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Some new results in Menger PbM-spaces for single-valued and multi-valued mappings via simulation functions with its application

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Abstract

The main goal of the present paper is to obtain several fixed point theorems for both multi-valued and single-valued mappings in Menger PbM-spaces, which is an extension of Menger PM-spaces. In this paper, we introduce different types of contractive mappings by introduce the notions of $(\alpha - \psi)$-E-type simulation function and $(\beta - \psi)$-E-type simulation function in Menger PbM-spaces which is a generalization of simulation functions introduced by Khojasteh et al. [18]. Furthermore, we present some nontrivial examples and an application to the existence of a solution of the Volterra-type integral equation.

Keywords: Menger PbM-spaces, Simulation function, self mappings, orbital admissible mappings, multi-valued mapping
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1 Introduction and preliminaries

In the last century, nonlinear functional analysis has experienced many advances. One of these improvements is the introduction of various spaces and is the proof of fixed point results in these spaces along with its applications in engineering science. One of these spaces is b-metric space. The notion of a b-metric space was defined by Bakhtin [3] and Czerwik [12]. Afterward, other authors discussed in b-metric spaces its properties and defined convergent, Cauchy sequence, and so on. Furthermore, they obtained several results in fixed points in this space with its applications to non-linear functional analysis for more details, we refer to [1, 8, 9, 7, 19, 23, 25, 24, 26, 27, 29, 30] and the references it contains.

In contrast, the concept of PM-spaces was introduced by Menger [21] in 1942. So, Sehgal and Bharucha-Reid [34] proved some fixed-point theorems in that space. Also, Schweizer and Sklar [33] discussed the properties of Menger PM-spaces. In the following, the fixed point theory in PM-spaces for single-valued and multi-valued mappings were extensively studied by many authors in [11, 10, 13, 14, 15, 17, 20, 28] and references therein.

Definition 1.1. [14] A function $f : \mathbb{R} \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{u\in R} f(u) = 0$. In addition, if $f(0) = 0$, then f is called a distance distribution function. Further-

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more, a distance distribution function f satisfying $\lim_{u\to+\infty} f(u) = 1$ is called a Menger distance distribution function. The set of all Menger distance distribution functions is denoted by D^+.

Definition 1.2. [11] A triangular norm (abbreviated, t-norm) is a binary operation $\mathcal{T}$ on [0,1], which satisfies the following conditions:

(a) $\mathcal{T}$ is associative and commutative,

(b) $\mathcal{T}$ is continuous,

(c) $\mathcal{T}(a,1) = a$ for all $a \in [0,1]$,

(d) $\mathcal{T}(a,b) \leq \mathcal{T}(c,d)$ whenever $a \leq c$ and $b \leq a$, for each $a,b,c,d \in [0,1]$.

Definition 1.3. [11] A triangular norm $\mathcal{T}$ is said to be of H-type (Hadzić type), if a family of functions $\{\mathcal{T}^n(t)\}$ is equicontinuous at $i = 1$; that is, there exists $\delta \in (0,1)$ such that $i > 1-\delta$ implies that $\mathcal{T}^n(i) > 1-\epsilon$ for all $\epsilon \in (0,1)$ and $(n \geq 1)$, where $\mathcal{T}^n : [0,1] \longrightarrow 0,1]$ is by $\mathcal{T}^1(i) = \mathcal{T}(i,i)$ and $\mathcal{T}^n(i) = \mathcal{T}(i, \mathcal{T}^{n-1}(i))$ for $n = 2,3,\cdots$.

Definition 1.4. [16] A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(\mathcal{W},\mathcal{G},\mathcal{T})$, where $\mathcal{W}$ is a nonempty set, $\mathcal{T}$ is a continuous i-norm, $\mathcal{G} : \mathcal{W}\times\mathcal{W} \to D^+$ is a mapping such that if $\mathcal{G}_{u,v}$ denotes the value of $\mathcal{G}$ at the pair (u,v), then the following conditions hold:

(PM1) $\mathcal{G}_{u,v}(i) = 1$ for all $i > 0 \Leftrightarrow u = v$,
(PM2) $\mathcal{G}_{u,v}(i) = \mathcal{G}_{v,u}(i)$ for all $u,v \in \mathcal{W}$ and $i > 0$,
(PM3) $\mathcal{G}_{u,z}(i+j) \geq \mathcal{T}(\mathcal{G}_{u,v}(i), \mathcal{G}_{v,z}(j))$ for all $u,v,z \in \mathcal{W}$ and $i,j \geq 0$.

In addition, in 2015, Hasanvand and Khanehgir [16] defined Menger PbM-space.

Definition 1.5. [16] Let α is a real number in $(0,1]$. A Menger probabilistic b-metric space (briefly, Menger PbM-space) is a triple $(\mathcal{W},\mathcal{G},\mathcal{T})$, where $\mathcal{W}$ is a nonempty set, $\mathcal{T}$ is a continuous t-norm, $\mathcal{G} : \mathcal{W}\times\mathcal{W} \to D^+$ is a mapping such that if $\mathcal{G}_{u,v}$ denotes the value of $\mathcal{G}$ at the pair (u,v), then the following conditions hold:

(PM1) $\mathcal{G}_{u,v}(i) = 1$, for all $i > 0 \Leftrightarrow u = v$;
(PM2) $\mathcal{G}_{u,v}(i) = \mathcal{G}_{v,u}(i)$, for all $u,v \in \mathcal{W}$ and $i > 0$;
(PM3) $\mathcal{G}_{u,z}(i+j) \geq \mathcal{T}(\mathcal{G}_{u,v}(\alpha i), \mathcal{G}_{v,z}(\alpha j))$, for all $u,v,z \in \mathcal{W}$ and $i,j \geq 0$.

