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Solvability of the infinite system of Hadamard-type boundary value problem in the double sequence space

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Abstract

In this work, we are interested in representing the solution of Hadamard type fractional differential equation by introducing the concept of double sequence space $2^c(\Delta)$. After that, we construct the Hausdorff measure of non-compactness on the space $2^c(\Delta)$. Furthermore, we see the existence of a solution of Hadamard-type fractional differential equation on the space $2^c(\Delta)$. After that, we demonstrate an example to see the applicability of our results.

Keywords: Measure of noncompactness (MNC), Fixed point theorem ($\bar{F}\bar{P}\bar{T}$), Hadamard type fractional differential equation ($\mathfrak{H}T\mathbb{F}DE$), Double sequence space $\mathtt{D}\mathcal{SS}$
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1 Introduction

Fractional differential equations involving fractional order derivatives have gained a lot of importance in recent years because of their variety of practical applications in different fields of science and engineering. Fixed point theory and MNC are widely used in solving different types of differential and integral equations, see [10, 11, 12, 13, 14, 16, 20, 30]. Moreover, MNC was first introduced by Kuratowski [25] in 1930. After that, G. Darbo [9] generalized Schauder's fixed point theorem with the help of Kuratowski's MNC.

In this paper, we want to establish the solvability of the Hadamard-type fractional differential equation:

$$H_{\mathrm{D}^{\mu^*}_{a_1^+}}\tilde{s}(t) + \mathbb{Y}(t) = 0,$$

$$\tilde{s}(a_1) = \tilde{s}^{/}(a_1) = 0, \qquad \tilde{s}(a_2) = \int_{a_1}^{a_2} \hat{g}(t)\tilde{s}(t)\frac{dt}{t},$$

where $a_1,\ a_2,\quad \mu^* \in \Re^+$ with $a_1 < a_2 < +\infty$, $2 < \mu^* < 3$, $H_{\mathrm{D}^{\mu^*}_{a^+}}$ is the Hadamard fractional derivative of order μ^*, and $\hat{g} : [a_1, a_2] \to \Re^+$ with $\hat{g}(t) \neq 0$, $t \in [a_1, a_2]$.

The theory of fractional differential equations deals with many scientific systems such as mathematical modelling systems, in the fields of biology, chemistry, physics, polymer rheology, economy, applied science etc. To see work on this follow the paper [17, 19, 21, 33].

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Now, here we introduce the Double sequence space whose initial work was established by Bromwich in 1965 [8]. Recently, Zeltser [36] studied both the theory of topological double sequence space and the theory of summability of double sequence space. Patterson and Savas [32] introduce the double lacunary statistical convergence. There are concepts of convergence of double sequence space that have been extended by several authors such as Pringsheim's sense, statistically bounded, regularly statistically null etc. To see recent work on convergence of double sequence space follow [15, 24, 31, 34].

2 Preliminaries

Now here we have recalled the definition of MNC which was introduced by Kuratowski and it has a very substantial role in the case of infinite dimensional normed space or metric space. In 1955, Darbo [9] introduce $\bar{F}\bar{P}\bar{T}$ which is the generalization form of Schauder $\bar{F}\bar{P}\bar{T}$ and Banach contraction principle. There are several types of well-known MNC. First we represent one of the substantial MNC is the Hausdorff MNC which is presented as:

For a bounded subset $\acute{\mathbb{H}}$ of a metric space $\ddot{F}$,

$$\chi(\acute{\mathbb{H}}) = \inf\{\grave{\epsilon} > 0 : \acute{\mathbb{H}} \text{ has finite } \grave{\epsilon} - \text{net in } \ddot{F}\}.$$

Another important MNC is Kuratowski, which is presented as (see [25]):

$$\gamma^*(\acute{\mathbb{H}}) = \inf\{\bar{\kappa}_1 > 0 | \acute{\mathbb{H}} = \bigcup_{l=1}^{m} \acute{\mathbb{H}}_l, diam(\acute{\mathbb{H}}_1) \leq \bar{\kappa}_1 \text{ for } 1 \leq l \leq m < \infty\},$$

where, $diam(\acute{\mathbb{H}}_l)$ indicates the diameter of the set $\acute{\mathbb{H}}_l$, which is given by

$$diam(\acute{\mathbb{H}}_l) = \sup\{d(s_1, s_2) : s_1, s_2 \in \acute{\mathbb{H}}_l\}.$$

The following definition will be used in the sequel.

Definition 2.1. [5] MNC in $\vec{\mathtt{C}}$ is a function $\vec{\mathcal{S}} : \mathfrak{K}_{\vec{\mathtt{C}}} \to \Re^+$ which satisfied the given conditions,

(i) $\vec{\mathcal{S}}\left(\dot{\mathcal{Z}}\right) = 0$ for relatively compact subset of $\vec{\mathtt{C}}$.

(ii) $\ker \vec{\mathcal{S}} = \left\{\dot{\mathcal{Z}} \in \mathfrak{K}_{\vec{\mathtt{C}}} : \vec{\mathcal{S}}\left(\dot{\mathcal{Z}}\right) = 0\right\} \neq \phi$ and $\ker \vec{\mathcal{S}} \subset \mathfrak{K}_{\vec{\mathtt{C}}}$.

(iii) $\dot{\mathcal{Z}} \subseteq \dot{\mathcal{Z}}_1 \implies \vec{\mathcal{S}}\left(\dot{\mathcal{Z}}\right) \leq \vec{\mathcal{S}}\left(\dot{\mathcal{Z}}_1\right)$.

(iv) $\vec{\mathcal{S}}\left(\bar{\dot{Z}}\right) = \vec{\mathcal{S}}\left(\dot{\mathcal{Z}}\right)$.

(v) $\vec{\mathcal{S}}\left(Conv\dot{\mathcal{Z}}\right) = \vec{\mathcal{S}}\left(\dot{\mathcal{Z}}\right)$.

(vi) $\vec{\mathcal{S}}\left(\mathbb{K}\dot{\mathcal{Z}} + \left(1 - \mathbb{K}\right)\dot{\mathcal{Z}}_1\right) \leq \mathbb{K}\,\vec{\mathcal{S}}\left(\dot{\mathcal{Z}}\right) + \left(1 - \mathbb{K}\right)\vec{\mathcal{S}}\left(\dot{\mathcal{Z}}_1\right)$ for $\mathbb{K} \in [0, 1]$.

(vii) if $\dot{\mathcal{Z}}_{\hat{h}} \in \mathfrak{K}_{\vec{\mathtt{C}}}$, $\dot{\mathcal{Z}}_{\hat{h}} = \dot{\mathcal{Z}}_{\hat{h}}$, $\dot{\mathcal{Z}}_{\hat{h}+1} \subset \dot{\mathcal{Z}}_{\hat{h}}$ for $\hat{h} = 1, 2, 3, 4, ...$ and $\lim_{\hat{h}\to\infty} \vec{\mathcal{S}}\left(\dot{\mathcal{Z}}_{\hat{h}}\right) = 0$ then $\bigcap_{\hat{h}=1}^{\infty} \dot{\mathcal{Z}}_{\hat{h}} \neq \phi$.

