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Discrete wave packets on non-Archimedean fields

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Abstract

In this paper, we introduce a comprehensive theory of discrete wave packet systems on non-Archemedean fields by exploiting the machinery of Fourier transforms. We also define discrete periodic wave packet transform. A characterization of the system to be a Parseval frame and discrete periodic wave packet frame for $\ell^2(\mathcal{Z})$ are obtained.

Keywords: Wave packet frame, Non-Archimedean field, Discrete Fourier transform, Parseval frame
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1 Introduction

Non-Archimedean fields have been used for the global space-time theory to unify microscopic and macroscopic physics. Using real-time and space-time coordinates in mathematical physics leads to problems with the Archimedean axiom on the microscopic level. According to the Archimedean axiom, "any given segment on the line can be surpassed by the successive addition of a smaller segment along the same line".This means that we can measure the arbitrarily small distances but a measurement of distances smaller than the Planck length is impossible. Volovich [25] proposes to base physics on a coalition of non-Archimedean normed fields and the classical fields. The p-series fields and p-adic fields are best known non-Archimedean normed fields. As $p \to \infty$, many of the fundamental properties of p-adic analysis approach their counterparts in classical analysis. Thus, p-adic analysis could provide a bridge from microscopic to macroscopic physics.

In the frame theory, the traditional study of function systems involves the Gabor systems. These systems were first introduced by Denis Gabor [14] by using a Gaussian distribution function as a window function to construct efficient, time-frequency localized expansions of finite-energy signals. Gabor systems $\{M_{mb}T_{na}\psi(x) : m, n \in \mathbb{Z}\}$ are generated by modulations and translations of a single function $\psi(x) \in L^2(\mathbb{R})$ and hence, can be viewed as the set of time-frequency shifts of $\psi(x)$ along the lattice $a\mathbb{Z} \times b\mathbb{Z}$ in $\mathbb{R}^2$. Based on this development, Duffin and Schaeffer [13] introduced frames for Hilbert spaces, while addressing some deep problems in the non-harmonic Fourier series. Frames did not generate much interest outside the non-harmonic Fourier series until the seminal work by Daubechies, Grossmann, and Meyer [11]. After their pioneer, the theory of frames began to be studied widely and deeply. An important problem in practice is to determine conditions for Gabor systems to be frames. Many results in this area, including necessary conditions and sufficient conditions have been established during the last two decades [1, 2, 3, 5, 7, 10].

Similar to the Gabor systems, wavelet systems $\left\{\psi_{j,k} := a^{j/2}\psi(a^j x - bk) : j, k \in \mathbb{Z}\right\}$ are generated by a set of dilations and translations of a given single window function ψ. Therefore, one of the fundamental problems in the study

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of wavelet frames is to find conditions on the wavelet function and the dilation and translation parameters so that the corresponding wavelet system forms a frame. Daubechies [10] proved the first result on the necessary and sufficient conditions for wavelet frames and then an improved result was given by Chui and Shi [8]. Here a natural question arises. Can we construct a family of functions which possesses simultaneously the advantages of both the above-defined function systems and which overcome their drawbacks? The answer is yes.

In the year 1978, during the pursuit of representations of functions, Córdoba and Fefferman [9] introduced the notion of wave packet systems by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. Then wave packet systems were successfully applied to solve the problems in physics, especially quantum mechanics. Ibnatovich [16] constructed a non-spreading, unnormalizable wave packet satisfying the Schrödinger equation and considered a modification of the Schrödinger equation which allows the normalization of the wave packet. Exploiting the machinery of the Bohmian model of quantum mechanics, Pan [20] presented a physically transparent picture of this spreading phenomenon that may not be available in the standard formalism of quantum mechanics and re-examined the way the wave packet spreads. In 2016, Kaur et al. [17] constructed the wave packet in the cubical billiard and observed its time evolution for various closed orbits. Lebate et al. [18] adopted the same expression to describe any collections of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R})$. In fact, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems.

There is a substantial body of work that has been concerned with the analysis of Gabor, wavelet and wave packet systems on a non-Archimedean field K. Recently, Shah and Ahmad [23] have introduced the concept of wave packet systems on non-archimedean fields and provide the complete characterizations for the wave packet system $\left\{D_{\mathfrak{p}}M_{u(m)b}T_{un(a)} : m, n \in \mathbb{N}_0\right\}$ to be a frame for $L^2(K)$ and presented and obtained the necessary and sufficient conditions about the wave packet systems to be wave packet Parseval frames on non-archimedian fields. Ahmad et.al [4] introduced the concept of periodic Gabor frames on non-Archimedean fields of positive characteristic and established a necessary and sufficient condition for a periodic Gabor system to be a Gabor frame for $L^2(\Omega)$. Furthermore, they present some equivalent characterizations of Parseval Gabor frames on non-Archimedean fields using some fundamental equations in the time domain.

In physics and applied mathematics, many algorithmic realizations of the wavelet systems are in discrete settings because of the discrete nature of all filters and input/output data. Using the discrete power growth space given by Pevnyi and Zheludev, Dou et al. [12] have constructed a type of spline wavelets. Han [15] gave a comprehensive theory of discrete framelets and wavelets via an algorithmic approach by directly studying a discrete framelet transform. Lu and Li [19] gave a simple characterization of the sequences for which purely shift-invariant systems are normalized tight frames for $\ell^2(\mathbb{Z}^d)$ and established a sufficient condition for those systems to be frames for $\ell^2(\mathbb{Z}^d)$ which is a variant of the classical wavelet systems to be frames for $L^2(\mathbb{R}^d)$. Xu et al. [26] propose a class of wave packet systems in a discrete setting and provide a complete characterization of these systems to be a Parseval frame. Motivated and inspired by the above work, we study discrete versions of periodic wave packet systems on non-Archimedean fields.

