1)$-times operator q-integrating from function ϕ. By Lemma 3.7, since $\phi \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$ thus $\int_a^{z_{m-1}} \phi(z_m)d_qz_m$ is a continuous function in z_{m-1}. Furthermore, under the hypothese of induction, $\int_a^{z_{m-2}}(\int_a^{z_{m-1}} \phi(z_m)d_qz_m)d_qz_{m-1}, ...,$

$\overbrace{\int_a^z(\int_a^{z_1}...\int_a^{z_{m-2}}(\int_a^{z_{m-1}}}^{(m)-times}\phi(z_m)d_qz_m)d_qz_{m-1}...)d_qz_1$ are continuous functions in $z_{m-2}, z_{m-3}, ..., z$, respectively. By Lemma 3.8, it follows easily that $\overbrace{\int_a^{z_{m+1}}(\int_a^{z}(\int_a^{z_1}...\int_a^{z_{m-2}}(\int_a^{z_{m-1}}}^{(m+1)-times}\phi(z_m)d_qz_m)d_qz_{m-1}...)d_qz_1)d_qz$ is a continuous function in z_{m+1}, which is our claim. □

Lemma 3.10. Suppose that $\phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ is a fuzzy valued function and ${}_q^F D\phi$ is continuous and q-integrable, then

$$\int_a^z {}_q^F D\phi(s)d_qs = \phi(z) \ominus_{gH} \phi(a).$$

Proof. Let ϕ be a ${}_q^F[i.gH]$-D function for $r \in [0, 1]$

$$\left[\int_a^z {}_q^F D\phi_{i.gH}(s)d_qs\right]_r = \left[\int_a^z D_q\underline{\phi}(s;r)d_qs, \int_a^z D_q\bar{\phi}(s;r)d_qs\right]. \tag{3.1}$$

In continuation, by using the definition of q-integral in Section 2 and assumption q-integrable of ${}_q^F D\phi_{gH}$, we have

$$\int_a^z D_q\bar{\phi}(s;r)d_qs = z(1-q)\sum_{i=0}^{\infty} q^i D_q\bar{\phi}(zq^i;r) - a(1-q)\sum_{i=0}^{\infty} q^i D_q\bar{\phi}(aq^i;r), \tag{3.2}$$

by the definition of q-derivative

$$D_q\bar{\phi}(tq^i;r) = \frac{\bar{\phi}(zq^i;r) - \bar{\phi}(zq^{i+1};r)}{(1-q)zq^i}, \tag{3.3}$$

Then

$$z(1-q)\sum_{i=0}^{\infty} q^i.\left[\frac{\bar{\phi}(zq^i;r) - \bar{\phi}(zq^{i+1};r)}{(1-q)zq^i}\right] = \sum_{i=0}^{\infty}\bar{\phi}(zq^i;r) - \bar{\phi}(zq^{i+1};r) \overset{0<q<1}{=} \bar{\phi}(z;r) - \bar{\phi}(0), \tag{3.4}$$

and also

$$a(1-q)D_q\bar{\phi}(aq^i;r) = \bar{\phi}(a;r) - \bar{\phi}(0). \tag{3.5}$$

Substituting Eqs. (3.4) and (3.5) into (3.2), we obtain

$$\int_a^z D_q\bar{\phi}(s;r)d_qs = \bar{\phi}(z;r) - \bar{\phi}(a;r). \tag{3.6}$$

It is found in the same trend

$$\int_a^z D_q\underline{\phi}(s;r)d_qs = \underline{\phi}(z;r) - \underline{\phi}(a;r). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.1) yields

$$\left[\int_a^z {}_q^F D\phi_{i.gH}(s)d_qs\right]_r = \left[\underline{\phi}(z;r) - \underline{\phi}(a;r), \bar{\phi}(z;r) - \bar{\phi}(a;r)\right] = [\phi(z) \ominus \phi(a)]_r.$$

It remains to exclude the case when ϕ is ${}_q^F[ii.gH]$-D. Similar arguments apply to the ${}_q^F[ii.gH]$-D

$$\left[\int_a^z {}_q^F D\phi_{ii.gH}(s)d_qs\right]_r = \left[\bar{\phi}(z;r) - \bar{\phi}(a;r), \underline{\phi}(z;r) - \underline{\phi}(a;r)\right] = [(-1) \odot (\phi(a) \ominus \phi(z))]_r.$$

which completes the proof.

Theorem 3.11. Consider $\phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ is fuzzy ${}_q^F[gH]$-D such that the type of ${}_q^F[gH]$-D of ϕ in $\mathbb{T}_\mu$ do not change. Then for $t \in \mathbb{T}_\mu$

I. If $\phi(s)$ is ${}_q^F[i.gH]$-D then ${}_q^F D\phi_{i.gH}(s)$ is q-integrable over $\mathbb{T}_\mu$ and

$$\phi(z) = \phi(a) \oplus \int_a^z {}_q^F D\phi_{i.gH}(s) d_q s.$$

II. If $\phi(s)$ is ${}_q^F[ii.gH]$-D then ${}_q^F D\phi_{ii.gH}(s)$ is q-integrable over $\mathbb{T}_\mu$ and

$$\phi(z) = \phi(a) \ominus (-1) \int_a^z {}_q^F D\phi_{ii.gH}(s) d_q s.$$

Lemma 3.12. Suppose that $\phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ is fuzzy ${}_q^F[gH]$-D and ${}_q^F D\phi_{gH}(s) \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$, then

$$\int_z^a {}_q^F D\phi_{i.gH}(s) d_q s = (-1) \odot \int_a^z {}_q^F D\phi_{ii.gH}(s) d_q s,$$

Proof. Since ${}_q^F D\phi_{i.gH}(s)$ is continuous, according to Lemma 3.7, it follows that ${}_q^F D\phi_{i.gH}(s)$ is q-integrable and using Lemma 3.10 yields

$$\begin{aligned}\left[\int_z^a {}_q^F D\phi_{i.gH}(s) d_q s\right]_r &= \left[\int_z^a D_q\underline{\phi}(s;r) d_q s, \int_z^a D_q\bar{\phi}(s;r) d_q s\right] \\ &= [\underline{\phi}(a;r) - \underline{\phi}(z;r), \bar{\phi}(a;r) - \bar{\phi}(z;r)] = [\phi(a) \ominus \phi(z)]_r,\end{aligned} \tag{3.8}$$

moreover

$$\left[\int_z^a {}_q^F D\phi_{ii.gH}(s) d_q s\right]_r = [\bar{\phi}(z;r) - \bar{\phi}(a;r), \underline{\phi}(z;r) - \underline{\phi}(a;r)] = [(-1) \odot (\phi(a) \ominus \phi(z))]_r. \tag{3.9}$$

By combining (3.8) with (3.9), the lemma follows.

