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Coefficient estimates for a subclass of analytic and bi-univalent functions by an integral operator

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Abstract

In this paper, we introduce and investigate a subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda, m, n, \alpha, \gamma)$ of bi-univalent functions in the open unit disk $\mathbb{U}$. Upper bounds for this class's second and third coefficients of functions are found. The results, which we have presented in this paper, would generalize and improve some recent works of several earlier authors.

Keywords: Analytic functions, Bi-univalent functions, Coefficient estimates, Starlike functions, Koebe One-Quarter Theorem
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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the following normalized form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, we denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. The *Koebe One-Quarter Theorem* [2] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Hence, every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

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where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$. The class consisting of bi-univalent functions are denoted by Σ.

Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient a_2 of functions $f \in \mathcal{S}$ gives the growth and distortion bounds as well as covering theorems.

Lewin [8] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [7] proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Tan [12] obtained the bound for $|a_2|$ namely $|a_2| \leq 1.485$ which is the best known estimate for functions in the class Σ. Recently there interest to study the bi-univalent functions class Σ (see [4, 6, 13, 14, 15]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{1, 2\})$ for each $f \in \Sigma$ is still an open problem.

Recently, Salman and Atshan [10] introduced two subclasses of Σ and obtained estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. Salman and Atshan [10] introduced the integral operator $\mathcal{J}^{\alpha}_{m,n}$ given by

$$\begin{aligned} \mathcal{J}^{\alpha}_{m,n} &: \Sigma \to \Sigma \\ \mathcal{J}^{\alpha}_{m,n} f(z) &= z + \sum_{j=2}^{\infty} \left(\frac{\beta(m+j, n+j)}{\beta(m+1, n+1)} \right)^{\alpha} a_j z^j \end{aligned}$$

where $\beta(m,n) = \int_0^1 \frac{t^{m+1}}{(1-t)^{1-n}} dt$, $m, n > 0$ and $\alpha \in \mathbb{N} \bigcup \{0\}$. We denote

$$(\mathcal{K}^j_{m,n})^{\alpha} = \left(\frac{\beta(m+j, n+j)}{\beta(m+1, n+1)} \right)^{\alpha},$$

therefore

$$\mathcal{J}^{\alpha}_{m,n} f(z) = z + \sum_{j=2}^{\infty} (\mathcal{K}^j_{m,n})^{\alpha} a_j z^j.$$

Definition 1.1 ([10]). A function f given by (1.1) is said to be in the class $\mathcal{H}^{\alpha}_{\Sigma}(\lambda, m, n, \tau)$, if the following conditions are satisfied:

$$f \in \Sigma \ , \quad \left| arg \left((1-\lambda) \frac{\mathcal{J}^{\alpha}_{m,n} f(z)}{z} + \lambda (\mathcal{J}^{\alpha}_{m,n} f(z))' \right) \right| < \frac{\tau\pi}{2}$$

and

$$\left| arg \left((1-\lambda) \frac{\mathcal{J}^{\alpha}_{m,n} g(w)}{w} + \lambda (\mathcal{J}^{\alpha}_{m,n} g(w))' \right) \right| < \frac{\tau\pi}{2},$$

where $z, w \in \mathbb{U}$, $m, n > 0$, $\alpha \in \mathbb{N} \bigcup \{0\}$, $\lambda \geq 1$, $0 < \tau \leq 1$ and the function g is given by (1.2).