The rest of the paper is tailored as follows. In Section 2, we recall some basic Fourier analysis on non-Archimedean fields and also some results which are required in the subsequent sections including the definitions of discrete periodic wave packet systems on such fields. In section 3, we construct the first-stage discrete periodic wave packet frame and establish a sufficient condition for the wave packet system to be a frame for $\ell^2(\mathcal{Z})$. In Section 4, we construct the J^{th}-stage discrete periodic wave packet frame for $\ell^2(\mathcal{Z})$. In Section 5, by using discrete periodic wave packet transform, the associated decomposition and reconstruction algorithms are established.

2 Preliminaries on Non-Archimedean Fields

2.1 Non-Archimedean Fields

A non-Archimedean field K is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. If K is of positive characteristic, then K is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let K be a fixed non-Archimedean field with the ring of integers $\mathfrak{D} = \{x \in K : |x| \leq 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non-Archimedean norm $|\cdot| : K \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|x\,y| = |x||y|$ for all $x, y \in K$;

(c) $|x+y| \le \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the ultra-metric inequality. Let $\mathfrak{B} = \{x \in K : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in K. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of K such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then one can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| < q^{-k}\}$ is also compact and open and is a subgroup of K^+. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that χ_u is any character on K^+, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [13], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\text{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \quad 0 \le a_k < p, \quad \text{and } k = 0, 1, \ldots, c-1,$$

We define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\, \mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.2}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Here after we use the notation $\chi_n = \chi_{u(n)}, n \ge 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.3}$$

2.2 Fourier Transforms on Non-Archimedean Fields

The Fourier transform of $f \in L^1(K)$ is denoted by $\hat{f}(\xi)$ and defined by

$$F\{f(x)\} = \hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx. \tag{2.4}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)} dx = \int_K f(x)\chi(-\xi x)\, dx.$$

The properties of Fourier transforms on non-Archimedean field K are much similar to those of on the classical field $\mathbb{R}$. In fact, the Fourier transform on non-Archimedean fields of positive characteristic have the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \le \|f\|_1$.
- If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(K)$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\chi_\xi(x)}\, dx, \tag{2.5}$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx. \tag{2.6}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} \big|f(x)\big|^2 dx = \sum_{n\in\mathbb{N}_0} \Big|\hat{f}\big(u(n)\big)\Big|^2. \tag{2.7}$$

Let $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$, where $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of $\mathfrak{D}$ in K^+. Then we define

$$\ell^2(\mathcal{Z}) = \left\{ z : \mathcal{Z} \to \mathbb{C} : \sum_{n\in\mathbb{N}_0} |z(u(n))|^2 < \infty \right\}$$

is a Hilbert space with inner product

$$\langle z, w\rangle = \sum_{n\in\mathbb{N}_0} z(u(n))\overline{w(u(n))}.$$

The following definitions are natural which are used in the subsequent sections:

Definition 2.1. The *Fourier transform* on $\ell^2(\mathcal{Z})$ is the map $\widehat{\ }: \ell^2(\mathcal{Z}) \to L^2(\mathfrak{D})$ defined for $z \in \ell^2(\mathcal{Z})$ by

$$\hat{z}(\xi) = \sum_{n\in\mathbb{N}_0} z(u(n))\chi_{u(n)}(\xi),$$

and the *Inverse Fourier transform* on $L^2(\mathfrak{D})$ is the map $\vee : L^2(\mathfrak{D}) \to \ell^2(\mathcal{Z})$ defined for $f \in L^2(\mathfrak{D})$ by

$$f^\vee(u(n)) = \big\langle f, \chi_{u(n)}\big\rangle = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx.$$

For $z \in \ell^2(\mathcal{Z})$, we have

$$\begin{aligned}
(\widehat{z})^\vee\,(u(n)) &= \big\langle \hat{z}, \chi_{u(n)}\big\rangle \\
&= \Big\langle \sum_{m\in\mathbb{N}_0} z(u(m))\chi_{u(m)}, \chi_{u(n)} \Big\rangle \\
&= \sum_{m\in\mathbb{N}_0} z(u(m))\,\big\langle \chi_{u(m)}, \chi_{u(n)}\big\rangle \\
&= z(u(n)).
\end{aligned}$$

Since $\big\{\chi_{u(n)} : n \in \mathbb{N}_0\big\}$ is an orthonormal basis for $L^2(\mathfrak{D})$. It is also clear that the function $\widehat{z}$ is an integral periodic function because for $m \in \mathbb{N}_0$, we have

$$\begin{aligned}
\widehat{z}(\xi + u(m)) &= \sum_{n\in\mathbb{N}_0} z(u(n))\chi_{u(n)}(\xi), \chi_{u(n)}(u(m)). \\
&= \sum_{n\in\mathbb{N}_0} z(u(n))\chi_{u(n)}(\xi) \\
&= \widehat{z}(\xi).
\end{aligned}$$

For $z, w \in \ell^2(\mathcal{Z})$, we have *Parseval's relation*:

$$\langle z, w\rangle = \sum_{n\in\mathbb{N}_0} z(u(n))\overline{w(u(n))} = \int_{\mathfrak{D}} \widehat{z}(\xi)\overline{\widehat{w}(\xi)}\, d\xi = \langle \widehat{z}, \widehat{w}\rangle\,,$$

and *Plancherel's formula*:

$$\|z\|^2 = \sum_{n\in\mathbb{N}_0} |z(u(n))|^2 = \int_{\mathfrak{D}} |\widehat{z}(\xi)|^2\, d\xi = \|\widehat{z}\|^2.$$

Definition 2.2. Let $\mathcal{H}$ denotes a separable Hilbert space and $\mathcal{P}$ be countable index set. A countable set $\{\varphi_{P} : p \in \mathcal{P}\}$ in $\mathcal{H}$ is called a frame $\mathcal{H}$ if there exists constants C and D, $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{p\in\mathcal{P}} |\langle f, \varphi_p\rangle|^2 \leq D\|f\|^2 \tag{2.1}$$

holds for all $f \in \mathcal{H}$. The greatest possible such C is the lower frame bound and the least possible such D is the upper frame bound. We say that $\{\varphi_p : p \in \mathcal{P}\}$ is a tight frame for $\mathcal{H}$ with frame bound C if $C = D$, and a normalized tight (Parseval) frame if $C = D = 1$.