The subfamily $ker\vec{\mathcal{S}}$, defined by (ii), represents kernel of measure $\vec{\mathcal{S}}$ and since $\vec{\mathcal{S}}(\dot{\mathcal{S}}_\infty) \leq \vec{\mathcal{S}}(\dot{\mathcal{Z}}_{\hat{h}})$ for any $\hat{h}$, we can say that $\vec{\mathcal{S}}(\dot{\mathcal{Z}}_\infty) = 0$. Then $\dot{\mathcal{Z}}_\infty = \bigcap_{\hat{h}=1}^{\infty} \dot{\mathcal{Z}}_{\hat{h}} \in ker\vec{\mathcal{S}}$.

Double sequence space($2^c(\Delta)$):

Let $\bar{\bar{T}} = (\hat{w}_{nk})$ define the double sequence. The zero single sequence will be represented by $\bar{\vartheta} = (\vartheta, \vartheta, ...)$ and the zero double sequence will represented by $2\bar{\vartheta}$. A double sequence $\bar{\bar{T}} = (\hat{w}_{nk})$ convergence in Pringsheim's sense or P$-$convergent to Z if $\lim_{n,k\to\infty}(\hat{w}_{nk}) = Z$ (denoted by $\mathrm{P} - \lim \bar{\bar{T}} = Z$). Some works on double sequence spaces were studied by Hardy [18], Moricz [27], Moricz and Rhoades [28].

Now here we represent the double sequence space $2^c(\Delta)$ over the normed space $(\mathtt{X}, ||.||)$:

$$2^c(\Delta) = \{< \hat{w}_{nk} >\in 2^\omega :< \Delta\hat{w}_{nk} >\in 2^c(\Delta)\},$$

where $\Delta\hat{w}_{nk} = \hat{w}_{nk} - \hat{w}_{n+1,k} - \hat{w}_{n,k+1} + \hat{w}_{n+1,k+1}$, for all n,k $\in \mathbf{N}$. If $(\mathtt{X}, ||.||)$ is normed linear space then $2^c(\Delta)$ also normed linear space with the norm as :

$$||\bar{\bar{T}}||_\Delta = \sup_n ||\hat{w}_{n1}|| + \sup_k ||\hat{w}_{1k}|| + \sup_{n,k} ||\Delta\hat{w}_{nk}||. \tag{2.1}$$

2.1 Some examples of double sequence space

Example 2.2 (Double sequence space defined by a modulus function [32]). Let $\mathbf{S}''$ be the set of all double sequences of complex numbers. Let $\mathrm{A} = (\hat{d}_{u,v,n,k})$ be a non negative four dimensional matrix of real entries with $\sup_{u,v} \sum_{n,k,=0,0}^{\infty,\infty} < \infty$, and we consider f as a modulus function, then

$$Q_0''(\mathrm{A},\mathrm{f}) = \left\{ \bar{\bar{T}} \in \mathbf{S}'' : \mathrm{P} - \lim_{u,v} \sum_{n,k=0,0}^{\infty,\infty} \hat{d}_{u,v,n,k} \mathrm{f}(|\hat{w}_{n,k}|) = 0 \right\}, \tag{2.2}$$

$$Q''(\mathrm{A},\mathrm{f}) = \left\{ \bar{\bar{T}} \in \mathbf{S}'' : \mathrm{P} - \lim_{u,v} \sum_{n,k=0,0}^{\infty,\infty} \hat{d}_{u,v,n,k} \mathrm{f}\left(|\hat{w}_{n,k} - Z|\right) = 0, \quad \text{for some } Z \right\}, \tag{2.3}$$

$$Q_\infty''(\mathrm{A},\mathrm{f}) = \left\{ \bar{\bar{T}} \in \mathbf{S}'' : \sup_{u,v} \sum_{n,k=0,0}^{\infty,\infty} \hat{d}_{u,v,n,k} \mathrm{f}\left(|\hat{w}_{n,k}|\right) < \infty \right\}. \tag{2.4}$$

Example 2.3 (Double sequence space $\mathfrak{l}_2(\mathsf{p},\mathrm{f},\mathrm{q},\mathrm{s})$ [6]). Let $\mathfrak{X}$ be a complex linear space with the zero element $\bar{\vartheta}$ and $\mathfrak{X} = (\mathfrak{X},\mathrm{q})$ be a semi- normed with the seminorm q. Let us denote $\mathbf{w}^2(\mathfrak{X})$ as the linear space of all double sequences $\bar{\bar{T}} = (\hat{w}_{nk})$ with $\hat{w}_{nk} \in \mathfrak{X}$. Let $\mathsf{p} = (\mathsf{p}_{nk})$ be a double sequence of strictly positive real numbers and f be a modulus function. Thus the double sequence space $\mathfrak{l}_2(\mathsf{p},\mathrm{f},\mathrm{q},\mathrm{s})$ can be expressed as :

$$\mathfrak{l}_2(\mathsf{p},\mathrm{f},\mathrm{q},\mathrm{s}) = \left\{ \bar{\bar{T}} \in \mathbf{w}^2(\mathfrak{X}) : \sum_{n,k=1}^{\infty} (nk)^{-\mathrm{s}} \left[\mathrm{f}(\mathrm{q}(\hat{w}_{nk}))\right]^{\mathsf{p}_{nk}} < \infty, \quad \mathrm{s} \geq 0 \right\}. \tag{2.5}$$

Theorem 2.4. [26] Suppose that $\complement$ is considered as a bounded subset of the Banach space $2^c(\Delta)$. Here it is defined the projector $\mathbb{P}_{nk} : 2^c(\Delta) \to 2^c(\Delta)$ as

$$\mathbb{P}_{nk}(\ddot{\eth}) = \left(\ddot{\eth}_{n1}, \ddot{\eth}_{n2}, \ldots.., \ddot{\eth}_{nk}, \ddot{\eth}_{1k}, \ddot{\eth}_{2k}, \ldots..\ddot{\eth}_{nk}, 2\bar{\vartheta}, 2\bar{\vartheta}, \ldots..\right),$$

where $\ddot{\eth} =< \ddot{\eth}_{nk} >\in 2^c(\Delta)$ for all n,k $\in \mathbf{N}$. Then the Hasudorff MNC is defined as

$$\chi(\complement) = \lim_{n,k\to\infty} \left\{ \sup_{\ddot{\eth}\in\complement} \left\{ \sup_n ||\ddot{\eth}_{n1}|| + \sup_k ||\ddot{\eth}_{1k}|| + \sup_{n,k} ||\Delta\ddot{\eth}_{nk}|| \right\} \right\}. \tag{2.6}$$

Definition 2.5. [2, 23] Let $\delta > 0$. Then the Hadamard fractional left integral of order $\breve{\mu} > 0$ of a function $\bar{\zeta} : [\delta,\infty) \to \Re$ is defined as

$$H_{I_{\delta+}^{\breve{\mu}}} \bar{\zeta}(\iota) = \frac{1}{\Gamma(\breve{\mu})} \int_\delta^\iota \left(ln\frac{\iota}{\breve{F}}\right) \bar{\zeta}(\breve{F}) \frac{d\breve{F}}{\breve{F}}.$$