Theorem 3.13. [22] Let ${}_q^F D^{(j)}\phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ and ${}_q^F D^{(j)}\phi \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F}),\ j = 1, ..., \nu$. For all $z \in \mathbb{T}_\mu$

I. Consider ${}_q^F D^{(j)}\phi,\ j = 1, ..., \nu$ are ${}_q^F[i.gH]$-D and without changing the type of ${}_q^F[gH]$-D on $\mathbb{T}_\mu$, then

$${}_q^F D^{(j-1)}\phi_{i.gH}(z) = {}_q^F D^{(j-1)}\phi_{i.gH}(a) \oplus \int_a^z {}_q^F D^{(j)}\phi_{i.gH}(s) d_q s.$$

II. If ${}_q^F D^{(j)}\phi,\ j = 1, ..., \nu$ are ${}_q^F[ii.gH]$-D and the type of ${}_q^F[gH]$-D do not change on $\mathbb{T}_\mu$, then

$${}_q^F D^{(j-1)}\phi_{ii.gH}(z) = {}_q^F D^{(j-1)}\phi_{ii.gH}(a) \oplus \int_a^z {}_q^F D^{(j)}\phi_{ii.gH}(s) d_q s.$$

III. Assume that ${}_q^F D^{(j)}\phi$ for $j = 1, \ldots, \nu$ exist and in each order of q-differentiation, the type of ${}_q^F[gH]$-D changes on $\mathbb{T}_\mu$(i.e. ${}_q^F[gH]$-D changes from ${}_q^F[i.gH]$ to ${}_q^F[ii.gH]$ and vice versa), then

$$\begin{cases} {}_q^F D^{(j-1)}\phi_{i.gH}(z) = {}_q^F D^{(j-1)}\phi_{i.gH}(a) \ominus (-1) \int_a^z {}_q^F D^{(j)}\phi_{ii.gH}(s) d_q s, & j \text{ is an even number}, \\ {}_q^F D^{(j-1)}\phi_{ii.gH}(z) = {}_q^F D^{(j-1)}\phi_{ii.gH}(a) \ominus (-1) \int_a^z {}_q^F D^{(j)}\phi_{i.gH}(s) d_q s, & j \text{ is an odd number}. \end{cases}$$

Having disposed of these preliminary contents, we can introduce fuzzy q-Taylor's expansion in the next section.

4 Fuzzy q-Taylor Theorem

In this section, we focus on fuzzy q-Taylor theorem around $'a'$ which is a suitable number. This theorem provides a criterion for approximating fuzzy valued functions. In order to get favorable results, it is necessary to put some restrictions:

Theorem 4.1. Let ${}_q^F D^{(j)}\phi \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F}),\ j = 0, ..., \nu$. For $a, z \in \mathbb{T}_\mu$,
I. If ${}_q^F D^{(j)}\phi,\ j = 0, 1, ..., \nu-1$ are ${}_q^F[i.gH]$-D, provided that the type of ${}_q^F[gH]$-D has no change on $\mathbb{T}_\mu$. Then

$$\begin{aligned}\phi(z) &= \phi(a) \oplus {}_q^F D\phi_{i.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \oplus {}_q^F D^{(2)}\phi_{i.gH}(a) \odot \frac{(z-a)_q^2}{\Gamma_q(3)} \oplus ... \oplus {}_q^F D^{(\nu-1)}\phi_{i.gH}(a) \odot \frac{(z-a)_q^{\nu-1}}{\Gamma_q(\nu)} \oplus R_\nu(a,z)\\ &= \phi(a) \oplus \sum_{j=1}^{\nu-1} \frac{(z-a)_q^j}{\Gamma_q(j+1)} \odot {}_q^F D^{(j)}\phi_{i.gH}(a) \oplus R_\nu(a,z),\end{aligned}$$

where $R_\nu(a,z) := \int_a^z \left(\int_a^{z_1} ... \int_a^{z_{\nu-2}} \left(\int_a^{z_{\nu-1}} {}_q^F D^{(\nu)}\phi_{i.gH}(z_\nu) d_q z_\nu\right) d_q z_{\nu-1} ... d_q z_2\right) d_q z_1$.

II. If ${}_q^F D^{(j)}\phi,\ j = 0, 1, ..., \nu-1$ are ${}_q^F[ii.gH]$-D, provided that the type of ${}_q^F[gH]$-D has no change on $\mathbb{T}_\mu$. Then

$$\begin{aligned}\phi(z) &= \phi(a) \ominus (-1){}_q^F D\phi_{ii.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \ominus (-1){}_q^F D^{(2)}\phi_{ii.gH}(a) \odot \frac{(z-a)_q^2}{\Gamma_q(3)}\\ &\ominus (-1) ... \ominus (-1){}_q^F D^{(\nu-1)}\phi_{ii.gH}(a) \odot \frac{(z-a)_q^{\nu-1}}{\Gamma_q(\nu)} \ominus (-1)R_\nu(a,z)\\ &= \phi(a) \ominus (-1)\sum_{j=1}^{\nu-1} \frac{(z-a)_q^j}{\Gamma_q(j+1)} \odot {}_q^F D^{(j)}\phi_{ii.gH}(a) \ominus (-1)R_\nu(a,z),\end{aligned}$$

where $R_\nu(a,z) := \int_a^z \left(\int_a^{z_1} ... \int_a^{z_{\nu-2}} \left(\int_a^{z_{\nu-1}} {}_q^F D^{(\nu)}\phi_{ii.gH}(z_\nu) d_q z_\nu\right) d_q z_{\nu-1} ... d_q z_2\right) d_q z_1$.

III. If ${}_q^F D^{(j)}\phi,\ j = 2k-1,\ k \in \mathbb{N}$ are ${}_q^F[i.gH]$-D and ${}_q^F D^{(j)}\phi,\ j = 2k,\ k \in \mathbb{N} \cup \{0\}$ are ${}_q^F[ii.gH]$-D, then

$$\begin{aligned}\phi(z) &= \phi(a) \ominus (-1){}_q^F D\phi_{ii.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \oplus {}_q^F D^{(2)}\phi_{i.gH}(a) \odot \frac{(z-a)_q^2}{\Gamma_q(3)} \ominus (-1) ... \ominus (-1)\\ &{}_q^F D^{(\frac{i}{2}-1)}\phi_{ii.gH}(a) \odot \frac{(z-a)_q^{\frac{i}{2}-1}}{\Gamma_q(\frac{i}{2})} \oplus {}_q^F D^{(\frac{i}{2})}\phi_{i.gH}(a) \odot \frac{(z-a)_q^{\frac{i}{2}}}{\Gamma_q(\frac{i}{2}+1)} \ominus (-1) ... \ominus (-1)R_\nu(a,z)\\ &= \phi(a) \ominus (-1) \sum_{\substack{j=1\\ j \text{ is odd}}}^{\nu-1} \frac{(z-a)_q^j}{\Gamma_q(j+1)} \odot {}_q^F D^{(j)}\phi_{ii.gH}(a) \oplus \sum_{\substack{j=1\\ j \text{ is even}}}^{\nu-1} \frac{(z-a)_q^j}{\Gamma_q(j+1)} \odot {}_q^F D^{(j)}\phi_{i.gH}(a) \ominus (-1)R_\nu(a,z),\end{aligned}$$

where $R_\nu(a,z) := \int_a^z \left(\int_a^{z_1} ... \int_a^{z_{\nu-2}} \left(\int_a^{z_{\nu-1}} {}_q^F D^{(\nu)}\phi_{ii.gH}(z_\nu) d_q z_\nu\right) d_q z_{\nu-1} ... d_q z_2\right) d_q z_1$.