Definition 2.3. For $m, k \in \mathbb{N}_0$, the *translation operator* $T_{u(m)} : \ell^2(\mathcal{Z}) \to \ell^2(\mathcal{Z})$ is defined by

$$T_{u(m)} z(u(n)) = z\left(u(n) - u(m)\right)$$

and the *modulation operator* $M_{u(k)} : \ell^2(\mathcal{Z}) \to \ell^2(\mathcal{Z})$ by

$$M_{u(k)} z\left(u(n)\right) = z\left(u(n)\right) \overline{\chi_{u(k)}}.$$

Definition 2.4. For $v_\ell^j \in \ell^2(\mathcal{Z}),\ 0 \leq \ell \leq L, 0 \leq j \leq J$, we call $\mathcal{W}(V)$ a J^{th} -stage wave packet system associated with $V = \{v_\ell : 0 \leq \ell \leq L\}$ if

$$\mathcal{W}(V) = \left\{\mathcal{W}_{k,m}^j v_\ell^j = T_{\mathfrak{p}^j u(m)} M_{u(k)} v_\ell^j : k \in \mathbb{N}_0, m \in \mathbb{N}_0, 0 \leq \ell \leq L, 0 \leq j \leq J\right\}. \tag{2.2}$$

For $J = 0$, the wave packet system in (2.2) becomes the Gabor system

$$\left\{\overline{\chi_{u(k)}(u(m)} v_\ell(\cdot - u(m)) : k \in \mathbb{N}_0, 0 \leq \ell \leq L\right\}.$$

On taking $k = 0$ in (2.2), the wave packet system degenerates into the wavelet system

$$\left\{v_\ell^j(\cdot - \mathfrak{p}^j u(m)) : m \in \mathbb{N}_0, 0 \leq \ell \leq L, j = 1, 2, ..., J\right\}.$$

Thus the Gabor system and wavelet system are the particular cases of wave packet systems.

3 First-Stage Discrete Periodic Wave Packet Frames

In this section, we provide the characterization of the first-stage wave packet system namely, the Gabor system $\left\{\mathcal{W}_{k,m} v_\ell = T_{\mathfrak{p}u(m)} M_{u(k)} v_\ell : k \in \mathbb{N}_0, m \in \mathbb{N}_0, 0 \leq \ell \leq L\right\}$ to be a Parseval frame for $\ell^2(\mathcal{Z})$.

Definition 3.1. If the system $\{\mathcal{W}_{k,m} v_\ell : k \in \mathbb{N}_0, m \in \mathbb{N}_0, 0 \leq \ell \leq L\}$ is a Parseval frame for $\ell^2(\mathcal{Z})$, then for each $z(u(n)) \in \ell^2(\mathcal{Z})$, the discrete periodic wave packet transforms of $z(u(n))$ can be calculated by the following formula

$$\left\{\langle z(u(n)), \mathcal{W}_{k,m} v_\ell\rangle : k \in \mathbb{N}_0, m \in \mathbb{N}_0, 0 \leq \ell \leq L\right\}.$$

The reconstruction formula from the above definition is as follows:

$$z(u(n)) = \sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} \sum_{m\in\mathbb{N}_0} \langle z(u(n)), \mathcal{W}_{k,m} v_\ell\rangle \mathcal{W}_{k,m} v_\ell.$$

Since $\langle z(u(n)), \mathcal{W}_{k,m} v_\ell \rangle = \sum_{n\in\mathbb{N}_0} z(u(n))\overline{[M_{u(k)} v_\ell]}(u(n)-\mathfrak{p}u(k))$ for $k\in\mathbb{N}_0, m\in\mathbb{N}_0, 0\le \ell\le L$. Therefore the following operators should be invoked. For $z, v\in \ell^2(\mathcal{Z})$, the *transition operator* $\mathcal{T}_{v,\mathfrak{p}}:\ell^2(\mathcal{Z})\to\ell^2(\mathcal{Z})$ is defined as

$$[\mathcal{T}_{v,\mathfrak{p}} z](u(m)) = \sum_{n\in\mathbb{N}_0} z(u(n))\overline{v(u(n)-\mathfrak{p}u(m))}, m\in\mathbb{N}_0. \tag{3.1}$$

Next we introduce the down sampling operator $\downarrow\ \mathfrak{p}:\ell^2(\mathcal{Z})\to\ell^2(\mathcal{Z})$ defined by

$$(z\downarrow\mathfrak{p})(u(n)) := (\mathfrak{p}u(n)),\ n\in\mathbb{N}_0. \tag{3.2}$$

It is convenient to use the notion $z(\mathfrak{p}\cdot)$ for $z\downarrow\mathfrak{p}$. Now the transition operator $\mathcal{T}_{v,\mathfrak{p}}$ in equation (3.1) can be equivalently expressed as $\mathcal{T}_{v,\mathfrak{p}} z = (\tilde{v} * z(u(n))\downarrow\mathfrak{p}$.