Definition 2.6. [2, 23] Let $\eta_* > 0$. Then the Hadamard fractional left derivative of a function $\bar{\zeta}_1 : [\eta_*,\infty) \to \Re$, $\hat{\iota}^{n-1}\bar{\zeta}_1^{\,n-1}(\ddot{\iota}) \in [\eta_*,\infty)$, n $\in \mathbb{N}$ of order $\breve{\mu} \in (n-1,n)$ is stated as

$$H_{D_{\eta_*^+}^{\breve{\mu}}} \bar{\zeta}_1(\ddot{\iota}) = \frac{1}{\Gamma(n-\breve{\mu})} \left(\ddot{\iota}\frac{d}{d\ddot{\iota}}\right)^n \int_{\eta_*}^{\ddot{\iota}} \left(ln\frac{\ddot{\iota}}{\breve{F}}\right)^{n-\breve{\mu}-1} \bar{\zeta}_1(\breve{F}) \frac{d\breve{F}}{\breve{F}}.$$

Theorem 2.7 (Shauder). [1] Assume that $\ddot{\mathfrak{F}}$ be a nonempty, closed, convex and bounded subset (NCCB) of a Banach Space $\acute{\mathbb{W}}$. Then every compact, continuous mapping $\mathfrak{O} : \ddot{\mathfrak{F}} \to \mathfrak{O}$ has at least one $\bar{F}\bar{P}$.

Theorem 2.8. (Darbo [9]) Assume that $\ddot{\mathfrak{F}}$ is a NCCB of a Banach Space $\acute{\mathbb{W}}$ and let $\mathfrak{O} : \ddot{\mathfrak{F}} \to \ddot{\mathfrak{F}}$ be a continuous mapping. Let us consider a constant $\tilde{e_1} \in [0,1)$ such that

$$\tilde{C}(\ddot{\mathfrak{F}}\omega) \leq \tilde{e_1}\, \tilde{C}(\omega), \omega \subseteq \ddot{\mathfrak{F}}.$$

Then $\mathfrak{O}$ has at least one fixed point in $\ddot{\mathfrak{F}}$.

Lemma 2.9 ([35]). Considering $\breve{\nu}^* \in (2,3)$, $\bar{\mathcal{Y}} \in L[a,b]$. Then the Hadamard-type fractional boundary value problem

$$H_{D_{a+}^{\breve{\nu}^*}}\hat{\zeta}(\ddot{\sigma}) + \bar{\mathcal{Y}}(\ddot{\sigma}) = 0, \qquad \ddot{\sigma} \in (a,b)$$

$$\hat{\zeta}(a) = \hat{\zeta}'(a) = 0, \qquad \hat{\zeta}(b) = \int_a^b \mathbf{r}(\ddot{\sigma})\hat{\zeta}(\ddot{\sigma})\frac{d\ddot{\sigma}}{\ddot{\sigma}},$$

has a solution

$$\hat{\zeta}(\ddot{\sigma}) = \int_a^b \bar{\mathfrak{H}}(\ddot{\sigma}, \vec{F}) + \bar{\mathcal{Y}}(\vec{F})\frac{d\vec{F}}{\vec{F}}, \qquad \ddot{\sigma} \in [a,b],$$

where

$$\bar{\mathfrak{H}}(\ddot{\sigma}, \vec{F}) = \hat{\mathsf{G}}(\ddot{\sigma}, \vec{F}) + \frac{(\ln\frac{\ddot{\sigma}}{a})^{\breve{\nu}^*-1}}{\bar{\Re}}\int_a^b \mathbf{r}(\ddot{\sigma})\hat{\mathsf{G}}(\ddot{\sigma}, \vec{F})\frac{d\ddot{\sigma}}{\ddot{\sigma}},$$

$$\hat{\mathsf{G}}(\ddot{\sigma}, \vec{F}) = \frac{1}{\Gamma(\breve{\nu}^*)(ln\frac{b}{a})^{\breve{\nu}^*-1}}\begin{Bmatrix} (\ln\frac{\ddot{\sigma}}{a})^{\breve{\nu}^*-1}(\ln\frac{b}{\vec{F}})^{\breve{\nu}^*-1} - (\ln\frac{\ddot{\sigma}}{\vec{F}})^{\breve{\nu}^*-1}(\ln\frac{b}{a})^{\breve{\nu}^*-1}, & a \le \vec{F} \le \ddot{\sigma} \le b, \\ (\ln\frac{\ddot{\sigma}}{a})^{\breve{\nu}^*-1}(\ln\frac{b}{\vec{F}})^{\breve{\nu}^*-1}, & a \le \ddot{\sigma} \le \vec{F} \le b \end{Bmatrix},$$

$$\bar{\Re} = \left(\ln\frac{b}{a}\right)^{\breve{\nu}^*-1} - \int_a^b \mathbf{r}(\ddot{\sigma})(\ln\frac{\ddot{\sigma}}{a})^{\breve{\nu}^*-1}\frac{d\ddot{\sigma}}{\ddot{\sigma}}.$$

3 Solution in Sequence space $2^c(\Delta)$

In this section, we are going to elaborate the solution of following $\mathfrak{H}T\mathbb{F}DE$ of boundary value problem

$$H_{D_{\eta+}^{\breve{\nu}^*}}\acute{\chi}_j(\tilde{\Omega}) + \mathbb{Y}_j(\tilde{\Omega}) = 0, \qquad \tilde{\Omega}(1,2) \tag{3.1}$$

$$\acute{\chi}_j(1) = \acute{\chi}_j'(1) = 0, \ \acute{\chi}_j(2) = \int_1^2 \mathbf{r}(\tilde{\Omega})\acute{\chi}_j(\tilde{\Omega})\frac{d\tilde{\Omega}}{\tilde{\Omega}}; \qquad j = 1,2,3...$$

Let us denote $\{\mathbb{Y}_j\}_{j=1}^{\infty} = \mathbb{Y}$ and $\{\acute{\chi}_j\}_{j=1}^{\infty} = \acute{\chi}$, then the infinite system equations (3.1) can be written as

$$H_{D_{\eta+}^{\breve{\nu}^*}}\acute{\chi}(\tilde{\Omega}) + \mathbb{Y}(\tilde{\Omega}) = 0, \qquad \tilde{\Omega}(1,2) \tag{3.2}$$

$$\acute{\chi}_j(1) = \acute{\chi}\prime_j(1) = 0, \ \acute{\chi}_j(2) = \int_1^2 \mathbf{r}(\tilde{\Omega})\acute{\chi}(\tilde{\Omega})\frac{d\tilde{\Omega}}{\tilde{\Omega}}; \qquad j = 1,2,3...$$

To check the existence of the solutions we consider the following assumptions:

(d1) Let $\mathbb{Y} = \mathbb{Y}_{nk} \in (\mathbb{J} \times \Re^{\infty}, \Re)$, the continuous operator $\mathbb{Y} : \mathbb{J} \times 2^c(\Delta) \to 2^c(\Delta)$ is presented as

$$\mathbb{Y}\acute{\chi}(\tilde{\Omega}) = \mathbb{Y}(\tilde{\Omega}, \acute{\chi}(\tilde{\Omega})) = < \mathbb{Y}_{nk}(\tilde{\Omega}, \acute{\chi}(\tilde{\Omega}) >,$$

where $\mathbb{J} = [1,2]$ and the family of functions $\left\{(\mathbb{Y}\acute{\chi})(\tilde{\Omega})\right\}_{\tilde{\Omega}\in\mathbb{J}}$ is equicontinuous at every point of $2^c(\Delta)$.