IV. For ${}_q^F D^{(j)}\phi \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F}),\ j \geq 3$, suppose that ϕ on $[z_0, \xi]$ is ${}_q^F[(ii)-gH]$-D and on $[\xi, z_\nu]$ is ${}_q^F[i.gH]$-D, in fact ξ is S.P (type II) for first order ${}_q^F[gH]$-derivative of ϕ. Moreover, for $a \in [z_0, \xi]$, let second order ${}_q^F[gH]$-derivative of ϕ in ξ_1 of $[a, \xi]$ have S.P (type I). On the other hand, the type of ${}_q^F[gH]$-D for ${}_q^F D^{(j)}\phi, j \leq \nu$ on $[\xi, z_\nu]$ do not change. So

$$\begin{aligned}\phi(z) &= \phi(a) \ominus (-1){}_q^F D\phi_{ii.gH}(a) \odot \frac{(\xi-a)_q}{\Gamma_q(2)} \oplus (-1){}_q^F D^{(2)}\phi_{i.gH}(a) \odot \frac{(a-\xi_1)_q}{\Gamma_q(2)}\\ &\odot \frac{(\xi-a)_q}{\Gamma_q(2)} \ominus (-1){}_q^F D^{(2)}\phi_{ii.gH}(\xi_1) \odot \left[\frac{(\xi-\xi_1)_q^2}{\Gamma_q(3)} - \frac{(a-\xi_1)_q^2}{\Gamma_q(3)}\right]\\ &\oplus {}_q^F D\phi_{i.gH}(\xi) \odot \frac{(z-\xi)_q}{\Gamma_q(2)} \oplus {}_q^F D^{(2)}\phi_{i.gH}(\xi) \odot \frac{(z-\xi)_q^2}{\Gamma_q(3)} \oplus \int_a^\xi \left(\int_a^{\xi_1} \left(\int_a^{z_2} {}_q^F D^{(3)}\phi_{i.gH}(z_4) d_q z_4\right) d_q z_2\right) d_q z_1\\ &\ominus (-1) \int_a^\xi \left(\int_{\xi_1}^{z_1} \left(\int_{\xi_1}^{z_3} {}_q^F D^{(3)}\phi_{ii.gH}(z_5) d_q z_5\right) d_q z_3\right) d_q z_1 \oplus \int_\xi^z \left(\int_\xi^{s_1} \left(\int_\xi^{s_2} {}_q^F D^{(3)}\phi_{i.gH}(s_3) d_q s_3\right) d_q s_2\right) d_q s_1.\end{aligned}$$

Proof. Under the assumption that ${}^F_q D^{(i)}\phi \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F}),\ i = 0, ..., \nu$, we can write ${}^F_q D^{(i)}\phi$ are q-integrable on $\mathbb{T}_\mu$,

I. Since ϕ is a continuous function and ${}^F_q[i.gH]$-D, by Theorem 3.11, we see that

$$\phi(z) = \phi(a) \oplus \int_a^z {}^F_q D\phi_{i.gH}(z_1) d_q z_1.$$

The Theorem 3.13 now yields

$${}^F_q D\phi_{i.gH}(z_1) = {}^F_q D\phi_{i.gH}(a) \oplus \int_a^{z_1} {}^F_q D^{(2)}\phi_{i.gH}(z_2) d_q z_2.$$

Therefore, gives

$$\begin{aligned}\int_a^z {}^F_q D\phi_{i.gH}(z_1) d_q z_1 &= \int_a^z {}^F_q D\phi_{i.gH}(a) d_q z_1 \oplus \int_a^z \left(\int_a^{z_1} {}^F_q D^{(2)}\phi_{i.gH}(z_2) d_q z_2\right) d_q z_1\\ &= {}^F_q D\phi_{i.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \oplus \int_a^z \left(\int_a^{z_1} {}^F_q D^{(2)}\phi_{i.gH}(z_2) d_q z_2\right) d_q z_1.\end{aligned}$$

Since the last double q-integral belongs to $\mathbb{R}_\mathcal{F}$ and using Lemma 3.9 finds

$$\phi(z) = \phi(a) \oplus {}^F_q D\phi_{i.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \oplus \int_a^z \left(\int_a^{z_1} {}^F_q D^{(2)}\phi_{i.gH}(z_2) d_q z_2\right) d_q z_1.$$

By similar argument gives

$${}^F_q D^{(2)}\phi_{i.gH}(z_2) = {}^F_q D^{(2)}\phi_{i.gH}(a) \oplus \int_a^{z_2} {}^F_q D^{(3)}\phi_{i.gH}(z_3) d_q z_3.$$

Applying operator q-integral to ${}^F_q D^{(2)}\phi_{i.gH}(z_2)$, we obtain

$$\int_a^{z_1} {}^F_q D^{(2)}\phi_{i.gH}(z_2) d_q z_2 = {}^F_q D^{(2)}\phi_{i.gH}(a) \odot \frac{(z_1-a)_q}{\Gamma_q(2)} \oplus \int_a^{z_1} \left(\int_a^{z_2} {}^F_q D^{(3)}\phi_{i.gH}(z_3) d_q z_3\right) d_q z_2,$$

furthermore, by the definition of q-integral, we get

$$\int_a^z \left(\int_a^{z_1} {}^F_q D^{(2)}\phi_{i.gH}(z_2) d_q z_2\right) d_q z_1 = {}^F_q D^{(2)}\phi_{i.gH}(a) \odot \frac{(z-a)_q^2}{\Gamma_q(3)} \oplus \int_a^z \left(\int_a^{z_1} \left(\int_a^{z_2} {}^F_q D^{(3)}\phi_{i.gH}(z_3) d_q z_3\right) d_q z_2\right) d_q z_1.$$

The last triple q-integral belongs to $\mathbb{R}_\mathcal{F}$, the Lemma 3.9 gets

$$\begin{aligned}\phi(z) = \phi(a) &\oplus {}^F_q D\phi_{i.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \oplus {}^F_q D^{(2)}\phi_{i.gH}(a) \odot \frac{(z-a)_q^2}{\Gamma_q(3)}\\ &\oplus \int_a^z \left(\int_a^{z_1} \left(\int_a^{z_2} {}^F_q D^{(3)}\phi_{i.gH}(z_3) d_q z_3\right) d_q z_2\right) d_q z_1.\end{aligned}$$

The high order of the last formula by Lemma 3.9 is a continuous function in terms of z so it belongs to $\mathbb{R}_\mathcal{F}$. With the same manner, we can demonstrate that part **I** is satisfied. The same proof obtains when we consider the assumption ${}^F_q[ii.gH]$-D of ϕ. So, the proof of part **II** has analysis similar to the proof of part **I** and the details are left to the reader.

III. Let ϕ be the ${}^F_q[ii.gH]$-D, using Theorem 3.11 we get

$$\phi(z) = \phi(a) \ominus (-1) \int_a^z {}^F_q D\phi_{ii.gH}(z_1) d_q z_1.$$

Based on the assumptions of theorem we know that ϕ is ${}^F_q[ii.gH]$-D and ${}^F_q D\phi$ is ${}^F_q[i.gH]$-D. Thus, the Theorem 3.13 gets

$${}^F_q D\phi_{ii.gH}(z_1) = {}^F_q D\phi_{ii.gH}(a) \ominus (-1) \int_a^{z_1} {}^F_q D^{(2)}\phi_{i.gH}(z_2) d_q z_2.$$

Now applying operator q-integral to ${}_q^F D\phi_{ii.gH}(z_1)$ gives

$$\int_a^z {}_q^F D\phi_{ii.gH}(z_1) d_q z_1 = {}_q^F D\phi_{ii.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \ominus (-1) \int_a^z \left(\int_a^{z_1} {}_q^F D^{(2)} \phi_{i.gH}(z_2) d_q z_2 \right) d_q z_1.$$