On combining the transition with modulation operators, we can obtain the following three types of sequences:

Case I:

$$[M_{u(k)}\mathcal{T}_{v,\mathfrak{p}} z](k,m) = \overline{\chi_{u(k)}}(u(m)) \sum_{n\in\mathbb{N}_0} z(u(n))\overline{v(u(n)-\mathfrak{p}u(m))},\quad k\in\mathbb{N}_0,\ m\in\mathbb{N}_0. \tag{3.3}$$

Case II:

$$[\mathcal{T}_{v,\mathfrak{p}} M_{u(k)} z](k,m) = \sum_{n\in\mathbb{N}_0} z(u(n))\overline{M_{-u(k)} T_{\mathfrak{p}u(m)} v(u(n))},\quad k\in\mathbb{N}_0,\ m\in\mathbb{N}_0. \tag{3.4}$$

Case III: The *pre-frame operator* $\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}:\ell^2(\mathcal{Z})\to\ell^2(\mathcal{Z}\times\mathcal{Z})$ is defined by

$$[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}} z](k,m) = \sum z(u(n))\overline{\mathcal{W}_{k,m} v(u(n))},\quad k\in\mathbb{N}_0,\quad m\in\mathbb{N}_0, \tag{3.5}$$

where $\mathcal{W}_{k,m} v = T_{\mathfrak{p}u(m)} M_{u(k)} v,\quad k\in\mathbb{N}_0,\ m\in\mathbb{N}_0$. A relation between cases II and III is as follows:

$$\mathcal{T}_{M_{u(k)}v}\mathfrak{p} z(u(n)) = \overline{\chi_{u(k)}(\mathfrak{p}u(m))}\mathcal{T}_{v,\mathfrak{p}} M_{-u(k)} z(u(n)),\quad k\in\mathbb{N}_0, m\in\mathbb{N}_0.$$

We know that if $\{\mathcal{W}_{k,m} v : k\in\mathbb{N}_0, m\in\mathbb{N}_0\}$ is a frame for $\ell^2(\mathcal{Z})$, the associated pre-frame operator $\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}$ is a bounded linear. Let $\mathcal{T}^*_{M_{u(k)}v,\mathfrak{p}}:\ell^2(\mathcal{Z}\times\mathcal{Z})\to\ell^2(\mathcal{Z})$ be adjoint operator of $\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}$. Therefore for each $\{w(k,m): k\in\mathbb{N}_0, m\in\mathbb{N}_0\}\in\ell^2(\mathcal{Z}\times\mathcal{Z})$, the conjugate operator $\mathcal{T}^*_{M_{u(k)}v,\mathfrak{p}}$ can be expressed as follows:

$$[\mathcal{T}^*_{M_{u(k)}v,\mathfrak{p}} w(k,m)](u(n)) \sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} w(k,m)\mathcal{W}_{k,m} v(u(n)),\quad n\in\mathbb{N}_0. \tag{3.6}$$

So the associated frame operator $\mathcal{S}:\ell^2(\mathcal{Z})\to\ell^2(\mathcal{Z})$ can be derived as follows:

$$\mathcal{S}_{z(u(n))} = \mathcal{T}^*_{M_{u(k)}v\mathfrak{p}} z(u(n)) = \sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\langle z(u(n)), \mathcal{W}_{k,m} v\rangle \mathcal{W}_{k,m} v(u(n)), n\in\mathbb{N}_0.$$

On the basis of the above definition of operators, we have the following theorem.

Theorem 3.2. Let $z, v\in\ell^2(\mathcal{Z})$, $\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}, \mathcal{T}^*_{E_{u(k)},\mathfrak{p}}$ and the frame operator $\mathcal{S}$ are defined as above. For a fixed $k\in\mathbb{N}_0$, if we write $[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}](k,\cdot)$ as

$$[\mathcal{T}_{M_k v,\mathfrak{p}}](k,m) := \sum_{w\in\mathbb{N}_0} z(w)\overline{\mathcal{W}_{k,m} v(w)},\quad m\in\mathbb{N}_0,$$

then

$$[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}} z(k,\cdot)]^\wedge(\zeta) = q\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{\hat{v}(\zeta+u(s)+u(k))},\quad \zeta\in\mathcal{Z}$$

and the DFT of $\mathcal{S}z(u(n))$ is given by

$$\widehat{\mathcal{S}}_z(\zeta) = q\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{\hat{v}(\zeta+u(s)+u(k))}\hat{v}(\zeta+u(k)),\quad \zeta\in\mathcal{Z}, \tag{3.7}$$

where $\Omega := \{0, , 2, ..., q-1\}$.

Proof . For each $z, v \in \ell^2(\mathcal{Z})$ and a fixed $k \in \mathbb{N}_0$, write h_k as

$$h_k(u(m)) := \sum_{w\in\mathbb{N}_0} z(w)\overline{T_{u(m)}M_{u(k)}v(w)}, \quad m \in \mathbb{N}_0.$$

It can be checked that $[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}z](k,m) = h_k(\mathfrak{p}u(m))$ for all $m \in \mathbb{N}_0$. On one hand, we have

$$\begin{aligned}
\hat{h}_k(\zeta) &= \sum_{k\in\mathbb{N}_0}\sum_{w\in\mathbb{N}_0} z(w)\overline{T_{u(m)}M_{u(k)}v(w)\chi_{u(m)}(\zeta)} \\
&= \sum_{m\in\mathbb{N}_0}\sum_{w\in\mathbb{N}_0} z(w)\overline{v(u(n))\chi_{u(w)}(\zeta)}\chi_{u(n)(\zeta+u(m))} \\
&= \left\{\sum_{w\in\mathbb{N}_0} z(w)\overline{\chi_{u(w)}(\zeta)}\right\}\left\{\sum_{n\in\mathbb{N}_0}\overline{v(u(n))}\chi_{u(n)}(\zeta+u(m))\right\} \\
&= \hat{z}(\zeta)\overline{\hat{v}(\zeta+u(m))}.
\end{aligned}$$

For $\zeta \in \mathcal{Z}$, we have

$$\begin{aligned}
\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{\hat{v}(\zeta+u(s)+(k))} &= \sum_{s\in\Omega}\hat{h}_k(\zeta+u(s)) \\
&= \sum_{s\in\Omega}\hat{h}_k(m)\overline{\chi_{u(m)}(\zeta+u(s))} \\
&= \sum_{m\in\mathbb{N}_0} h_k(m)\overline{\chi_{u(m)}(\zeta)}\left\{\sum_{s\in\Omega}\overline{\chi_{u(s)}(u(m))}\right\},
\end{aligned}$$

which implies that

$$\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{\hat{v}(\zeta+u(s)+u(m))} = q\sum_{m\in\mathbb{N}_0} h_k(\mathfrak{p}u(m))\overline{\chi_{\mathfrak{p}u(m)}(\zeta)}.$$