(d2) For each $\grave{\hbar} \in \mathbb{J}$ and $\ddot{\eth} \in 2^c(\Delta)$ the following inequilities hold:

$$\left|\mathbb{Y}_{nk}(\grave{\hbar}, \ddot{\eth}(\grave{\hbar}))\right| \le \left|\ddot{\wp}_{nk}(\grave{\hbar})\right| + \left|\psi_{nk}(\grave{\hbar})\right|\left|\ddot{\eth}_{nk}(\grave{\hbar})\right|,$$

$$\left|\Delta\mathbb{Y}_{nk}(\grave{\hbar}, \ddot{\eth}(\grave{\hbar}))\right| \le \left|\ddot{\wp}_{nk}(\grave{\hbar})\right| + \left|\psi_{nk}(\grave{\hbar})\right|\left|\Delta\ddot{\eth}_{nk}(\grave{\hbar})\right|,$$

where $\ddot{\wp}_{nk}(\grave{\hbar})$, $\psi_{nk}(\grave{\hbar})$ are real continuous functions such that $\ddot{\wp}_{nk}(\grave{\hbar})$, $\psi_{nk}(\grave{\hbar})$ are uniformly converging and equibounded on $\mathbb{J}$. Also we denote

$$\sup_{\grave{\hbar}\in\mathbb{J}}\sup_{n,k}|\ddot{\wp}_{nk}(\grave{\hbar})| = \Upsilon_1, \qquad \sup_{\grave{\hbar}\in\mathbb{J}}\sup_{n,k}|\psi_{nk}(\grave{\hbar})| = \Upsilon_2.$$

Theorem 3.1. If the system of equations (3.1) is satisfied the assumption $(d1)-(d2)$ with the condition $\bar{\bar{B}}\Upsilon_2 \leq 1$, where $\bar{\bar{B}} = \acute{\acute{U}}\log 2$ and $\acute{\acute{U}} = \max_{\tilde{\Omega}\in\mathbb{J}} \Im(\tilde{\Omega}, \bar{F})$ then given system (3.1) has at least one solution $\acute{\chi} =< \acute{\chi}_{nk} >\in \mathbf{C}(\mathbb{J}, 2^c(\Delta))$.

Proof . Let us take $\acute{\chi} = (\acute{\chi}_{nk})$, which is the double sequence function that satisfies the boundary value problem and suppose that each $(\acute{\chi}_{nk})$ be continuous for all n,k in $\mathbb{N}$ on $\mathbb{J}$. First we define the operator $\bar{\Theta} : \mathbf{C}(\mathbb{J}, 2^c(\Delta)) \longrightarrow \mathbf{C}(\mathbb{J}, 2^c(\Delta))$ by

$$(\bar{\Theta}\acute{\chi})(\tilde{\Omega}) = \int_1^2 \Im(\tilde{\Omega}, \bar{F})\mathbb{Y}(\bar{F}, \acute{\chi}(\bar{F}))\frac{d\bar{F}}{\bar{F}}.$$

Our aim to see $\bar{\Theta}$ is bounded. Now, applying (2.1) and our assumption, we have for a fixed arbitrary $\tilde{\Omega} \in \mathbb{J}$,

$$\begin{aligned}
\| (\bar{\Theta}\acute{\chi})(\tilde{\Omega}) \|_\Delta = & \| \int_1^2 \Im(\tilde{\Omega}, \bar{F})\mathbb{Y}(\bar{F},)\frac{d\bar{F}}{\bar{F}} \|_\Delta \\
&= \sup_n \| \int_1^2 \Im(\tilde{\Omega}, \bar{F})\mathbb{Y}_{n1}(\bar{F}, \acute{\chi}(\bar{F}))\frac{d\bar{F}}{\bar{F}} \| + \sup_k \| \int_1^2 \Im(\tilde{\Omega}, \bar{F})\mathbb{Y}_{1k}(\bar{F}, \acute{\chi}(\bar{F}))\frac{d\bar{F}}{\bar{F}} \| \\
&\quad + \sup_{n,k} \| \int_1^2 \Im(\tilde{\Omega}, \bar{F})\Delta\mathbb{Y}_{n,k}(\bar{F}, \acute{\chi}(\bar{F}))\frac{d\bar{F}}{\bar{F}} \| \\
&\leq \int_1^2 \Im(\tilde{\Omega}, \bar{F})[\sup_n |\mathbb{Y}_{n1}(\bar{F}, \acute{\chi}(\bar{F}))| + \sup_k |\mathbb{Y}_{1k}(\bar{F}, \acute{\chi}(\bar{F}))| + \sup_{n,k} |\Delta\mathbb{Y}_{nk}(\bar{F}, \acute{\chi}(\bar{F}))|]\frac{d\bar{F}}{\bar{F}} \\
&\leq \int_1^2 \sup_{\tilde{\Omega}\in\mathbb{J}} \Im(\tilde{\Omega}, \bar{F})[| \sup_n \{|\ddot{\wp}_{n1}(\bar{F})| + |\Psi_{n1}(\bar{F})||\acute{\chi}_{n1}(\bar{F})|\} + \sup_k \{|\ddot{\wp}_{1k}(\bar{F})| + |\Psi_{1k}(\bar{F})||\acute{\chi}_{1k}(\bar{F})|\} \\
&\quad + \sup_{n,k} \{|\ddot{\wp}_{nk}(\bar{F})| + |\Psi_{nk}(\bar{F})||\Delta\acute{\chi}_{1k}(\bar{F})|\}]\frac{d\bar{F}}{\bar{F}} \\
&\leq \acute{\acute{U}} \int_1^2 \sup_{\tilde{\Omega}\in\mathbb{J}} \big[\sup_n |\ddot{\wp}_{n1}(\tilde{\Omega})| + \sup_n |\Psi_{n1}(\tilde{\Omega})| \sup_n |\acute{\chi}_{n1}(\tilde{\Omega})| + \sup_k |\ddot{\wp}_{1k}(\tilde{\Omega})| + \sup_n |\Psi_{1k}(\tilde{\Omega})| \sup_k |\acute{\chi}_{1k}(\tilde{\Omega})| \\
&\quad + \sup_{n,k} |\ddot{\wp}_{nk}(\tilde{\Omega})| + \sup_{n,k} |\Psi_{nk}(\tilde{\Omega})| \sup_{n,k} |\Delta\acute{\chi}_{nk}(\tilde{\Omega})|]\frac{d\bar{F}}{\bar{F}} \\
&\leq \acute{\acute{U}} \int_1^2 \sup_{\tilde{\Omega}\in\mathbb{J}} \big[\sup_{n,k} |\ddot{\wp}_{n,k}(\tilde{\Omega})| + \sup_{n,k} |\Psi_{n,k}(\tilde{\Omega})| \sup_n |\acute{\chi}_{n1}(\tilde{\Omega})| + \sup_{n,k} |\ddot{\wp}_{n,k}(\tilde{\Omega})| + \sup_{n,k} |\Psi_{n,k}(\tilde{\Omega})| \sup_k |\acute{\chi}_{1k}(\tilde{\Omega})| \\
&\quad + \sup_{n,k} |\ddot{\wp}_{n,k}(\tilde{\Omega})| + \sup_{n,k} |\Psi_{n,k}(\tilde{\Omega})| \sup_{n,k} |\Delta\acute{\chi}_{nk}(\tilde{\Omega})|]\frac{d\bar{F}}{\bar{F}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sup_{\tilde{\Omega}\in\mathbb{J}} \| (\bar{\Theta}\acute{\chi})(\tilde{\Omega}) \|_\Delta \leq & \acute{\acute{U}} \int_1^2 [\Upsilon_1 + \Upsilon_2 \sup_n |\acute{\chi}_{n1}(\tilde{\Omega})| + \Upsilon_1 + \Upsilon_2 \sup_k |\acute{\chi}_{1k}(\tilde{\Omega})| + \Upsilon_1 + \Upsilon_2 \sup_{n,k} |\Delta\acute{\chi}_{nk}(\tilde{\Omega})|]\frac{d\bar{F}}{\bar{F}} \\
&\leq \int_1^2 \frac{\acute{\acute{U}}}{\bar{F}}[3\Upsilon_1 + \Upsilon_2||\acute{\chi}||]d\bar{F} = [3\Upsilon_1 + \Upsilon_2||\acute{\chi}||]\acute{\acute{U}}\log 2.
\end{aligned}$$