Using Lemma 3.9, the double q-integral can be obtained which belongs to $\mathbb{R}_{\mathcal{F}}$. So

$$\phi(z) = \phi(a) \ominus (-1) {}_q^F D\phi_{ii.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \oplus \int_a^z \left(\int_a^{z_1} {}_q^F D^{(2)} \phi_{i.gH}(z_2) d_q z_2 \right) d_q z_1.$$

Similarly, since ${}_q^F D\phi$ is ${}_q^F[i.gH]$-D, ${}_q^F D^{(2)}\phi$ is ${}_q^F[ii.gH]$-D and we get

$${}_q^F D^{(2)} \phi_{i.gH}(z_2) = {}_q^F D^{(2)} \phi_{i.gH}(a) \ominus (-1) \int_a^{z_2} {}_q^F D^{(3)} \phi_{ii.gH}(z_3) d_q z_3,$$

and

$$\int_a^{z_1} {}_q^F D^{(2)} \phi_{i.gH}(z_2) d_q z_2 = {}_q^F D^{(2)} \phi_{i.gH}(a) \odot \frac{(z_1-a)_q}{\Gamma_q(2)} \ominus (-1) \int_a^{z_1} \left(\int_a^{z_2} {}_q^F D^{(3)} \phi_{ii.gH}(z_3) d_q z_3 \right) d_q z_2.$$

Now, applying q-operator $\int_a^z$, gives

$$\begin{aligned} \int_a^z \left(\int_a^{z_1} {}_q^F D^{(2)} \phi_{i.gH}(z_2) d_q z_2 \right) d_q z_1 &= {}_q^F D^{(2)} \phi_{i.gH}(a) \odot \frac{(z-a)_q^2}{\Gamma_q(3)} \\ &\ominus (-1) \int_a^z \left(\int_a^{z_1} \left(\int_a^{z_2} {}_q^F D^{(3)} \phi_{ii.gH}(z_3) d_q z_3 \right) d_q z_2 \right) d_q z_1. \end{aligned}$$

Since the Lemma 3.9 is satisfied, the last triple q-integral belongs to $\mathbb{R}_{\mathcal{F}}$. Then, we have

$$\begin{aligned} \phi(z) = \phi(a) &\ominus (-1) {}_q^F D\phi_{ii.gH}(a) \odot \frac{(z-a)_q}{\Gamma_q(2)} \oplus {}_q^F D^{(2)} \phi_{i.gH}(a) \odot \frac{(z-a)_q^2}{\Gamma_q(3)} \\ &\ominus (-1) \int_a^z \left(\int_a^{z_1} \left(\int_a^{z_2} {}_q^F D^{(3)} \phi_{ii.gH}(z_3) d_q z_3 \right) d_q z_2 \right) d_q z_1. \end{aligned}$$

Using the same process, the proof for this ${}_q^F[gH]$-D can be done.

IV. Let ϕ be the ${}_q^F[ii.gH]$-D in $[a, \xi]$, now Theorem 3.11 yields

$$\phi(\xi) = \phi(a) \ominus (-1) \int_a^{\xi} {}_q^F D\phi_{ii.gH}(z_1) d_q z_1, \tag{4.1}$$

and in the interval $[\xi, z_\nu]$, ϕ is ${}_q^F[i.gH]$-D, so it is conclude for $z \in [\xi, z_\nu]$

$$\phi(z) = \phi(\xi) \oplus \int_{\xi}^z {}_q^F D\phi_{i.gH}(s_1) d_q s_1. \tag{4.2}$$

We know that ξ is a S.P for ${}_q^F[gH]$-differentiability ϕ thus by replacing (4.1) in (4.2) we get

$$\phi(z) = \phi(a) \ominus (-1) \int_a^{\xi} {}_q^F D\phi_{ii.gH}(z_1) d_q z_1 \oplus \int_{\xi}^z {}_q^F D\phi_{i.gH}(s_1) d_q s_1. \tag{4.3}$$

Consider the first q-integral on RHS of the Eq.(4.3):

By noting to assumptions of the theorem, the ${}_q^F[gH]$-derivative of the function ϕ has the S.P ξ_1 of type I. So, ${}_q^F D\phi_{ii.gH}$ is ${}_q^F[i.gH]$-D on $[a, \xi_1]$, then type of ${}_q^F[gH]$-differentiability change. By these conditions, the Theorem 3.13, admits that

$${}_q^F D\phi_{ii.gH}(\xi_1) = {}_q^F D\phi_{ii.gH}(a) \ominus (-1) \int_a^{\xi_1} {}_q^F D^{(2)} \phi_{i.gH}(z_2) d_q z_2. \tag{4.4}$$

On the other hand, we know that ${}^{F}_{q}D\phi_{ii.gH}$ is ${}^{F}_{q}[ii.gH]$-D on $[\xi_1,\xi]$ and the type of ${}^{F}_{q}[gH]$-differentiability do not change. Thus, for $z_1 \in [\xi_1,\xi]$ from Theorem 3.13, it follows that

$$ {}^{F}_{q}D\phi_{ii.gH}(z_1) = {}^{F}_{q}D\phi_{ii.gH})(\xi_1) \oplus \int_{\xi_1}^{z_1} {}^{F}_{q}D^{(2)}\phi_{ii.gH}(z_3)d_qz_3. \tag{4.5} $$

Replacing Eq.(4.4) into Eq.(4.5) gives

$$ {}^{F}_{q}D\phi_{ii.gH}(z_1) = {}^{F}_{q}D\phi_{ii.gH}(a) \ominus (-1)\int_{a}^{\xi_1} {}^{F}_{q}D^{(2)}\phi_{i.gH}(z_2)d_qz_2 \oplus \int_{\xi_1}^{z_1} {}^{F}_{q}D^{(2)}\phi_{ii.gH}(z_3)d_qz_3. \tag{4.6} $$

Thus,

$$ {}^{F}_{q}D^{(2)}\phi_{i.gH}(z_2) = {}^{F}_{q}D^{(2)}\phi_{i.gH}(a) \oplus \int_{a}^{z_2} {}^{F}_{q}D^{(3)}\phi_{i.gH}(z_4)d_qz_4, \tag{4.7} $$

Then

$$ \int_{a}^{\xi_1} {}^{F}_{q}D^{(2)}\phi_{i.gH}(z_2)d_qz_2 = {}^{F}_{q}D^{(2)}\phi_{i.gH}(a) \odot \frac{(\xi_1-a)_q}{\Gamma_q(2)} \oplus \int_{a}^{\xi_1}\left(\int_{a}^{z_2} {}^{F}_{q}D^{(3)}\phi_{i.gH}(z_4)d_qz_4\right)d_qz_2, \tag{4.8} $$

and

$$ {}^{F}_{q}D^{(2)}\phi_{ii.gH}(z_3) = {}^{F}_{q}D^{(2)}\phi_{ii.gH}(\xi_1) \oplus \int_{\xi_1}^{z_3} {}^{F}_{q}D^{(3)}\phi_{ii.gH}(z_5)d_qz_5. \tag{4.9} $$

Then

$$ \int_{\xi_1}^{z_1} {}^{F}_{q}D^{(2)}\phi_{ii.gH}(z_3)d_qz_3 = {}^{F}_{q}D^{(2)}\phi_{ii.gH}(\xi_1) \odot \frac{(z_1-\xi_1)_q}{\Gamma_q(2)} \oplus \int_{\xi_1}^{z_1}\left(\int_{\xi_1}^{z_3} {}^{F}_{q}D^{(3)}\phi_{ii.gH}(z_5)d_qz_5\right)d_qz_3. \tag{4.10} $$