On the other hand, we have

$$[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}z(k,\cdot)]^\wedge(\zeta) = \sum_{k\in\mathbb{N}_0}[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}z](k,m)\overline{\chi_{u(m)}(\zeta)} = \sum_{m\in\mathbb{N}_0} h_k(\mathfrak{p}u(m))\overline{\chi_{\mathfrak{p}u(m)}(\zeta)}.$$

Thus

$$[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}z(k,\cdot]^\wedge\zeta = q\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{\hat{v}(\zeta+u(s)+u(k))}.$$

For $\zeta \in \mathcal{Z}$, we have

$$\begin{aligned}
[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}w(k,m)*]^\wedge\zeta &= \sum_{n\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} w(k,m)\mathcal{W}_{k,m}v(u(n))\overline{\chi_{u(n)}(\zeta)} \\
&= \sum_{k\in\mathbb{N}_0}\left(\sum_{m\in\mathbb{N}_0} w(t,k)\overline{\chi_{u(m)}(\zeta)}\right)\left(v(u(n))\overline{\chi_{u(n)}(\zeta+u(m))}\right) \\
&= \sum_{k\in\mathbb{N}_0}[w(k,\cdot)]^\wedge(\zeta)\hat{v}(\zeta+u(k)).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\hat{\mathcal{S}}_z(\zeta) &= \left(\mathcal{T}^*_{M_{u(k)}v,\mathfrak{p}}z(k,\cdot)\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}z\right)^\wedge(\zeta) \\
&= \sum_{k\in\mathbb{N}_0}[\mathcal{T}_{M_{u(k)}v,\mathfrak{p}}z(k,\cdot)]^\wedge(\zeta)\hat{v}(\zeta+u(k)) \\
&= q\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{\hat{v}(\zeta+u(s)+u(k)}\hat{v}(\zeta+u(k)).
\end{aligned}$$

This completes the proof of the theorem. □

Now we proceed to state and prove a theorem which is a characterization of the system

$$\{\mathcal{W}_{k,m}v_\ell : k \in \mathbb{N}_0, m \in \mathbb{N}_0, 0 \leq \ell \leq L\}$$

to be a Parseval frame for $\ell^2(\mathcal{Z})$.

Theorem 3.3. Let $v_\ell \in \ell^2(\mathcal{Z}), 0 \leq \ell \leq L$. Define $\mathcal{T}_{M_{u(k)}v_\ell,\mathfrak{p}}$ and $\mathcal{T}^*_{M_{u(k)}v_\ell,\mathfrak{p}}$ by means of the equations (3.5) and (3.6), respectively. then the following conditions are equivalent:

(i) The system $\{W_{k,m}v_\ell : k \in \mathbb{N}_0, 0 \leq \ell \leq L\}$ is a Parseval frame for $\ell^2(\mathcal{Z})$;

(ii) For $z \in \ell^2(\mathcal{Z})$,

$$\sum_{\ell=0}^{L} \mathcal{T}^*_{M_{u(k)}v_\ell,\mathfrak{p}} \mathcal{T}_{M_{u(k)}v_\ell,\mathfrak{p}} z = z; \tag{3.8}$$

(iii) For any $s \in \Omega, \zeta \in \mathcal{Z}$,

$$\sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} \hat{v}_\ell(\zeta + u(k))\overline{\hat{v}_\ell(\zeta + t + u(s))} = q\delta(s), \tag{3.9}$$

where δ is the Dirac sequence.

Proof . We first show the equivalence between (i) and (ii). The system $\{W_{k,m}v_\ell : k \in \mathbb{N}_0, m \in \mathbb{N}_0, 0 \leq \ell \leq L\}$ is a Parseval frame for $\ell^2(\mathcal{Z})$ and hence is equivalent to

$$z = \sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} \sum_{m\in\mathbb{N}_0} \langle z, \mathcal{W}_{k,m}v_\ell\rangle \mathcal{W}_{k,m}v_\ell = \sum_{\ell=0}^{L} \mathcal{T}^*_{E_{u(k)}v_\ell,\mathfrak{p}} z, \quad z \in \ell^2(\mathcal{Z}).$$

Next we show that (iii) implies (i). By equation (3.7), for all $z \in \ell^2(\mathcal{Z}),\ \zeta \in \mathcal{Z}$, equation (3.8) is equivalent to

$$\hat{z}(\zeta) = \widehat{\mathcal{S}}z(\zeta) = q \sum_{s\in\Omega} \hat{z}(\zeta + u(s)) \sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} \hat{v}(\zeta + u(k))\, \overline{\hat{v}_\ell(\zeta + u(k) + u(s))} \tag{3.10}$$

Finally, to obtain the equivalence, we will establish that (ii) implies (iii). For a fixed $\zeta_0 \in \mathcal{Z}$, define $z_1 \in \ell^2(\mathcal{Z})$ as follows:

$$\hat{z}_1(\zeta_0) = \begin{cases} 1, & \zeta = \zeta_0 \\ 0, & \zeta \neq \zeta_0. \end{cases}$$

Plugging z_1 and ζ_0 into equation (3.10) leads to

$$\begin{aligned} \widehat{z}_1(\zeta_0) &= q\widehat{z}_1(\zeta_0) \sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} |\widehat{v}_\ell(\zeta_0 + u(k))|^2 + q \sum_{s\in\Omega\setminus\{0\}} \widehat{z}_1(\zeta_0 + u(s)) \left(\sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} \hat{v}_\ell(\zeta_0 + u(k))\overline{\hat{v}_\ell(\zeta_0 + u(k) + u(s))} \right) \\ &= q\widehat{z}_1(\zeta_0) \sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} |\hat{v}_\ell(\zeta_0 + u(k))|^2. \end{aligned}$$