Theorem 1.2 ([10]). Let f given by (1.1) be in the class $\mathcal{H}^{\alpha}_{\Sigma}(\lambda, m, n, \tau)$ $(m, n > 0,\ \alpha \in \mathbb{N} \bigcup \{0\},\ \lambda \geq 1,\ 0 < \tau \leq 1)$. Then

$$|a_2| \leq \frac{2\tau}{\sqrt{2\tau(1+2\lambda)(\mathcal{K}^3_{m,n})^{\alpha} + (1-\tau)(1+\lambda)^2(\mathcal{K}^2_{m,n})^{2\alpha}}}$$

and

$$|a_3| \leq \frac{4\tau^2}{(1+\lambda)^2(\mathcal{K}^2_{m,n})^{2\alpha}} + \frac{2\tau}{(1+2\lambda)(\mathcal{K}^3_{m,n})^{\alpha}}.$$

Definition 1.3 ([10]). A function f given by (1.1) is said to be in the class $\mathcal{H}^{\alpha}_{\Sigma}(\lambda, m, n, \delta)$ if the following conditions are satisfied:

$$f \in \Sigma \ , \quad \mathfrak{Re} \left((1-\lambda) \frac{\mathcal{J}^{\alpha}_{m,n} f(z)}{z} + \lambda (\mathcal{J}^{\alpha}_{m,n} f(z))' \right) > \delta$$

and

$$\mathfrak{Re}\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}g(w)}{w}+\lambda(\mathcal{J}^{\alpha}_{m,n}g(w))'\right)>\delta,$$

where $z, w \in \mathbb{U}$, $m, n > 0$, $\alpha \in \mathbb{N}\bigcup\{0\}$, $\lambda \geq 1$, $0 \leq \delta < 1$ and the function g is given by (1.2).

Theorem 1.4 ([10]). Let f given by (1.1) be in the class $\mathcal{H}^{\alpha}_{\Sigma}(\lambda, m, n, \delta)$ $(m, n > 0,\ \alpha \in \mathbb{N}\bigcup\{0\},\ \lambda \geq 1,\ 0 \leq \delta < 1)$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\delta)}{(1+2\lambda)(\mathcal{K}^{3}_{m,n})^{\alpha}}}$$

and

$$|a_3| \leq \frac{4(1-\delta)^2}{(1+\lambda)^2(\mathcal{K}^{2}_{m,n})^{2\alpha}}+\frac{2(1-\delta)}{(1+2\lambda)(\mathcal{K}^{3}_{m,n})^{\alpha}}.$$

The purpose of the this paper is to introduce new subclass $\mathcal{G}^{h,p}_{\Sigma}(\lambda, m, n, \alpha, \gamma)$ of bi-univalent functions class Σ. Moreover, we obtain estimates on initial coefficients $|a_2|$ and $|a_3|$ for functions in this class. The results presented in this paper would generalize and improve some recent works of Salaman and Atshan [10], Frasin [3], Frasin and Aouf [4] and Srivastava et al. [11].

2 Coefficient Estimates

In this section, we introduce and investigate the general subclass $\mathcal{G}^{h,p}_{\Sigma}(\lambda, m, n, \alpha, \gamma)$.

Definition 2.1. Let $h, p : \mathbb{U} \to \mathbb{C}$ be analytic functions and

$$\min\{\mathfrak{Re}(h(z)), \mathfrak{Re}(p(z))\} > 0 \quad (z \in \mathbb{U}) \quad and \quad h(0) = p(0) = 1.$$

A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{G}^{h,p}_{\Sigma}(\lambda, m, n, \alpha, \gamma)$, if the following conditions are satisfied:

$$f \in \Sigma, \left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}f(z)}{z}+\lambda(\mathcal{J}^{\alpha}_{m,n}f(z))'+\gamma z(\mathcal{J}^{\alpha}_{m,n}f(z))''\right) \in h(\mathbb{U}) \tag{2.1}$$

and

$$\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}g(w)}{w}+\lambda(\mathcal{J}^{\alpha}_{m,n}g(w))'+\gamma w(\mathcal{J}^{\alpha}_{m,n}g(w))''\right) \in p(\mathbb{U}), \tag{2.2}$$

where $z, w \in \mathbb{U}$, $\alpha \in \mathbb{N}\bigcup\{0\}$, $m, n > 0$, $\lambda \geq 1$, $\gamma \geq 0$ and $g = f^{-1}$.