Applying Eqs.(4.8), (4.10) and (4.6) we get

$$ \begin{aligned} {}^{F}_{q}D\phi_{ii.gH}(z_1) = {}^{F}_{q}D\phi_{ii.gH}(a) \ominus {}^{F}_{q}D^{(2)}\phi_{i.gH}(a) \odot \frac{(a-\xi_1)_q}{\Gamma_q(2)} \oplus {}^{F}_{q}D^{(2)}\phi_{ii.gH}(\xi_1) \\ \odot \frac{(z_1-\xi_1)_q}{\Gamma_q(2)} \ominus (-1)\int_{a}^{\xi_1}\left(\int_{a}^{z_2} {}^{F}_{q}D^{(3)}\phi_{i.gH}(z_4)d_qz_4\right)d_qz_2 \\ \oplus \int_{\xi_1}^{z_1}\left(\int_{\xi_1}^{z_3} {}^{F}_{q}D^{(3)}\phi_{ii.gH}(z_5)d_qz_5\right)d_qz_3. \end{aligned} $$

Finally, in the RHS of Eq.(4.3) the first q-integral can be obtained as

$$ \begin{aligned} \int_{a}^{\xi} {}^{F}_{q}D\phi_{ii.gH}(z_1)d_qz_1 = {}^{F}_{q}D\phi_{ii.gH}(a) \odot \frac{(\xi-a)_q}{\Gamma_q(2)} \ominus {}^{F}_{q}D^{(2)}\phi_{i.gH}(a) \odot \frac{(a-\xi_1)_q}{\Gamma_q(2)} \\ \odot \frac{(\xi-a)_q}{\Gamma_q(2)} \oplus {}^{F}_{q}D^{2}\phi_{ii.gH}(\xi_1) \odot \left[\frac{(\xi-\xi_1)_q^2}{\Gamma_q(3)} - \frac{(a-\xi_1)_q^2}{\Gamma_q(3)}\right] \\ \ominus (-1)\int_{a}^{\xi}\left(\int_{a}^{\xi_1}\left(\int_{a}^{z_2} {}^{F}_{q}D^{(3)}\phi_{i.gH}(z_4)d_qz_4\right)d_qz_2\right)d_qz_1 \\ \oplus \int_{a}^{\xi}\left(\int_{\xi_1}^{z_1}\left(\int_{\xi_1}^{z_3} {}^{F}_{q}D^{(3)}\phi_{ii.gH}(z_5)d_qz_5\right)d_qz_3\right)d_qz_1. \end{aligned} \tag{4.11} $$

In order to find the second q-integral on RHS of the Eq.(4.3). Now, for the first q-integral, we have:

Due to the hypothesis of the theorem, ${}_q^F D^{(i)}\phi_{i.gH},\ i = 2,3$ are ${}_q^F[i.gH]$-D on $[\xi, z_\nu]$, and the type of ${}_q^F[gH]$-differentiability do not change. By Theorem 3.13 we deduce that

$$ {}_q^F D\phi_{i.gH}(s_1) = {}_q^F D\phi_{i.gH}(\xi) \oplus \int_\xi^{s_1} {}_q^F D^{(2)}\phi_{i.gH}(s_2)d_q s_2, \tag{4.12} $$

and

$$ {}_q^F D^{(2)}\phi_{i.gH}(s_2) = {}_q^F D^{(2)}\phi_{i.gH}(\xi) \oplus \int_\xi^{s_2} {}_q^F D^{(3)}\phi_{i.gH}(s_3)d_q s_3. \tag{4.13} $$

Then

$$ \int_\xi^{s_1} {}_q^F D^{(2)}\phi_{i.gH}(s_2)d_q s_2 = {}_q^F D^{(2)}\phi_{i.gH}(\xi) \odot \frac{(s_1-\xi)_q}{\Gamma_q(2)} \oplus \int_\xi^{s_1} \left(\int_\xi^{s_2} {}_q^F D^{(3)}\phi_{i.gH}(s_3)d_q s_3 \right) d_q s_2. \tag{4.14} $$

Substituting (4.14) into (4.12) we obtain

$$ {}_q^F D\phi_{i.gH}(s_1) = {}_q^F D\phi_{i.gH}(\xi) \oplus {}_q^F D^{(2)}\phi_{i.gH}(\xi) \odot \frac{(s_1-\xi)_q}{\Gamma_q(2)} \oplus \int_\xi^{s_1} \left(\int_\xi^{s_2} {}_q^F D^{(3)}\phi_{i.gH}(s_3)d_q s_3 \right) d_q s_2. $$

Thus, the second q-integral on RHS of the Eq.(4.3) can be obtained

$$ \begin{aligned} \int_\xi^{z} {}_q^F D\phi_{i.gH}(s_1)d_q s_1 = {}_q^F D\phi_{i.gH}(\xi) \odot \frac{(z-\xi)_q}{\Gamma_q(2)} \oplus {}_q^\phi D^{(2)}\phi_{i.gH}(\xi) \odot \frac{(z-\xi)_q^2}{\Gamma_q(3)} \\ \oplus \int_\xi^{z} \left(\int_\xi^{s_1} \left(\int_\xi^{s_2} {}_q^F D^{(3)}\phi_{i.gH}(s_3)d_q s_3 \right) d_q s_2 \right) d_q s_1. \end{aligned} \tag{4.15} $$

In Eq.(4.3): By substituting Eqs.(4.11) and (4.15), in Eq.(4.3) we imply that establishes the formula of final part and this is precisely the assertion of the fuzzy q-Taylor theorem.

Lemma 4.2. Let ${}_q^F D^{(\nu)}\phi \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$ and $z \in \mathbb{T}_\mu$. For finding the fuzzy q-Taylor's expansion of ϕ around $z_0 \in \mathbb{T}_\mu$, we have

I. If ${}_q^F D^{(j)}\phi,\ j = 0,1,...,\nu-1$ are ${}_q^F[i.gH]$-D, provided that type of ${}_q^F[gH]$-D has no change on $\mathbb{T}_\mu$. Then there exists $\eta \in \mathbb{T}_\mu$ s. t.

$$ \phi(z) = \phi(z_0) \oplus \sum_{j=1}^{\nu-1} \left[\frac{(z-z_0)_q^j}{\Gamma_q(j+1)} \odot {}_q^F D^{(j)}\phi_{i.gH}(z_0) \right] \oplus \frac{(z-z_0)_q^\nu}{\Gamma_q(\nu+1)} \odot {}_q^F D^{(\nu)}\phi_{i.gH}(\eta). $$

II. If ${}_q^F D^{(j)}\phi,\ j = 0,1,...,\nu-1$ are ${}_q^F[ii.gH]$-D, provided that type of ${}_q^F[gH]$-D has no change on $\mathbb{T}_\mu$. Then there exists $\eta \in \mathbb{T}_\mu$ s. t.

$$ \phi(z) = \phi(z_0) \ominus (-1) \sum_{j=1}^{\nu-1} \left[\frac{(z-z_0)_q^j}{\Gamma_q(j+1)} \odot {}_q^F D^{(j)}\phi_{ii.gH}(z_0) \right] \ominus (-1) \frac{(z-z_0)_q^\nu}{\Gamma_q(\nu+1)} \odot {}_q^F D^{(\nu)}\phi_{ii.gH}(\eta). $$