So we have $\displaystyle\sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} |\hat{v}_\ell(\zeta_0 + u(k))|^2 = q,\quad \zeta \in \mathcal{Z}$ and equation (3.10) can be reduced by

$$\sum_{s\in\Omega\setminus\{0\}} \hat{z}(\zeta + u(s)) \left(\sum_{\ell=0}^{L} \sum_{k\in\mathbb{N}_0} \hat{v}_\ell(\zeta + u(k))\overline{\hat{v}_\ell(\zeta + u(k) + u(s))} \right) = 0, \quad \zeta \in \mathcal{Z}. \tag{3.11}$$

For $\zeta_0 \in \mathcal{Z}$ and $m_0 \in \Omega\setminus\{0\}$, define $z_2 \in \ell^2(\mathcal{Z})$ as follows:

$$\widehat{z}(\zeta + u(s)) = \begin{cases} 1, & (\zeta, s) = (\zeta_0, s_0) \\ \\ 0, & (\zeta, s) \neq (\zeta_0, s_0). \end{cases}$$

Replacing z by z_2 in equation (3.11), we have

$$\sum_{\ell=0}^{L}\sum_{k\in\mathbb{N}_0}\widehat{v}_\ell(\zeta+u(k))\overline{\widehat{v}_\ell(\zeta+u(k)+u(s))}=0,\quad \zeta\in\mathcal{Z},\ s\in\Omega\backslash\{0\}.$$

This completes the proof of equation (3.9). □

The filter bank $\{v_\ell, 0\le\ell\le L\}$ in the above result is the generalization of the classical extension principles given by Ron and Shen [22].

Definition 3.4. Let $u, v_\ell\in\ell^2(\mathcal{Z}), 1\le\ell\le L$. A Parseval frame for $\ell^2(\mathcal{Z})$ of the form

$$\mathcal{W}_{k,m,\ell}=\{\mathcal{W}_{k,m}u\}_{k\in\mathbb{N}_0,m\in\mathbb{N}_0}\cup\{\mathcal{W}_{k,m}v_\ell\}_{k\in\mathbb{N}_0,m\in\mathbb{N}_0,1\le\ell\le L}\tag{3.12}$$

is called a first-stage discrete periodic wave packet frame for $\ell^2(\mathcal{Z})$.

If $\mathcal{W}_{k,m,\ell}$ is a first-stage discrete periodic wave packet frame for $\ell^2(\mathbb{N}_0)$, then, for each $z\in\ell^2(\mathcal{Z})$, the first-stage discrete periodic wave packet transform of $z(u(n))$ can be computed by

$$\{\langle z,\mathcal{W}_{k,m}u\rangle\}_{k\in\mathbb{N}_0,m\in\mathbb{N}_0}\cup\{\langle z,\mathcal{W}_{k,m}v_\ell\rangle\}_{k\in\mathbb{N}_0,m\in\mathbb{N}_0,1\le\ell\le L}.$$

Therefore, the reconstruction formula is as follows:

$$z=\sum_{k\in\mathbb{N}_o}\sum_{m\in\mathbb{N}_0}\langle z,\mathcal{W}_{k,m}u\rangle\mathcal{W}_{k,m}u+\sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\langle z,\mathcal{W}_{k,m}v_\ell\rangle\mathcal{W}_{k,m}v_\ell.$$

4 J^{th}-Stage Discrete Periodic Wave Packet Frames

In this section our main is to construct the J^{th}-stage discrete periodic wave packet frames for $\ell^2(\mathcal{Z})$ by using iteration of the filter sequence.

Definition 4.1. A sequence of vectors $u^1, v^1_\ell, u^2, v^2_\ell, ..., u^J, 1\le\ell\le L$ is a J^{th}-stage Parseval frame filter sequence, if for each $1\le j\le J$,

$$u^j, v^j_1, v^j_2, ..., v^j_L\in\ell^2(\mathcal{Z})$$

and for each $s\in\Omega, \zeta\in\mathcal{Z}$,

$$\hat{u^j}(\zeta)\overline{\hat{u^j}(\zeta+u(s))}+\sum_{\ell=1}^{L}\hat{v^j}_\ell(\zeta)\overline{\hat{v^j}_\ell(\zeta+u(s))}=q\delta(s),\quad 1\le j\le J,\tag{4.1}$$

Suppose that for $1\le j\le J$, $u^j, v^j_1, v^j_2, ..., v^j_L\in\ell^2(\mathcal{Z})$ form a J^{th}-stage Parseval frame filter sequence. Let $g^1_t=u^1$, $f^1_{\ell,k}=v^1_\ell, 1\le\ell\le L$ for a fixed $k\in\mathbb{N}_0$, g^j_ℓ and $f^j_{\ell,t}$ are defined by

$$g^j_k=\sum_{m\in\mathbb{N}_0}g^{j-1}_k(u(n)-\mathfrak{p}^{j-1}u(m))\chi_{\mathfrak{p}^{j-1}u(k)}(u(m))u^j(m),\quad n\in\mathbb{N}_0\tag{4.2}$$

and

$$f^j_{\ell,k}=\sum_{m\in\mathbb{N}_0}g^{j-1}_k(u(n)-\mathfrak{p}^{j-1}u(m))\chi_{\mathfrak{p}^{j-1}u(k)}(u(m))v^j_\ell(k),\ 1\le\ell\le L,\ n\in\mathbb{N}_0,\tag{4.3}$$

where $g^j_k, f^j_{1,k}, ..., f^j_{L,k}\in\ell^2(\mathbb{N}_0)$. We call $\left\{\mathcal{W}^j_{k,m}g^j_k\right\}_{k\in\mathbb{N}_0,k\in\mathbb{N}_0,1\le j\le J}$ $\left(or\left\{\mathcal{W}^j_{k,m}f^j_{\ell,k}\right\}_{k\in\mathbb{N}_0,m\in\mathbb{N}_0,\ 1\le\ell\le L,1\le j\le J}\right)$ the wave packet system associated with $g^j_k\big(f^j_{\ell,k}\big)$ where

$$\left\{\mathcal{W}^j_{k,m}g^j_k=\mathcal{T}_{\mathfrak{p}^j u(m)}M_{u(k)}g^j_k: k\in\mathbb{N}_0, m\in\mathbb{N}_0, 1\le j\le J\right\}\tag{4.4}$$

and

$$\{\mathcal{W}^j_{k,m}f^j_{\ell,k}=\mathcal{T}_{\mathfrak{p}^j u(m)}M_{u(k)}f^j_{\ell,k}: k\in\mathbb{N}_0, m\in\mathbb{N}_0, 1\le\ell\le L, 1\le j\le J\}.\tag{4.5}$$