Remark 2.2. There are many choices of the functions h, p and the parameters α , λ and γ which would provide interesting subclasses of bi-univalent functions. For example, if we let

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\tau} \qquad (0 < \tau \leq 1)$$

it can be easily verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1.

(1) By setting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\tau}$, we have

$$\mathcal{G}^{h,p}_{\Sigma}(\lambda, m, n, \alpha, \gamma)\Big|_{h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\tau}} = \mathcal{G}^{\tau}_{\Sigma}(\lambda, m, n, \alpha, \gamma)$$

where the class $\mathcal{G}^{\tau}_{\Sigma}(\lambda, m, n, \alpha, \gamma)$ consists of functions $f \in \Sigma$ satisfying the following conditions:

$$\left|arg\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}f(z)}{z}+\lambda(\mathcal{J}^{\alpha}_{m,n}f(z))'+\gamma z(\mathcal{J}^{\alpha}_{m,n}f(z))''\right)\right| < \frac{\tau\pi}{2}$$

and

$$\left|arg\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}g(w)}{w}+\lambda(\mathcal{J}^{\alpha}_{m,n}g(w))'+\gamma w(\mathcal{J}^{\alpha}_{m,n}g(w))''\right)\right| < \frac{\tau\pi}{2}.$$

(2) By setting $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\tau}$ and $\gamma=0$ in Definition 2.1, the class $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ reduces to the class $\mathcal{H}_{\Sigma}^{\alpha}(\lambda,m,n,\tau)$.

(3) By setting $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\tau}$ and $\gamma=\alpha=0$ in Definition 2.1, the class $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ reduces to the class $\beta_{\Sigma}(\tau,\lambda)$ which was considered by Frain and Aouf [4, Definition 2.1].

(4) By setting $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\tau}$, $\gamma=\alpha=0$ and $\lambda=1$ in Definition 2.1, the class $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ reduces to the class $\mathcal{H}_{\Sigma}^{\tau}$ which was introduced by Srivastava et al. [11, Definition 1].

(5) By setting $h(z)=p(z)=\left(\frac{1+z}{1-z}\right)^{\tau}$, $\lambda=1$ and $\alpha=0$ in Definition 2.1, the class $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ reduces to the class $\mathcal{H}_{\Sigma}(\tau,\gamma)$ which was introduced by Frasin [3, Definition 2.1].

Also, if we let

$$h(z)=p(z)=\frac{1+(1-2\delta)z}{1-z}\qquad(0\le\delta<1)$$

it can be easily verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1.

(6) By setting $h(z)=p(z)=\frac{1+(1-2\delta)z}{1-z}$, we have

$$\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)\Big|_{h(z)=p(z)=\frac{1+(1-2\delta)z}{1-z}}=\mathcal{G}_{\Sigma}^{\delta}(\lambda,m,n,\alpha,\gamma)$$

where the class $\mathcal{G}_{\Sigma}^{\delta}(\lambda,m,n,\alpha,\gamma)$ consists of functions $f\in\Sigma$ satisfying the following conditions:

$$\mathfrak{Re}\left((1-\lambda)\frac{\mathcal{J}_{m,n}^{\alpha}f(z)}{z}+\lambda(\mathcal{J}_{m,n}^{\alpha}f(z))'+\gamma z(\mathcal{J}_{m,n}^{\alpha}f(z))''\right)>\delta$$

and

$$\mathfrak{Re}\left((1-\lambda)\frac{\mathcal{J}_{m,n}^{\alpha}g(w)}{w}+\lambda(\mathcal{J}_{m,n}^{\alpha}g(w))'+\gamma w(\mathcal{J}_{m,n}^{\alpha}g(w))''\right)>\delta.$$

(7) By setting $h(z)=p(z)=\frac{1+(1-2\delta)z}{1-z}$ and $\gamma=0$ in Definition 2.1, the class $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ reduces to the class $\mathcal{H}_{\Sigma}^{\alpha}(\lambda,m,n,\tau)$.