III. If ${}_q^F D^{(j)}\phi,\ j = 2k-1,\ k = 1,...,\frac{\nu-1}{2}$ are ${}_q^F[i.gH]$-D and ${}_q^F D^{(j)}\phi,\ j = 2k,\ k = 0,1,...,\frac{\nu-1}{2}$ are ${}_q^F[ii.gH]$-D where $\nu-1$ is an even number, then there exists $\eta \in \mathbb{T}_\mu$ s. t.

$$ \begin{aligned} \phi(z) = &\phi(z_0) \ominus (-1) \sum_{\substack{j=1 \\ j \text{ is odd}}}^{\nu-1} \left[\frac{(z-z_0)_q^j}{\Gamma_q(j+1)} \odot {}_q^F D^{(j)}\phi_{ii.gH}(z_0) \right] \oplus \sum_{\substack{j=1 \\ j \text{ is even}}}^{\nu-1} \left[\frac{(z-z_0)_q^j}{\Gamma_q(j+1)} \odot {}_q^F D^{(j)}\phi_{i.gH}(z_0) \right] \\ &\ominus (-1) \frac{(z-z_0)_q^\nu}{\Gamma_q(\nu+1)} \odot {}_q^F D^{(\nu)}\phi_{ii.gH}(\eta). \end{aligned} $$

IV. If ϕ has a S.P at $\xi \in \mathbb{T}_\mu$ of type II (i.e. ${}_q^F[gH]$-D changes from ${}_q^F[ii.gH]$ to ${}_q^F[i.gH]$) and suppose that the type of ${}_q^F[gH]$-D for ${}_q^F D^{(j)}\phi,\ j=1,...,\nu-1$ are ${}_q^F[i.gH]$-D. Then there exists $\eta \in \mathbb{T}_\mu$ s. t.

$$\phi(z)=\begin{cases} \phi(z_0)\ominus(z_0-z)\odot {}_q^F D\phi_{ii.gH}(z_0)\oplus\sum_{j=2}^{\nu-1}\left[\frac{(z-z_0)_q^j}{\Gamma_q(j+1)}\odot {}_q^F D^{(j)}\phi_{i.gH}(z_0)\right] & \\ \oplus\frac{(z-z_0)_q^\nu}{\Gamma_q(\nu+1)}\odot {}_q^F D^{(\nu)}\phi_{i.gH}(\eta), & 0<z\le\xi, \\ \phi(a)\oplus\sum_{j=1}^{\nu-1}\left[\frac{(z-z_0)_q^j}{\Gamma_q(j+1)}\odot {}_q^F D^{(j)}\phi_{i.gH}(z_0)\right]\oplus\frac{(z-z_0)_q^\nu}{\Gamma_q(\nu+1)}\odot {}_q^F D^{(\nu)}\phi_{i.gH}(\eta), & \xi\le z. \end{cases}$$

V. If ϕ has a S.P at $\xi \in \mathbb{T}_\mu$ of type I (i.e. ${}_q^F[gH]$-D changes from ${}_q^F[i.gH]$ to ${}_q^F[ii.gH]$) and suppose that the type of ${}_q^F[gH]$-D for ${}_q^F D^{(j)}\phi,\ j=1,...,\nu-1$ are ${}_q^F[ii.gH]$-D. Then there exists $\eta \in \mathbb{T}_\mu$ s. t.

$$\phi(z)=\begin{cases} \phi(z_0)\oplus(z-z_0)\odot {}_q^F D\phi_{i.gH}(z_0)\ominus(-1)\sum_{j=2}^{\nu-1}\left[\frac{(z-z_0)_q^j}{\Gamma_q(j+1)}\odot {}_q^F D^{(j)}\phi_{ii.gH}(z_0)\right] & \\ \ominus(-1)\frac{(z-z_0)_q^\nu}{\Gamma_q(\nu+1)}\odot {}_q^F D^{(\nu)}\phi_{ii.gH}(\eta), & 0<z\le\xi, \\ \phi(z_0)\ominus(-1)\sum_{j=1}^{\nu-1}\left[\frac{(z-z_0)_q^j}{\Gamma_q(j+1)}\odot {}_q^F D^{(j)}\phi_{ii.gH}(z_0)\right]\ominus(-1)\frac{(z-z_0)_q^\nu}{\Gamma_q(\nu+1)}\odot {}_q^F D^{(\nu)}\phi_{ii.gH}(\eta), & \xi\le z. \end{cases}$$

Proof. Given that the preceding theorem the proof is straightforward.

At the end of this section, we illustrates the importance of fuzzy q-Taylor's expansion study by expressing two basic applications.

4.1 Applications of the fuzzy q-Taylor's expansion

1 In the early 19th century, most mathematicians believed that a continuous function is always differentiable (have derivatives everywhere) except at a significant set of points on a domain. However, further exploration between the dependence of continuity and differentiability propelled to the finding of continuous nowhere differentiable functions. Probably the first example of a continuous nowhere differentiable function on an interval is Bolzano function(≈1830). Afterwards many functions published with that particular condition including; Cellerier function (≈ 1860), Riemann function (≈ 1861), Weierstrass function (1872), Darboux function (1873), Peano function (1890) and etc. The existence of such functions increased the importance of examining q-Taylor's expansion.

As we mentioned before, the most important advantage of using the fuzzy q-Taylor's expansion is to find the fuzzy q-Taylor's expansion for fuzzy-valued continuous functions that are non-differentiable (or non-derivable) in the classical sense. In fact, without paying attention to non-differentiable (in conventional expression) of function in some points, the fuzzy q-Taylor's expansion is able to introduce an approximation for continuous function in the same points of non-differentiable. Fuzzy Weierstrass function is one the examples of a pathological fuzzy-valued function in a mathematical science. It is defined from the real line to the fuzzy numbers set which its particular property, being differentiable nowhere but continuous everywhere. This function is defined as a Fourier series in the following form

$$\phi(z)=c\odot\sum_{\nu=0}^{\infty}a^\nu\cos(b^\nu\pi z),\quad \text{where}\quad 7\leqslant b\in\mathbb{Z}^+_{2\nu-1},\ 0<a<1,\ c\in\mathbb{R}_\mathcal{F}, ab>1+3\pi/2.$$

In this paper, it is not our purpose to study of fuzzy q-Taylor's expansion of fuzzy Weierstrass function, rather we leave it to the reader to verify which this result is not far from being conclusive.

2 In approximation theory, fuzzy Taylor series expansion is a powerful tool. This expansion is the basis of many numerical methods that plays a basic role for studying the local behavior of a suitable fuzzy function. One of the important applications of fuzzy Taylor's expansion is the solving of the fuzzy differential equation (specially the fuzzy initial value problems). According to this description, solving the fuzzy initial value q-problem (i.e., includes q-derivative) will also be important. Let us consider that the fuzzy initial value q-problem as follows

$$\begin{cases} {}_q^F Dy(z)=\phi(t,y(z)), \quad z\in\mathbb{T}_\mu, \\ y(0)=y_0\in\mathbb{R}_\mathcal{F}, \end{cases}$$

---

denotes the positive odd integer number.

where $\phi : \mathbb{T}_\mu \times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ is continuous and $y(z)$ is an unknown fuzzy function of crisp variable $t \in \mathbb{T}_\mu$. Furthermore, ${}^F_q Dy(z)$ is the ${}^F_q[gH]$-derivative $y(z)$ with the finite set of S.Ps. For solving the same problem above, one of the most basic proposed numerical methods can be the use of the fuzzy q-Taylor's expansion that introduces its approximate solution. Since this method obtains an approximation in the form of a fuzzy q-Taylor's series around zero, by regulating the step length used in the q-Taylor's expansion and the increase in the number of expansion sentences, the error of the approximate solution can be reduced, obviously.