Definition 4.2. Suppose that $u^j, v_1^j, v_2^j, ..., v_L^j \in \ell^2(\mathcal{Z})$ form a J^{th}-stage Parseval frame filter sequence, $1 \leq j \leq J$. For a fixed $k \in \mathbb{N}_0$, define g_ℓ^j and $f_{\ell,k}^j$ by equations (4.2) and (4.3), respectively. A Parseval frame for $\ell^2(\mathcal{Z})$ of the form

$$\{\mathcal{W}_{k,m}^j f_{\ell,k}^j\}_{k\in\mathbb{N}_0, m\in\mathbb{N}_0, 1\leq \ell\leq L, 1\leq j\leq J} \cup \{\mathcal{W}_{k,m}^J g_k^J\}_{k\in\mathbb{N}_0, m\in\mathbb{N}_0} \tag{4.6}$$

is called a J^{th}-stage discrete periodic wave packet frame for $\ell^2(\mathcal{Z})$.

Our main goal here is to show that $f_{\ell,k}^1, f_{\ell,k}^2, ..., f_{\ell,k}^J, g_k^J \in \ell^2(\mathcal{Z}), 1 \leq \ell \leq L$, generate a J^{th}-stage discrete periodic wave packet frame for $\ell^2(\mathcal{Z})$. The key step is contained in the next theorem. For that we first state and prove the following lemma.

Lemma 4.3. For a fixed $k \in \mathbb{N}_0$, define g_ℓ^j and $f_{\ell,k}^j$ by expansions (4.2) and (4.3), respectively. Then

$$\widehat{g_\ell^j}(s) = \widehat{g_\ell^{j-1}}(s)\hat{u^j}(u(s) - u(k)), \quad s, k \in \mathbb{N}_0, \tag{4.7}$$

and

$$\widehat{f_{\ell,k}^j}(s) = \widehat{g_\ell^{j-1}}(s)\widehat{g_\ell^{j-1}}(s)\widehat{v_\ell^j}(u(s) - u(k)), \quad s, k \in \mathbb{N}_0, \quad 1 \leq \ell \leq L. \tag{4.8}$$

Proof . We first introduce the upsampling operator $\uparrow \mathfrak{p} : \ell^2(\mathcal{Z}) \to \ell^2(\mathcal{Z})$ defined by

$$(z \uparrow \mathfrak{p})(s) := \begin{cases} z(\mathfrak{p}^{-1}u(m)), & m \in \mathbb{N}_0; \\ 0, & \text{otherwise.} \end{cases}$$

It is convenient to use the notion $z(\mathfrak{p}^{-1})$ for $z \uparrow \mathfrak{p}$. It can be checked that

$$\left(u^j \uparrow \mathfrak{p}^{j-1}\right)(m) := \begin{cases} u^j(\mathfrak{p}^{-j+1}m), & m \in \mathbb{N}_0 \ \ and \ \ \mathfrak{p}^{j-1}|m; \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (4.2) that, for $n \in \mathbb{N}_0$,

$$\begin{aligned} g_k^j(n) &= \sum_{m\in\mathbb{N}_0} g_k^{j-1}(u(n) - \mathfrak{p}^{j-1}u(m))\chi_{\mathfrak{p}^{j-1}u(k)}(u(m))u^j(m) \\ &= \sum_{m\in\mathbb{N}_0} g_k^{j-1}(u(n) - u(m))\chi_{\mathfrak{p}^{j-1}u(k)}(u(m))(u^j \uparrow \mathfrak{p}^{j-1})(m) \\ &= g_k^{j-1} * [M_{-u(k)}(u^j \uparrow \mathfrak{p}^{j-1})](n). \end{aligned}$$

Since $u^j \in \ell^2(\mathcal{Z})$ and

$$\begin{aligned} (u^j \uparrow \mathfrak{p}^{j-1})\hat{}\,(s) &= \sum_{m\in\mathbb{N}_0} (u^j \uparrow \mathfrak{p}^{j-1})(k)\chi_{\mathfrak{p}^{j-1}u(m)}(u(s)) \\ &= \sum_{m\in\mathbb{N}_0} u^j(m)\chi_{\mathfrak{p}^{j-1}u(m)}(u(s)) = \widehat{u^j}(m), \end{aligned}$$

it follows $[M_{-u(k)}(u^j \uparrow \mathfrak{p}^{j-1})]\hat{}\,(s) = T_{u(k)}\widehat{u^j}(s) = \widehat{u^j}(u(s) - u(k))$. Therefore, we have

$$\widehat{g_k^j}(s) = \widehat{g_k^{j-1}}(s)[M_{-u(k)}(u^j \uparrow \mathfrak{p}^{j-1})]\hat{}\,(s) = \widehat{g_k^{j-1}}(s)\widehat{u^j}(u(s) - u(k)).$$

Similarly, for $s, k \in \mathbb{N}_0$,

$$(v_\ell^j \uparrow \mathfrak{p}^{j-1})\hat{}\,(s) = \widehat{v_\ell^j}(u(s) - u(k)), \quad \text{and} \quad \widehat{f_{\ell,k}^j}(s) = \widehat{g_k^{j-1}}(s)\hat{v_\ell^j}(u(s) - u(k)), \quad 1 \leq \ell \leq L.$$