(8) By putting $h(z)=p(z)=\frac{1+(1-2\delta)z}{1-z}$ and $\gamma=\alpha=0$ in Definition 2.1, the class $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ reduces to the class $\beta_{\Sigma}(\delta,\lambda)$ that was studied by Frain and Aouf [4, Definition 3.1].

(9) By putting $h(z)=p(z)=\frac{1+(1-2\delta)z}{1-z}$, $\gamma=\alpha=0$ and $\lambda=1$ in Definition 2.1, the $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ class reduces to the class $\mathcal{H}_{\Sigma}(\delta)$ that was introduced by Srivastava et al. [11, Definition 2].

(10) By setting $h(z)=p(z)=\frac{1+(1-2\delta)z}{1-z}$, $\alpha=0$ and $\lambda=1$ in Definition 2.1, the class $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ reduces to the class $\mathcal{H}_{\Sigma}(\delta,\gamma)$ which was introduced by Frasin [3, Definition 3.1].

Now, we derive the estimates of the coefficients $|a_2|$ and $|a_3|$ for class $\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$.

Theorem 2.3. If $f\in\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ $(m,n>0,\ \alpha\in\mathbb{N}\bigcup\{0\},\ \lambda\ge1,\ \gamma\ge0)$. Then

$$|a_2|\le\min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\lambda+2\gamma)^2(\mathcal{K}_{m,n}^{2})^{2\alpha}}},\sqrt{\frac{|h''(0)|+|p''(0)|}{4(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^{3})^{\alpha}}}\right\}\tag{2.3}$$

and

$$|a_3|\le\min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\lambda+2\gamma)^2(\mathcal{K}_{m,n}^{2})^{2\alpha}}+\frac{|h''(0)|+|p''(0)|}{4(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^{3})^{\alpha}},\frac{|h''(0)|}{2(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^{3})^{\alpha}}\right\}.\tag{2.4}$$

Proof . Since $f\in\mathcal{G}_{\Sigma}^{h,p}(\lambda,m,n,\alpha,\gamma)$ and $g=f^{-1}$. It follows from (2.1) and (2.2) that

$$(1-\lambda)\frac{\mathcal{J}_{m,n}^{\alpha}f(z)}{z}+\lambda(\mathcal{J}_{m,n}^{\alpha}f(z))'+\gamma z(\mathcal{J}_{m,n}^{\alpha}f(z))''=h(z)\tag{2.5}$$

and

$$(1-\lambda)\frac{\mathcal{J}_{m,n}^{\alpha}g(w)}{w}+\lambda(\mathcal{J}_{m,n}^{\alpha}g(w))'+\gamma w(\mathcal{J}_{m,n}^{\alpha}g(w))''=p(w),\tag{2.6}$$

respectively, where h and p satisfy the conditions of Definition 2.1. Also, the functions h and p have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots \tag{2.7}$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + p_3 w^2 + \cdots , \tag{2.8}$$

respectively. Now, by equating the coefficients in (2.6) and (2.7), we get

$$(1+\lambda+2\gamma)(\mathcal{K}^2_{m,n})^{\alpha} a_2 = h_1, \tag{2.9}$$

$$(1+2\lambda+6\gamma)(\mathcal{K}^3_{m,n})^{\alpha} a_3 = h_2, \tag{2.10}$$

$$-(1+\lambda+2\gamma)(\mathcal{K}^2_{m,n})^{\alpha} a_2 = p_1 \tag{2.11}$$

and

$$(1+2\lambda+6\gamma)(\mathcal{K}^3_{m,n})^{\alpha}(2a_2^2 - a_3) = p_2. \tag{2.12}$$

From (2.9) and (2.11), we have

$$h_1 = -p_1 \tag{2.13}$$

and

$$2(1+\lambda+2\gamma)^2(\mathcal{K}^2_{m,n})^{2\alpha} a_2^2 = h_1^2 + p_1^2. \tag{2.14}$$