Hence we can write

$$\| (\bar{\Theta}\acute{\chi})(\tilde{\Omega}) \|_{\mathbf{C}(\mathbb{J}, 2^c(\Delta))} \leq [3\Upsilon_1 + \Upsilon_2||\acute{\chi}||]\bar{\bar{B}},$$

where $\bar{\bar{B}} = \acute{\acute{U}}\log 2$. From this we can say that $\bar{\Theta}$ is bounded. Now, we construct a set $\vec{\mathfrak{G}}_{2^c(\Delta)} = \vec{\mathfrak{G}}_{2^c(\Delta)}(\acute{\xi}, \bar{s_0}) = \Big\{\acute{\xi} \in \mathbf{C}(\mathbb{J}, 2^c(\Delta)) : ||\acute{\xi}|| < \bar{s_0}\Big\}$, which is closed , convex and bounded. Now, our aim is to see $\bar{\Theta}$ is continuous on

$\mathbf{C}(\mathbb{J},\vec{\mathfrak{G}})$. For $\acute{\xi},\vec{\rho}_1\in\mathbf{C}(\mathbb{J},2^c(\Delta))$ and for $\tilde{\Omega}\in\mathbb{J}$, we have

$$\begin{aligned}\parallel\left(\bar{\Theta}\xi\right)(\tilde{\Omega})-\left(\bar{\Theta}\vec{\rho}_1\right)(\tilde{\Omega})\parallel_\Delta&=\parallel\int_1^2\Im(\tilde{\Omega},\bar{F})\mathbb{Y}(\bar{F},\acute{\xi}(\bar{F}))\frac{d\bar{F}}{\bar{F}}-\int_1^2\Im(\tilde{\Omega},\bar{F})\mathbb{Y}(\bar{F},\vec{\rho}_1(\bar{F}))\frac{d\bar{F}}{\bar{F}}\parallel_\Delta\\&=\parallel\int_1^2\Im(\tilde{\Omega},\bar{F})\left[\mathbb{Y}(\bar{F},\acute{\xi}(\bar{F}))-\mathbb{Y}(\bar{F},\vec{\rho}_1(\bar{F}))\right]\frac{d\bar{F}}{\bar{F}}\parallel_\Delta\\&=\sup_n\parallel\int_1^2\Im(\tilde{\Omega},\bar{F})\left[\mathbb{Y}_{n1}(\bar{F},\acute{\xi}(\bar{F}))-\mathbb{Y}_{n1}(\bar{F},\vec{\rho}_1(\bar{F}))\right]\frac{d\bar{F}}{\bar{F}}\parallel\\&\quad+\sup_k\parallel\int_1^2\Im(\tilde{\Omega},\bar{F})\left[\mathbb{Y}_{1k}(\bar{F},\acute{\xi}(\bar{F}))-\mathbb{Y}_{1k}(\bar{F},\vec{\rho}_1(\bar{F}))\right]\frac{d\bar{F}}{\bar{F}}\parallel\\&\quad+\sup_{n,k}\parallel\int_1^2\Im(\tilde{\Omega},\bar{F})\left[\Delta\left(\mathbb{Y}_{nk}(\bar{F},\acute{\xi}(\bar{F}))-\mathbb{Y}_{nk}(\bar{F},\vec{\rho}_1(\bar{F}))\right)\right]\frac{d\bar{F}}{\bar{F}}\parallel\\&\leq\int_1^2\frac{\acute{U}}{\bar{F}}d\bar{F}\parallel\mathbb{Y}\acute{\xi}-\mathbb{Y}\vec{\rho}_1\parallel_\Delta\\&\leq\bar{\bar{B}}\parallel\mathbb{Y}\acute{\xi}-\mathbb{Y}\vec{\rho}_1\parallel,\end{aligned}$$

where $\bar{\bar{B}}=\acute{U}\log 2$. Since the function $\mathbb{Y}(\bar{F},\acute{\xi}(\bar{F}))$ is equicontinuous in $2^c(\Delta)$, given any $\bar{\epsilon}>0$, there we find $\delta_*>0$ so that

$$\parallel\mathbb{Y}\acute{\xi}-\mathbb{Y}\vec{\rho}_1\parallel<\frac{\bar{\epsilon}}{\bar{\bar{B}}}for\parallel\acute{\xi}-\vec{\rho_1}\parallel<\delta_*.$$

Hence we have

$$\parallel\left(\bar{\Theta}\acute{\xi}\right)(\tilde{\Omega})-\left(\bar{\Theta}\vec{\rho}_1\right)(\tilde{\Omega})\parallel_\Delta\leq\bar{\bar{B}}\frac{\bar{\epsilon}}{\bar{\bar{B}}}<\bar{\epsilon}.$$