This completes the proof of the lemma. □

Theorem 4.4. Suppose $u^j, v_1^j, v_2^j, ..., v_L^j \in \ell^2(\mathcal{Z})$ form a J^{th}-stage Parseval frame filter sequence, $1 \leq j \leq J$. For $1 \leq j \leq J$ and $k \in \mathbb{N}_0$, define g_ℓ^j and $f_{\ell,k}^j$ by the equations (4.2) and (4.3), respectively, where $g_\ell^j, f_{1,k}^j, ..., f_{L,k}^j \in \ell^2(\mathcal{Z})$. Then, for all $z \in \ell^2(\mathcal{Z})$,

$$\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} \langle z, \mathcal{W}_{k,m}^{j-1}g_k^{j-1}\langle \mathcal{W}_{k,m}^{j-1}g_k^{j-1}\rangle\mathcal{W}_{k,m}^{j-1}g_k^{j-1} = \sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\langle z, \mathcal{W}_{k,m}^j g_k^j\rangle\mathcal{W}_{k,m}^j g_k^j + \sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\langle z, \mathcal{W}_{k,m}^j f_{\ell,k}^j\rangle\mathcal{W}_{k,m}^j f_{\ell,k}^j, \tag{4.9}$$

Proof . By equation (3.4) and (4.7)

$$\Big[\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\langle z,\mathcal{W}^j_{k,m}g^j_k\rangle\mathcal{W}^j_{k,m}g^j_k\Big]^{\widehat{}}(\zeta) = q\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{\hat{g}^j_k(\zeta+u(s)+u(t))}\hat{g}^j_k(\zeta+u(k))$$
$$= q^j\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{g^{j-1}_k(\zeta+u(s)+u(k))}\widehat{g^{j-1}_k}(\zeta+u(k))\overline{u^j(\zeta+u(s))}\hat{u^j}(\zeta).$$

In a similar manner, by equation (3.4) and (4.8)

$$\Big[\sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\langle z,\mathcal{W}^j_{k,m}f^j_{\ell,k}\rangle\mathcal{W}^j_{k,m}f^j_{\ell,k}\Big]^{\widehat{}}(\zeta)$$
$$= q^j\sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_)}\sum_{s\in\Omega}\hat{z}(\zeta+u(s))\overline{\widehat{g^{j-1}_k}(\zeta+u(s)+u(k))}\widehat{g^{j-1}_k}(\zeta+u(k))\overline{\widehat{v^j_\ell}(\zeta+u(s))}\widehat{v^j_\ell}(\zeta)$$

Therefore

$$\blacksquare := \Big[\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\langle z,\mathcal{W}^j_{k,m}g^j_k\rangle\mathcal{W}^j_{k,m}g^j_k+\sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0}\langle z,\mathcal{W}^j_{k,m}f^j_{\ell,k}\rangle\mathcal{W}^j_{k,m}f^j_{\ell,k}\Big]^{\widehat{}}(\zeta)$$
$$= q^j\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega}\widehat{z}(\zeta+u(s))g^{j-1}_k(\zeta+\widehat{u(s)}+u(k))\widehat{g^{j-1}_k}(\zeta+u(k))\times\Big[\overline{\widehat{u^j}(\zeta+u(s))}\widehat{u^j}(\zeta)+\sum_{\ell=1}^{L}\overline{\widehat{v^j_\ell}(\zeta+u(s))}\widehat{v^j_\ell}(\zeta)\Big].$$

Using the fact that all the sequences u^j and v^j are periodic $q^{j-1}N$ and equation (4.1), we have

$$\blacksquare := q^j\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega}\sum_{n=1}^{q^j-1}\hat{z}(\zeta+u(s)+u(n))\overline{\hat{g}^{j-1}_\ell(\zeta+u(s)+u(n)+u(k))}$$
$$\times\widehat{g^{j-1}_k}(\zeta+u(k))\Big[\overline{\widehat{u^j}(\zeta+u(s)+u(n))}\widehat{u^j(\zeta)}+\sum_{\ell=1}^{L}\overline{\widehat{v^j_\ell}(\zeta+u(s)+u(n))}\widehat{v^j_\ell}(\zeta)\Big]$$
$$= q^j\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega_{M_{j-1}}}\sum_{n=1}^{q^j-1}\hat{z}(\zeta+u(s)+u(n))\overline{g^{j-1}_k(\zeta+u(s)+u(n)+u(k))}$$
$$\times\widehat{g^{j-1}_k}(\zeta+u(k))\Big[\overline{\widehat{u^j}(\zeta+u(n))}\widehat{u^j}(\zeta)+\sum_{\ell=1}^{L}\overline{\widehat{v}^j_\ell(\zeta+u(n))}\widehat{v^j_\ell}(\zeta)\Big]$$
$$= q^j\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega}\sum_{n=1}^{q^j-1}\widehat{z}(\zeta+u(s)+u(n))\overline{\widehat{g^j_\ell}(\zeta+u(s)+u(n)+u(k))}\times\widehat{g^{j-1}_k}(\zeta+u(k))\,[q\delta(n)]$$
$$= q^{j-1}\sum_{k\in\mathbb{N}_0}\sum_{s\in\Omega}\widehat{z}(\zeta+u(s))\overline{\widehat{g}^{j-1}_k(\zeta+u(s)+u(k))}\widehat{g^{j-1}_k}(\zeta+u(k))$$
$$= \Big[\sum_{k\in\mathbb{N}_0}\sum_{m=1}^{q^j-1}\langle z,\mathcal{W}^{j-1}_{k,m}g^{j-1}_k\rangle\mathcal{W}^{j-1}_{k,m}g^{j-1}_k\Big](\zeta).$$

By applying the inversion DFT to $\blacksquare$, equation (4.9) can be derived. This completes the proof of the theorem. □

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