By using (2.10) and (2.12), we obtain

$$2(1+2\lambda+6\gamma)(\mathcal{K}^3_{m,n})^{\alpha} a_2^2 = p_2 + h_2. \tag{2.15}$$

Consequently, from (2.14) and (2.15), we get

$$a_2^2 = \frac{h_1^2 + p_1^2}{2(1+\lambda+2\gamma)^2(\mathcal{K}^2_{m,n})^{2\alpha}} \tag{2.16}$$

and

$$a_2^2 = \frac{h_2 + p_2}{2(1+2\lambda+6\gamma)(\mathcal{K}^3_{m,n})^{\alpha}}. \tag{2.17}$$

Therefore, we find from the (2.16) and (2.17) that

$$|a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\lambda+2\gamma)^2(\mathcal{K}^2_{m,n})^{2\alpha}}$$

and

$$|a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{4(1+2\lambda+6\gamma)(\mathcal{K}^3_{m,n})^{\alpha}}.$$

So, we get the desired estimate on the coefficient $|a_2|$ asserted. Next, in order to find the bound of the coefficient $|a_3|$, by subtracting (2.12) from (2.10), we get

$$a_3 = a_2^2 + \frac{h_2 - p_2}{2(1+2\lambda+6\gamma)(\mathcal{K}^3_{m,n})^{\alpha}}. \tag{2.18}$$

Upon substituting the value of a_2^2 from (2.17) into (2.18), it follows that

$$a_3 = \frac{h_2}{(1+2\lambda+6\gamma)(\mathcal{K}^3_{m,n})^{\alpha}}. \tag{2.19}$$

Therefore, we obtain

$$|a_3| \leq \frac{|h''(0)|}{2(1+2\lambda+6\gamma)(\mathcal{K}^3_{m,n})^{\alpha}}.$$

On the other hand, upon substituting the value of a_2^2 from (2.16) into (2.18), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{2(1+\lambda+2\gamma)^2(\mathcal{K}_{m,n}^2)^{2\alpha}} + \frac{h_2 - p_2}{2(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}, \tag{2.20}$$

Therefore, we get

$$|a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\lambda+2\gamma)^2(\mathcal{K}_{m,n}^2)^{2\alpha}} + \frac{|h''(0)| + |p''(0)|}{4(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}.$$

This completes the proof. □

3 Corollaries and Consequences

By choosing

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\tau} \qquad (0 < \tau \leq 1,\ z \in \mathbb{U})$$

in Theorem 2.3, we have the following result.

Corollary 3.1. Let the function f be given by (1.1) in the class $\mathcal{G}_{\Sigma}^{\tau}(\lambda, m, n, \alpha, \gamma)$ where $m, n > 0$, $\alpha \in \mathbb{N}\bigcup\{0\}$, $\lambda \geq 1$ and $0 < \tau \leq 1$. Then

$$|a_2| \leq \min\left\{\frac{2\tau}{(1+\lambda+2\gamma)(\mathcal{K}_{m,n}^2)^{\alpha}}, \tau\sqrt{\frac{2}{(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}}\right\}$$

and

$$\begin{aligned} |a_3| &\leq \min\left\{\frac{4\tau^2}{(1+\lambda+2\gamma)^2(\mathcal{K}_{m,n}^2)^{2\alpha}} + \frac{2\tau^2}{(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}, \frac{2\tau^2}{(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}\right\} \\ &= \frac{2\tau^2}{(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}. \end{aligned}$$

By taking $\gamma = 0$ in Corollary 3.1, we have the following result.

Corollary 3.2. Let the function f be given by (1.1) in the class $\mathcal{H}_{\Sigma}^{\alpha}(\lambda, m, n, \tau)$ where $m, n > 0$, $\alpha \in \mathbb{N}\bigcup\{0\}$, $\lambda \geq 1$ and $0 < \tau \leq 1$. Then

$$|a_2| \leq \min\left\{\frac{2\tau}{(1+\lambda)(\mathcal{K}_{m,n}^2)^{\alpha}}, \tau\sqrt{\frac{2}{(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\alpha}}}\right\}$$

and

$$|a_3| \leq \frac{2\tau^2}{(1+2\lambda)(\mathcal{K}_{m,n}^3)^{\alpha}}.$$

Remark 3.3. The bound on $|a_3|$ given in Corollary 3.2 is better than that given in Theorem 1.2.