Thus we can conclude that $\bar{\Theta}$ is continuous. Next, our aim to see $\bar{\Theta}$ is continuous in $\mathbb{J}$. For arbitrarily fix $\Bbbk_1$ and $\Bbbk_2$ in $\mathbb{J}$

$$\begin{aligned}&\parallel\left(\bar{\Theta}\acute{\xi}\right)(\Bbbk_1)-\left(\bar{\Theta}\acute{\xi}\right)(\Bbbk_2)\parallel_\Delta\\&=\parallel\int_1^2\Im(\Bbbk_1,\bar{F})\mathbb{Y}(\bar{F},\acute{\xi}(\bar{F}))\frac{d\bar{F}}{\bar{F}}-\int_1^2\Im(\Bbbk_2,\bar{F})\mathbb{Y}(\bar{F},\acute{\xi}(\bar{F}))\frac{d\bar{F}}{\bar{F}}\parallel_\Delta\\&=\parallel\int_1^2\left[\Im(\Bbbk_1,\bar{F})-\Im(\Bbbk_2,\bar{F})\right]\mathbb{Y}(\bar{F},\acute{\xi}(\bar{F}))\frac{d\bar{F}}{\bar{F}}\parallel_\Delta\\&=\sup_n\parallel\int_1^2\left[\Im(\Bbbk_1,\bar{F})-\Im(\Bbbk_2,\bar{F})\right]\mathbb{Y}_{n1}(\bar{F},\acute{\xi}(\bar{F}))\frac{d\bar{F}}{\bar{F}}\parallel+\sup_k\parallel\int_1^2\left[\Im(\Bbbk_1,\bar{F})-\Im(\Bbbk_2,\bar{F})\right]\mathbb{Y}_{1k}(\bar{F},\acute{\xi}(\bar{F}))\frac{d\bar{F}}{\bar{F}}\parallel\\&+\sup_{n,k}\parallel\int_1^2\left[\Im(\Bbbk_1,\bar{F})-\Im(\Bbbk_2,\bar{F})\right]\Delta\mathbb{Y}_{nk}(\bar{F},\acute{\xi}(\bar{F}))\frac{d\bar{F}}{\bar{F}}\parallel\\&\leq\int_1^2\left[\Im(\Bbbk_1,\bar{F})-\Im(\Bbbk_2,\bar{F})\right]\left[\sup_n\left|\mathbb{Y}_{n1}(\bar{F},\acute{\xi}(\bar{F}))\right|+\sup_k\left|\mathbb{Y}_{1k}(\bar{F},\acute{\xi}(\bar{F}))\right|+\sup_{n,k}\left|\Delta\mathbb{Y}_{nk}(\bar{F},\acute{\xi}(\bar{F}))\right|\right]\frac{d\bar{F}}{\bar{F}}\\&\leq\int_1^2\left[\left|\Im(\Bbbk_1,\bar{F})-\Im(\Bbbk_2,\bar{F})\right|\right]\parallel\mathbb{Y}\acute{\xi}\parallel_{\mathbf{C}(\mathbb{J},2^c(\Delta))}\frac{d\bar{F}}{\bar{F}}.\end{aligned}$$

Now, applying the continuity of $\Im(\Bbbk_1,\bar{F})$, we have for given $\bar{\epsilon}>0$, we can find $\delta_*>0$ such that if $\parallel\Bbbk_1-\Bbbk_2\parallel<\delta_*$ then $|\Im(\Bbbk_1,\bar{F})-\Im(\Bbbk_2,\bar{F})|<\frac{\bar{\epsilon}}{||\mathbb{Y}\acute{\xi}||_{\mathbf{C}(\mathbb{J},2^c(\Delta)}\log 2}$. Hence, we get

$$\begin{aligned}\parallel\left(\bar{\Theta}\acute{\xi}\right)(\Bbbk_1)-\left(\bar{\Theta}\acute{\xi}\right)(\Bbbk_2)\parallel_\Delta&\leq\parallel\mathbb{Y}\acute{\xi}\parallel_{\mathbf{C}(\mathbb{J},2^c(\Delta))}\int_1^2|\Im(\Bbbk_1,\bar{F})-\Im(\Bbbk_2,\bar{F})|\frac{d\bar{F}}{\bar{F}}\\&\leq\parallel\mathbb{Y}\acute{\xi}\parallel_{\mathbf{C}(\mathbb{J},2^c(\Delta))}\frac{\bar{\epsilon}}{||\mathbb{Y}\acute{\xi}||_{\mathbf{C}(\mathbb{J},2^c(\Delta)}\log 2}\log 2.\end{aligned}$$

Hence, we can conclude that $\parallel\left(\bar{\Theta}\acute{\xi}\right)(\Bbbk_1)-\left(\bar{\Theta}\acute{\xi}\right)(\Bbbk_2)\parallel_\Delta<\bar{\epsilon}$. Therefore, $\bar{\Theta}$ is continuous on $\mathbb{J}$. Lastly, we want to establish that $\bar{\Theta}$ is a Darbo condensing operator with respect to Hausdorff $\mathcal{MNC}$ χ on the space $\mathbf{C}(\mathbb{J},2^c(\Delta))$. Now, we represent the Hausdorff $\mathcal{MNC}$ for $\vec{\mathfrak{G}}_{2^c(\Delta)}$ which is a subset of $\mathbf{C}(\mathbb{J}, 2^c(\Delta))$ and it is defined as

$$\chi_{\mathbf{C}(\mathbb{J},2^c(\Delta))}\left(\vec{\mathfrak{G}}_{2^c(\Delta)}\right) = \sup_{\wp^*\in\mathbb{J}} \chi_{2^c(\Delta)}\left(\vec{\mathfrak{G}}_{2^c(\Delta)}(\wp^*)\right).$$