In the future works, we will examine the details of these applications that no attempt has been made here to develop.

5 Numerical Simulations

Here, we will investigate the some general examples and according to the previous section, present the q-Taylor expansion of these fuzzy functions. The figurs associated with the examples plotted using Wolfram Mathematica 9.0 software.

Example 5.1. We know that, a function may not be equal to its fuzzy q-Taylor's expansion, even if its fuzzy q-Taylor's expansion converges at every point. A function that is equal to its fuzzy q-Taylor's expansion in an open interval (or a disc in the complex plane) is defined as an analytic function in that interval. Consider that the following ${}^F_q[i.gH]$-D analytic function $\phi(z) = (0, 1, 1.5) \odot z^\nu$, where ν is a positive integer and z_0 is arbitrary point that belongs to $\mathbb{T}_\mu$. For $i = 1, 2, ..., j,\ j \leq \nu,\ {}^F_q D^{(i)}\phi$ are exist and ${}^F_q[i.gH]$-D. An easy computation yields that

$$\begin{aligned}
&[D_q\underline{\phi}(z;r), D_q\bar{\phi}(z;r)] = [r, 1.5 - 0.5r][\nu]_q z^{\nu-1},\\
&[D_q^{(2)}\underline{\phi}(z;r), D_q^{(2)}\bar{\phi}(z;r)] = [r, 1.5 - 0.5r][\nu]_q[\nu - 1]_q z^{\nu-2},\\
&\qquad\vdots\\
&[D_q^{(j)}\underline{\phi}(z;r), D_q^{(j)}\bar{\phi}(z;r)] = [r, 1.5 - 0.5r][\nu]_q[\nu - 1]_q...[\nu - j + 1]_q z^{\nu-j}.
\end{aligned}$$

We know that ϕ is fuzzy valued function and ${}^F_q[i.gH]$-D that the type of ${}^F_q[gH]$-D has no change on $\mathbb{T}_\mu$. Now, the Theorem 4.1 implies, the fuzzy q-Taylor's expansion of ϕ around z_0 is as follows

$$\begin{aligned}
[\phi(z)]_r =&[r, 1.5 - 0.5r]z_0^\nu + [r, 1.5 - 0.5r]\frac{(z - z_0)_q}{\Gamma_q(2)}[\nu]_q z_0^{\nu-1} + [r, 1.5 - 0.5r]\frac{(z - z_0)_q^2}{\Gamma_q(3)}[\nu]_q[\nu - 1]_q z_0^{\nu-2} + \cdots\\
&+ [r, 1.5 - 0.5r]\frac{(z - z_0)_q^j}{\Gamma_q(j + 1)}[\nu]_q[\nu - 1]_q \cdots [\nu - j + 1]_q z_0^{\nu-j} + ... + [r, 1.5 - 0.5r]\frac{(z - z_0)_q^\nu}{\Gamma_q(\nu + 1)}[\nu]_q[\nu - 1]_q...[2]_q[1]_q.
\end{aligned}$$

To simplify we can rewrite this expansion in the following form

$$\phi(z) = (0, 1, 1.5) \odot \sum_{k=0}^{\nu} \binom{\nu}{k}_q (z - z_0)_q^k\ z_0^{\nu-k},$$

where $\binom{\nu}{k}_q$ is called q-binomial coefficient. The plot of this example is observed in Figure 1(a).

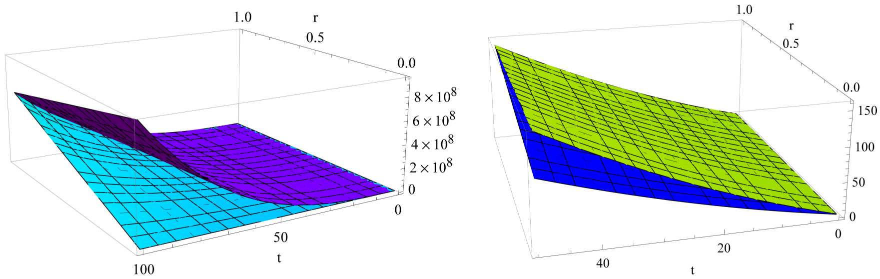
In the next example, we will build a binomial formula involving the q-binomial coefficient.

Example 5.2. Consider the following fuzzy valued function

$$\begin{cases} \phi : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F},\\ \phi(z) = k \odot (c + z)_q^\nu, \quad k \in \mathbb{R}_\mathcal{F}, \end{cases}$$

where ν is a non-negative integer, c be any constant and $k = (1.3, 2, 2.1)$ is fuzzy number. Using the fuzzy q-Taylor formula for $j \leq \nu$, let us calculate the expansion of ϕ around $z_0 = 0$. For this purpose, since ϕ is ${}^F_q[i.gH]$-D function and the type of ${}^F_q[gH]$-differentiability has no change on $\mathbb{T}_\mu$, we first compute ${}^F_q D^{(i)}\phi$ for $i = 1, 2, ..., j \leq \nu$, so

$$\begin{aligned}
&[D_q\underline{\phi}(z;r), D_q\bar{\phi}(z;r)] = [1.3 + 0.7r, 2.1 - 0.1r][\nu]_q(c + qz)_q^{\nu-1},\\
&[D_q^{(2)}\underline{\phi}(z;r), D_q^{(2)}\bar{\phi}(z;r)] = [1.3 + 0.7r, 2.1 - 0.1r][\nu]_q[\nu - 1]_q\ q(c + q^2z)_q^{\nu-2},\\
&\qquad\vdots\\
&[D_q^{(j)}\underline{\phi}(z;r), D_q^{(j)}\bar{\phi}(z;r)] = [1.3 + 0.7r, 2.1 - 0.1r][\nu]_q[\nu - 1]_q...[\nu - j + 1]_q\ q^{\frac{j(j-1)}{2}}(c + q^jz)_q^{\nu-j},
\end{aligned}$$



(a) (b)

Figure 1: (a) Graph of the fuzzy q-Taylor's expansion of the function ϕ for $q = 0.6,\ \nu = 3,\ z_0 = 2$ and $0 < z < 100$ in Example 5.1 (where the lower branch is Cyan and the upper branch is Blue). (b) Graph of the fuzzy q-Taylor expansion of the function ϕ for $q = 0.01,\ \nu = 6,\ c = 1$ and $0 < z < 50$ in Example 5.2 (where the lower branch is Blue and the upper branch is Green).

It satisfies all the other conditions for using of the Theorem 4.1. In continuation, substituting the above relations in the fuzzy q-Taylor formula, we deduce

$$\phi(z) = (1.3, 2, 2.1) \odot \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{\frac{k(k-1)}{2}} c^{\nu-k} z^k,$$

which this coefficient is called the Gauss's q-binomial formula. The fuzzy q-Taylor's formula of this example presented in Figure 1(b).