By taking $\alpha = 0$ in Corollary 3.2, we have the following result.

Corollary 3.4. Let the function f be given by (1.1) in the class $\beta_{\Sigma}(\tau, \lambda)$ where $\lambda \geq 1$ and $0 < \tau \leq 1$. Then

$$|a_2| \leq \begin{cases} \tau\sqrt{\frac{2}{1+2\lambda}}, & 1 \leq \lambda \leq 1+\sqrt{2} \\ \frac{2\tau}{1+\lambda}, & \lambda \geq 1+\sqrt{2} \end{cases}$$

and

$$|a_3| \leq \frac{2\tau^2}{1+2\lambda}.$$

Remark 3.5. The bound on $|a_2|$ given in Corollary 3.4 is better that than given by Frasin and Aouf [4, Theorem 2.2]. Because

(i) If $1 \leq \lambda \leq 1+\sqrt{2}$, we get
$$\tau\sqrt{\frac{2}{1+2\lambda}} \leq \frac{2\tau}{\sqrt{(1+\lambda)^2+\tau(1+2\lambda-\lambda^2)}}.$$

(ii) If $\lambda \geq 1+\sqrt{2}$, we have
$$\frac{2\tau}{1+\lambda} \leq \frac{2\tau}{\sqrt{(1+\lambda)^2+\tau(1+2\lambda-\lambda^2)}}.$$

Also, the bound on $|a_3|$ given in Corollary 3.4 is better that than given by Frasin and Aouf [4, Theorem 2.2].

By taking $\lambda = 1$ in Corollary 3.4, we have the following result.

Corollary 3.6. Let the function f be given by (1.1) in the class $\mathcal{H}_{\Sigma}^{\tau}$ where $0 < \tau \leq 1$. Then
$$|a_2| \leq \tau\sqrt{\frac{2}{3}} \qquad \text{and} \qquad |a_3| \leq \frac{2\tau^2}{3}.$$

Remark 3.7. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.6 are better than those given by Srivastava et al. [11, Theorem 1].

By taking $\lambda = 1$ and $\alpha = 0$ in Corollary 3.1, we have the following result.

Corollary 3.8. Let the function f be given by (1.1) in the class $\mathcal{H}_{\Sigma}(\tau,\gamma)$ where $\gamma \geq 0$ and $0 < \tau \leq 1$. Then
$$|a_2| \leq \begin{cases} \tau\sqrt{\frac{2}{3(1+2\gamma)}}\ , & 0 < \gamma \leq \frac{1+\sqrt{3}}{2} \\ \frac{\tau}{1+\gamma}\ , & \gamma \geq \frac{1+\sqrt{3}}{2} \end{cases}$$
and
$$|a_3| \leq \frac{2\tau^2}{3(1+2\gamma)}.$$

Remark 3.9. The bound on $|a_3|$ given in Corollary 3.8 is better than that given by Frain [3, Theorem 2.2].

By letting
$$h(z) = p(z) = \frac{1+(1-2\delta)z}{1-z} \qquad (0 \leq \delta < 1,\ z \in \mathbb{U})$$
in Theorem 2.3, we deduce the following corollary.