Now,

$$\begin{aligned}
&\chi_{2^c(\Delta)}\left[\left(\bar{\Theta}\vec{\mathfrak{G}}_{2^c(\Delta)}\right)(\wp^*)\right]\\
=&\lim_{n,k\to\infty}\Big[\sup_{\mho^*\in\vec{\mathfrak{G}}_{2^c(\Delta)}}\Big(\sup_n\|\int_1^2\Im(\wp^*,\bar{F})\mathbb{Y}_{n1}\bar{F},\mho_1(\bar{F}))\frac{d\bar{F}}{\bar{F}}\|+\sup_k\|\int_1^2\Im(\wp^*,\bar{F})\mathbb{Y}_{1k}(\bar{F},\mho_1(\bar{F}))\frac{d\bar{F}}{\bar{F}}\|\\
&+\sup_{n,k}\|\int_1^2\Im(\wp^*,\bar{F})\Delta\mathbb{Y}_{nk}(\bar{F},\mho_1(\bar{F}))\frac{d\bar{F}}{\bar{F}}\|\Big)\Big]\\
\le&\lim_{n,k\to\infty}\Big[\sup_{\mho^*\in\vec{\mathfrak{G}}_{2^c(\Delta)}}\Big(\int_1^2\Im(\wp^*,\bar{F})[\sup_n|\mathbb{Y}_{n1}(\bar{F},\mho^*(\bar{F}))|+\sup_k|\mathbb{Y}_{1k}(\bar{F},\mho^*(\bar{F}))|+\sup_{n,k}|\Delta\mathbb{Y}_{nk}(\bar{F},\mho^*(\bar{F}))|]\frac{d\bar{F}}{\bar{F}}\Big)\Big]\\
\le&\lim_{n,k\to\infty}\Big[\sup_{\mho^*\in\vec{\mathfrak{G}}_{2^c(\Delta)}}\Big(\int_1^2\Im(\wp^*,\bar{F})\big[\sup_n|\ddot{\wp}_{n1}(\bar{F})|+\sup_n|\psi_{n1}(\bar{F})|\sup_n|\mho^*_{n1}(\bar{F})|+\sup_k|\ddot{\wp}_{1k}(\bar{F})|\\
&+\sup_k|\psi_{1k}(\bar{F})|\sup_k|\mho^*_{1k}(\bar{F})|+\sup_{n,k}|\ddot{\wp}_{nk}(\bar{F})|+\sup_{n,k}|\psi_{nk}(\bar{F})|\sup_{nk}|\Delta\mho^*_{nk}(\bar{F})|\big]\frac{d\bar{F}}{\bar{F}}\Big)\Big]\\
\le&\lim_{n,k\to\infty}\Big[\sup_{\mho^*\in\vec{\mathfrak{G}}_{2^c(\Delta)}}\Big(\int_1^2\Im(\wp^*,\bar{F})[\sup_{\wp^*\in\mathbb{J}}(\sup_{n,k}|\psi_{nk}(\wp^*)|\sup_n|\mho^*_{n1}(\wp^*)|+\sup_{n,k}|\psi_{nk}(\wp^*)|\sup_k|\mho^*_{1k}(\wp^*)|\\
&+\sup_{n,k}|\psi_{nk}(\wp^*)|\sup_{n,k}|\Delta\mho^*_{nk}(\wp^*)|]\frac{d\bar{F}}{\bar{F}}\Big)\Big]\\
\le&\lim_{n,k\to\infty}\left\{\sup_{\mho^*\in\vec{\mathfrak{G}}_{2^c(\Delta)}}\left(\int_1^2\max_{\wp^*\in\mathbb{J}}\Im(\wp^*,\bar{F})\left[\Upsilon_2\big(\sup_n|\mho^*_{n1}(\wp^*)|+\sup_k|\mho^*_{1k}(\wp^*)|+\sup_{n,k}|\Delta\mho^*_{nk}(\wp^*)|\big)\right]\frac{d\bar{F}}{\bar{F}}\right)\right\}\\
\le&\lim_{n,k\to\infty}\left\{\sup_{\mho^*\in\vec{\mathfrak{G}}_{2^c(\Delta)}}\left(\int_1^2\Im(\wp^*,\bar{F})\left[\Upsilon_2\sup_{\wp^*\in\mathbb{J}}||\mho^*(\wp^*)||_\Delta\right]\frac{d\bar{F}}{\bar{F}}\right)\right\}\\
\le&\Upsilon_2\le\lim_{n,k\to\infty}\left\{\sup_{\mho^*\in\vec{\mathfrak{G}}_{2^c(\Delta)}}\int_1^2\frac{\acute{\acute{U}}}{\bar{F}}d\bar{F}\sup_{\wp^*\in\mathbb{J}}||\mho^*(\wp^*)||_\Delta\right\}\\
\le&\bar{\bar{B}}\Upsilon_2\lim_{n,k\to\infty}\left\{\sup_{\mho^*\in\vec{\mathfrak{G}}_{2^c(\Delta)}}\sup_{\wp^*\in\mathbb{J}}||\mho^*(\wp^*)||_\Delta\right\}\\
\le&\bar{\bar{B}}\Upsilon_2\chi_{\mathbf{C}(\mathbb{J},2^c(\Delta))}\left(\vec{\mathfrak{G}}_{2^c(\Delta)}\right).
\end{aligned}$$

So, we can write

$$\sup_{\wp^*\in\mathbb{J}}\chi_{2^c(\Delta)}\left[\left(\bar{\Theta}\vec{\mathfrak{G}}_{2^c(\Delta)}\right)(\wp^*)\right]\le\bar{\bar{B}}\Upsilon_2\chi_{\mathbf{C}(\mathbb{J},2^c(\Delta))}\left(\vec{\mathfrak{G}}_{2^c(\Delta)}\right)$$

$$\chi_{\mathbf{C}(\mathbb{J},2^c(\Delta))}\left(\bar{\Theta}\vec{\mathfrak{G}}_{2^c(\Delta)}\right)\le\bar{\bar{B}}\Upsilon_2\chi_{\mathbf{C}(\mathbb{J},2^c(\Delta))}\left(\vec{\mathfrak{G}}_{2^c(\Delta)}\right).$$

As $\bar{\bar{B}}\Upsilon_2 < 1$, which implies that $\bar{\Theta}$ is a Darbo condensing operator with Darbo constant $\bar{\bar{B}}\Upsilon_2$. Therefore, $\bar{\Theta}$ has at least one point in $\vec{\mathfrak{G}}_{2^c(\Delta)} \subset 2^c(\Delta)$, which gives the solution of the system of equations (3.1). □

4 Example

Now, we represent an example to illustrate the above theorem. Let us consider the following Hadamard type fractional differential boundary value problem :

$$H_{\mathrm{D}_{1+}^{\check{\nu}^*}}\mathfrak{L}(\bar{c})+\mathbf{Y}(\bar{c})=0, \tag{4.1}$$

with

$$\mathfrak{L}(1) = \mathfrak{L}^{/}(1) = 0, \qquad \mathfrak{L}(2) = \int_1^2 \hat{p}(\bar{c})\mathfrak{L}(\bar{c})\frac{d\bar{c}}{\bar{c}},$$

where $\mathbf{Y}(\bar{c}) = \mathbf{Y}(\acute{F}) = \mathbf{Y}_{nk}(\acute{F}, \mathfrak{L}(\acute{F})$ and

$$\mathbf{Y}_{nk}(\acute{F}, \mathfrak{L}(\acute{F}) = \sum_{n=k}^{\infty} \frac{1}{(1+k^3)n^3}\cos\pi(n+k) + \sum_{n=u,k=v} \frac{\cos(\acute{F}^2+1)}{n+1}\mathfrak{L}_{uv}.$$