Example 5.3. Consider that $\phi : \mathbb{T}_\mu \to \mathbb{R}_{\mathcal{F}},\quad \phi(z) = (0, 0.5, 1) \odot (c - z)_q^\nu$. In order to find the fuzzy q-Taylor's expansion, on account of the Theorem 4.1, we have

$$\overset{First-\ {}_q^F D}{\rightsquigarrow} \ [\,{}_q^F D\phi(z)]_r = (-1)[\nu]_q (c - qz)_q^{\nu-1}[0.5r, 1 - 0.5r],$$

$$\overset{Second-\ {}_q^F D}{\rightsquigarrow} \ [\,{}_q^F D^{(2)}\phi(z)]_r = [\nu]_q[\nu + 1]_q\ q(c - q^2 z)_q^{\nu-2}[0.5r, 1 - 0.5r],$$

$$\vdots$$

$$\overset{(j)th-\ {}_q^F D}{\rightsquigarrow} \ [\,{}_q^F D^{(j)}\phi(z)]_r = (-1)[\nu]_q ... [\nu + j - 1]_q q^{\frac{j(j-1)}{2}} (c - q^j z)_q^{\nu-j}[0.5r, 1 - 0.5r],$$

$$\overset{(j+1)th-\ {}_q^F D}{\rightsquigarrow} \ [\,{}_q^F D^{(j+1)}\phi(z)]_r = [\nu]_q ... [\nu + j - 1]_q[\nu + j]_q q^{\frac{j(j+1)}{2}} (c - q^{j+1} z)_q^{\nu-j-1}[0.5r, 1 - 0.5r].$$

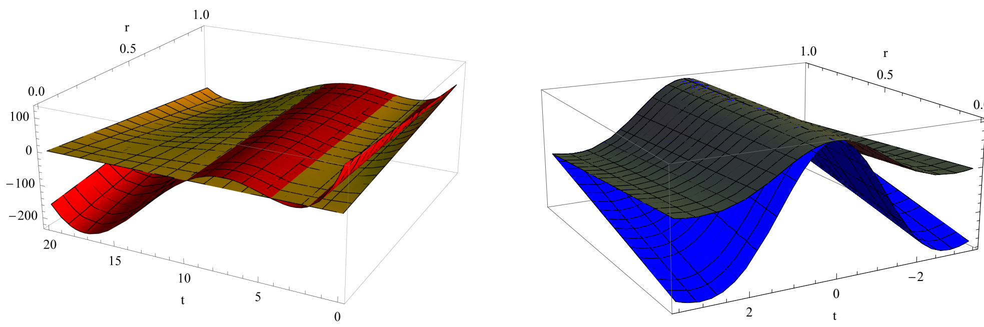
It is obvious that if $c < z_0$ or $c \geq z_0$, the type of ${}_q^F[gH]$-differentiability for ${}_q^F D^{(j)}\phi,\ j \leq \nu$ on $\mathbb{T}_\mu$ will change i.e., the fuzzy function ϕ in Example 5.3 has a some S.Ps. Now, Theorem 4.1 leads to

$$\phi(z) = (0, 0.5, 1) \odot \Bigg[(c - z_0)_q^\nu \ominus (-1) \sum_{k=1}^{\nu-1} \frac{(c - z_0)_q^k}{\Gamma_q(k + 1)} [\nu]_q[\nu + 1]_q ... [\nu + k - 1]_q$$
$$q^{\frac{k(k-1)}{2}} (c - q^k z_0)_q^{\nu-k} \ominus (-1) \frac{(c - z_0)_q^\nu}{\Gamma_q(\nu + 1)} [\nu]_q[\nu + 1]_q ... [2\nu - 2]_q[2\nu - 1]_q\ q^{\frac{\nu(\nu-1)}{2}}\Bigg].$$

We rewrite this formula then the fuzzy q-Taylor's expansion of $\phi(z)$ will be as follows

$$\phi(z) = (0, 0.5, 1) \odot (c - z_0)_q^\nu \ominus (-1) \sum_{k=1}^{\nu} \left[\binom{\nu + k - 1}{k}_q q^{\frac{k(k-1)}{2}} (c - z_0)_q^\nu \odot (0, 0.5, 1)\right].$$

Figure 2(a) shows the fuzzy q-Taylor's expansion of the function $\phi(z)$ around $z_0 = 10$ that is one of the S.Ps.



(a) (b)

Figure 2: (a) Graph of the function ϕ for $q = 0.45,\ \nu = 7,\ c = 2$ and $0 < z < 20$ in Example 5.3 (where the lower branch is Brown and the upper branch is Red). (b) Graph of the function ϕ for $q = 0.76$ and $-3.5 < z < 3.5$ in Example 5.4 (where the lower branch is Gray and the upper branch is Blue)

Example 5.4. Let us consider that $\phi(z) = (0.2, 1, 1.6) \odot \cos_q(z)$. We want to write the fuzzy q-Taylor's expansion of the function ϕ around z_0 which is a suitable number. As a result, we find that

$$\begin{aligned}
&\overset{First-\ {}^F_qD\ and\ (j-1)th-\ {}^F_qD}{\rightsquigarrow} [\,{}^F_qD\phi(z)]_r = [0.2+0.8r, 1.6-0.6r](-1)\sin_q(z),\\
&\overset{Second-\ {}^F_qD\ and\ (j)th-\ {}^F_qD}{\rightsquigarrow} [\,{}^F_qD^{(2)}\phi(z)]_r = [0.2+0.8r, 1.6-0.6r](-1)\cos_q(z),\\
&\overset{(j+1)th-\ {}^F_qD}{\rightsquigarrow} [\,{}^F_qD^{(j+1)}\phi(z)]_r = [0.2+0.8r, 1.6-0.6r]\sin_q(z),\\
&\overset{(j+2)th-\ {}^F_qD}{\rightsquigarrow} [\,{}^F_qD^{(j+2)}\phi(z)]_r = [0.2+0.8r, 1.6-0.6r]\cos_q(z).
\end{aligned}$$

It obvious that ϕ has the number of S.Ps. The conditions of Theorem 4.1 are established, then there exists $\eta \in \mathbb{T}_\mu$

$$\begin{aligned}
\phi(z) = (0.2, 1, 1.6) \odot \Bigg[\cos_q(z_0) \ominus (-1)\bigg((z-z_0)_q \sin_q(z_0) \oplus \frac{(z-z_0)_q^2}{[2]_q!}\cos_q(z_0)\bigg)\\
\oplus \bigg(\frac{(z-z_0)_q^3}{[3]_q!}\sin_q(z_0) \oplus \frac{(z-z_0)_q^4}{[4]_q!}\cos_q(z_0)\bigg) \ominus (-1)... \ominus (-1)\frac{(z-\xi)_q^\nu}{[\nu]_q!}\sin_q(\eta)\Bigg].
\end{aligned}$$

To illustrate the behaviour of the fuzzy q-Taylor's expansion of the fuzzy valued function ϕ around $z_0 = 1$, please refer to Figure 2(b).

6 Conclusions

In classical (or conventional) computer, the standard calculus is used as a discrete tool. Currently, it is studied a quantum computer is faster than the classical computer numerically. Perhaps it can be said the one of most important difference between the quantum of the classic is how to store information. In a quantum computer, it is stored as q-bits (quantum bits) while in a classical computer, information is stored as bits. Other difference between them is due to the definition of derivative, in a way the q-calculus works without the definition of limit, it applies the most desirable to a computer. As a result, efficiently quantum computers may be able to solve problems which are not practically possible on classical computers. The development of this calculus is contributing to the implementation of quantum computing.

In this paper, our main idea is to introduce appropriate formulas for fuzzy q-derivative and fuzzy q-integral based on generalized Hukuhara difference. Then definitions and basic theorems related to them has been asserted and proved. In the meanwhile, we propose fuzzy q-Taylor's method for the approximation of the fuzzy functions or the fuzzy q-functions investigated in different scenarios according to the type of differentiability. Finally, fuzzy q-Taylor's expansion of four examples of fuzzy functions is obtained.

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