Corollary 3.10. Let the function f be given by (1.1) in the class $\mathcal{G}_{\Sigma}^{\delta}(\lambda,m,n,\alpha,\gamma)$ where $m,n > 0$, $\alpha \in \mathbb{N}\bigcup\{0\}$, $\lambda \geq 1$ and $0 \leq \delta < 1$. Then
$$|a_2| \leq \min\left\{\frac{2(1-\delta)}{(1+\lambda+2\gamma)(\mathcal{K}_{m,n}^2)^{\alpha}}, \sqrt{\frac{2(1-\delta)}{(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}}\right\}$$
and
$$\begin{aligned} |a_3| &\leq \min\left\{\frac{2(1-\delta)^2}{(1+\lambda+2\gamma)^2(\mathcal{K}_{m,n}^2)^{2\alpha}} + \frac{2(1-\delta)}{(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}, \frac{2(1-\delta)}{(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}\right\} \\ &= \frac{2(1-\delta)}{(1+2\lambda+6\gamma)(\mathcal{K}_{m,n}^3)^{\alpha}}. \end{aligned}$$

By taking $\gamma = 0$ in Corollary 3.10, we have the following result.

Corollary 3.11. Let the function f be given by (1.1) in the class $\mathcal{H}_{\Sigma}^{\alpha}(\lambda, m, n, \delta)$ where $m, n > 0$, $\alpha \in \mathbb{N}\bigcup\{0\}$, $\lambda \geq 1$ and $0 \leq \delta < 1$. Then

$$|a_2| \leq \min\left\{\frac{2(1-\delta)}{(1+\lambda)(\mathcal{K}_{m,n}^2)^\alpha}, \sqrt{\frac{2(1-\delta)}{(1+2\lambda)(\mathcal{K}_{m,n}^3)^\alpha}}\right\}$$

and

$$|a_3| \leq \frac{2(1-\delta)}{(1+2\lambda)(\mathcal{K}_{m,n}^3)^\alpha}.$$

Remark 3.12. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.11 are better than that given in Theorem 1.4.

By taking $\alpha = 0$ in Corollary 3.11, we have the following result.

Corollary 3.13. Let the function f be given by (1.1) in the class $\beta_\Sigma(\delta, \lambda)$ where $\lambda \geq 1$ and $0 \leq \delta < 1$. Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\delta)}{1+2\lambda}}\ , & 0 \leq \delta \leq \frac{1}{2}\left(1 - \frac{\lambda^2}{1+2\lambda}\right) \\ \frac{2(1-\delta)}{1+\lambda}\ , & \frac{1}{2}\left(1 - \frac{\lambda^2}{1+2\lambda}\right) \leq \delta < 1 \end{cases}$$

and

$$|a_3| \leq \frac{2(1-\delta)}{1+2\lambda}.$$

Remark 3.14. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.13 are better than those given By Frasin [4, Theorem 3.2].

By taking $\lambda = 1$ in Corollary 3.13, we have the following result.

Corollary 3.15. Let the function f be given by (1.1) in the class $\mathcal{H}_\Sigma(\delta)$ where $0 \leq \delta < 1$. Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\delta)}{3}}\ , & 0 \leq \delta \leq \frac{1}{3} \\ (1-\delta)\ , & \frac{1}{3} \leq \delta < 1 \end{cases}$$

and

$$|a_3| \leq \frac{2(1-\delta)}{3}.$$

Remark 3.16. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.15 are better than those given by Srivastava et al. [11, Theorem 2].

By taking $\lambda = 1$ and $\alpha = 0$ in Corollary 3.10, we have the following result.

Corollary 3.17. Let the function f be given by (1.1) in the class $\mathcal{H}_\Sigma(\delta, \gamma)$ where $\gamma \geq 0$ and $0 \leq \delta < 1$. Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\delta)}{3(1+2\gamma)}}\ , & 0 \leq \delta \leq \frac{1}{3}\left(1 - \frac{2\gamma^2}{1+2\gamma}\right) \\ \frac{1-\delta}{1+\gamma}\ , & \frac{1}{3}\left(1 - \frac{2\gamma^2}{1+2\gamma}\right) \leq \delta < 1 \end{cases}$$

and

$$|a_3| \leq \frac{2(1-\delta)}{3(1+2\gamma)}.$$

Remark 3.18. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.17 are better than those given by Frasin [3, Theorem 3.2].

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