Here $\breve{\nu^*} = 2.5$, $\hat{p}(\bar{c}) = \ln \bar{c}$,

$$\Im(\bar{c}, \acute{F}) = \hat{G}(\bar{c}, \acute{F}) + \frac{(ln\bar{c})^{\breve{\nu^*}-1}}{\bar{\Re}}\int_1^2 \hat{p}(\bar{c})\hat{G}(\bar{c}, \acute{F})\frac{d\bar{c}}{\bar{c}},$$

$$\begin{aligned}\bar{\Re} =&(\ln 2)^{2.5-1} - \int_1^2 \hat{p}(\bar{c})(\ln \bar{c})^{2.5-1}\frac{d\bar{c}}{\bar{c}} = 0.6^{1.5} - \int_1^2 (\ln \bar{c})^{2.5}\frac{d\bar{c}}{\bar{c}}\\ =&0.46 - (\ln 1.5)^{2.5}\ln \bar{c}]_1^2 = 0.46 - 0.104 \times 0.301\\ =&0.429\end{aligned}$$

and

$$\hat{G}(\bar{c}, \acute{F}) = \frac{1}{\Gamma(2.5)(\ln 2)^{1.5}}\left\{\begin{array}{cc} (\ln \bar{c})^{1.5}(\ln \frac{2}{\acute{F}})^{1.5} - (\ln \frac{\bar{c}}{\acute{F}})^{1.5}(\ln 2)^{1.5} & 1 \le \acute{F} \le \bar{c} \le 2 \\ (\ln \bar{c})^{1.5}(\ln \frac{2}{\acute{F}})^{1.5}, & 1 \le \bar{c} \le \acute{F} \le 2 \end{array}\right\}.$$

Now, our aim is to see $\mathbf{Y}$ is equicontinuous.

$$\begin{aligned}\parallel \mathbf{Y}(\acute{F}, \mathfrak{L}(\acute{F})) - \mathbf{Y}(\acute{F}, D(\acute{F})) \parallel_{\Delta} =& \parallel \sum_{n=k}^{\infty} \frac{1}{(1+k^3)n^3}\cos\pi(n+k) + \sum_{n=u,k=v}\frac{\cos(\acute{F}^2+1)}{n(n+1)}\mathfrak{L}_{uv}\\ &- \sum_{n=k}^{\infty} \frac{1}{(1+k^3)n^3}\cos\pi(n+k) - \sum_{n=u,k=v}\frac{\cos(\acute{F}^2+1)}{n(n+1)}D_{uv} \parallel_{\Delta}\\ =& \parallel \sum_{n=u,k=v}\frac{\cos(\acute{F}^2+1)}{n(n+1)}[\mathfrak{L}_{uv} - D_{uv}] \parallel_{\Delta}\\ \le& \sum_{n=u,k=v}\frac{1}{n(n+1)} \parallel \mathfrak{L}_{uv} - D_{uv} \parallel_{\Delta}\\ \le& \frac{\pi^2}{6} \parallel \mathfrak{L}(\acute{F}) - D(\acute{F}) \parallel_{\Delta}.\end{aligned}$$

For the continuity of the space $2^c(\Delta)$, given any $\tilde{\varepsilon} > 0$, there exist any $\grave{\rho}^* > 0$ such that

$$\parallel \mathfrak{L}(\acute{F}) - D(\acute{F}) \parallel_{\Delta} \le \grave{\rho}^* = \frac{6\tilde{\varepsilon}}{\pi^2},$$

for $\mathfrak{L}_{nk}(\acute{F})$,$D_{nk}(\acute{F}) \in 2^c(\Delta)$. Hence, we have $\parallel \mathbf{Y}(\acute{F}, \mathfrak{L}(\acute{F})) - \mathbf{Y}(\acute{F}, D(\acute{F})) \parallel_{\Delta} \le \tilde{\varepsilon}$. So, we get the result, i.e assumption $(d1)$ satisfied. Now for the assumption $(d2)$, let $\ddot{\wp}_{nk}(\acute{F}) = \frac{1}{n^3}\sum_{n=k}^{\infty}\frac{1}{(1+k^4}$ and $\psi_{nk}(\acute{F}) = \frac{\cos(\acute{F}^2+1)}{6}$ be continuous, so that $\ddot{\wp}_{nk}(\acute{F})$ converges uniformly to $\frac{\pi^4}{90}$ and $\psi_{nk}(\acute{F})$ is equibounded on $\mathbb{J}$. Also, we have

$$\Upsilon_1 = \sup_{\acute{F}\in\mathbb{J}}\sup_{n,k}\frac{1}{n^3}\sum_{n=k}^{\infty}\frac{1}{1+k^4} = \frac{\pi^4}{90}, \qquad \Upsilon_2 = \sup_{\acute{F}\in\mathbb{J}}\sup_{n,k}\frac{1}{n^3}\sum_{n=k}^{\infty}\frac{\cos(\acute{F}^2+1)}{6} = \frac{1}{6}.$$

Now for any $\acute{F} \in \mathbb{J}$, if $\mathfrak{L}_{nk}(\acute{F}) \in 2^c(\Delta)$, then $\Delta\mathbf{Y}_{nk}(\acute{F}, \mathfrak{L}(\acute{F})) \in 2^c(\Delta)$, so we have

$$\begin{aligned}\left|\mathbf{Y}_{nk}(\acute{F}, \mathfrak{L}(\acute{F}))\right| \le& \sum_{n=k}^{\infty}\frac{1}{(1+k^4)n^3}|\cos\pi(n+k)| + \sum_{n=u,k=v}|\frac{\cos(\acute{F}^2+1)}{n(n+1)}||\mathfrak{L}_{uv}(\acute{F})|\\ \le& \sum_{n=k}^{\infty}\frac{1}{(1+k^4)n^3} + \frac{\cos(\acute{F}^2+1)}{6}|\mathfrak{L}_{uv}(\acute{F})|\\ \le& |\ddot{\wp}_{nk}(\acute{F})| + |\psi_{nk}(\acute{F})||\mathfrak{L}_{uv}(\acute{F})|.\end{aligned}$$

Also, we have,

$$\begin{aligned}\left|\Delta \mathbf{Y}_{nk}(\acute{F},\mathfrak{L}(\acute{F}))\right| \leq & \frac{1}{n^3}\sum_{n=k}^{\infty}\frac{1}{1+k^4}|\cos\pi(n+k)| + \sum_{n=u,k=v}\frac{\cos(\acute{F}^2+1)}{n(n+1)}||\Delta\mathfrak{L}_{uv}(\acute{F})| \\ \leq & |\ddot{\wp}_{nk}(\acute{F})| + |\psi_{nk}(\acute{F})||\Delta\mathfrak{L}_{uv}(\acute{F})|.\end{aligned}$$

Lastly, we have to show $\Upsilon_2\bar{\bar{B}} \leq 1$. Here $\Upsilon_2 = \frac{1}{6}$, and

$$\bar{\bar{B}} = \int_1^2 \frac{\acute{\acute{U}} d\acute{F}}{\acute{F}} = \acute{\acute{U}}\log 2 = \acute{\acute{U}}[\log \acute{F}]_1^2 \leq 2\times 0.30 = 0.60,$$

where we assume that $\acute{\acute{U}} = \max_{\bar{c}\in\mathbb{J}} \Im(\bar{c},\acute{F}) \leq 2$. Hence we have, $\Upsilon_2\bar{\bar{B}} \leq \frac{1}{6}\times 0.60 = 0.1 \leq 1$. Since, it satisfied all the hypothesizes of the theorem (3.1), we can establish that the equation (4.1) has at least one solution which belong to $\mathbf{C}(\mathbb{J}, 2^c(\Delta))$.

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