Example 1.6. [16] Suppose $\mathcal{W} = \mathbb{R}^+$, $\mathcal{T}(u,v) = \min\{u,v\}$ and define $\mathcal{G} : \mathcal{W}\times\mathcal{W} \to D^+$ by

$$\mathcal{G}_{u,v}(i) = \begin{cases} \frac{i}{i+|u-v|^2}, & \text{if } i > 0 \\ 0 & \text{otherwise,} \end{cases}$$

for all $u,v \in \mathcal{W}$. Then $(\mathcal{W},\mathcal{G},\mathcal{T})$ is a complete Menger PbM-space with $\alpha = \frac{1}{2}$.

Also, Hasanvand and Khanehgir proved a fixed point theorem for a single-value operator.

Theorem 1.7. [16] Let $(\mathcal{W},\mathcal{G},\mathcal{T})$ be a complete Menger PbM-space with coefficient α, which satisfies $\mathcal{T}(a,a) \geq a$ with $a \in [0,1]$. Also, let $f : \mathcal{W} \to \mathcal{W}$ be a generalized $\beta-\gamma$-type contractive mapping satisfying the following conditions:

(i) f is (β,γ)-admissible;

(ii) there exists $u_0 \in \mathcal{W}$ such that $\beta(u_0, fu_0, i) \leq 1$ and $\gamma(u_0, fu_0, i) \geq 1$ for all $i > 0$;

(iii) for all $n \in \mathbb{N}$ and for all $i > 0$, if $\{u_n\}$ is a sequence in $\mathcal{W}$ such that $\beta(u_{n-1}, u_n, i) \leq 1$ and $\gamma(u_n, u_{n+1}, i) \geq 1$ and $u_n \to u$ as $n \to \infty$, then $\beta(u_{n-1}, u, i) \leq 1$ and $\gamma(u_n, fu, i) \geq 1$.

Then f has a fixed point.

Definition 1.8. [6] A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a ϕ-function if it satisfies the following conditions:

i) $\psi(i) = 0$ if and only if $i = 0$;

ii) $\psi(i)$ is strictly monotone increasing and $\psi(i) \to \infty$ as $i \to \infty$;

iii) ψ is left-continuous in $(0, \infty)$;

iv) ψ is continuous at 0.

Lemma 1.9. [13] Suppose $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ is a Menger PbM-space with coefficient α and ψ is a Φ-function. If we have $\mathcal{G}_{u,v}(\alpha^k \psi(i)) \geq \mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c}))$ for $u, v \in \mathcal{W}$, $c \in (0, 1)$, $k \in \mathbb{N}$ and for all $i > 0$, then $\mathcal{G}_{u,v}(i) = 1$.

The collection of all ϕ-functions will be denoted by Ψ. In order to unify several existing fixed point results in the literature, Khojasteh et al. [18] introduced a mapping namely simulation function and the notion of Z-contraction with respect to c. The Z-contraction generalized the Banach contraction and unify several known type of contractions involving the combination of $d(Tx, Ty)$ and $d(x, y)$ and satisfies some particular conditions in complete metric spaces.

Definition 1.10. A simulation function is a mapping $c : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

1) $c(m, n) < n - m$ for all $n, m > 0$;

2) if $\{m_i\}$,$\{n_i\}$ are sequences in $(0, \infty)$ such that $\lim_{i\to\infty} m_i = \lim_{i\to\infty} n_i > 0$, then

$$\limsup_{i\to\infty} c(m_i, n_i) < 0.$$

Also, Z is considered the family of all simulation functions.

Example 1.11. [18] Suppose $f_l : [0, \infty) \to [0, \infty)$ be continuous function such that $f_l(u) = 0$ if and only if, $u = 0$. For $l = 1, 2$ we define the mappings $c : [0, \infty) \times [0, \infty) \to \mathbb{R}$, as follows,

$$c(m, n) = f_1(n) - f_2(m),$$

for all $m, n \in [0, \infty)$, where $f_1, f_2 : [0, \infty) \to [0, \infty)$ are two continuous functions such that $f_1(u) = f_2(u) = 0$ iff $u = 0$ and $f_1(i) < i \leq f_2(i)$ for all $i > 0$.

Throughout this paper, the family of all nonempty closed and bounded subsets of $\mathcal{W}$ is denoted by $CB(\mathcal{W})$, the family of all nonempty subsets of $\mathcal{W}$ by $N(\mathcal{W})$. Further, for notions such as convergence and Cauchy sequences, completeness and examples in Menger PbM-space, we refer to [16].

Definition 1.12. [28] Suppose $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ is a Menger PM-space. Function $H : CB(\mathcal{W}) \times CB(\mathcal{W}) \longrightarrow [0, 1]$ for all $i \geq 0$ define by

$$H_{A,B}(i) = \sup_{j\leq i} \mathcal{T}(\inf_{p\in A} \mathcal{G}_{p,B}(j), \inf_{q\in B} \mathcal{G}_{A,q}(j)),$$

is said to be the distance between A and B, where $\mathcal{G}_{u,A}(i) = \sup_{p\in A} \mathcal{G}_{p,u}(i)$ is the distance between a point and a set in PM-space.

Definition 1.13. [28] Suppose $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ is a Menger PM-space.

(i) A subset $D \subset \mathcal{W}$ is said to be approximative if $P_{u,D}(i) = \{p \in D : \mathcal{G}_{u,D}(i) = \mathcal{G}_{p,u}(i)\}$ for all $u \in \mathcal{W}$, has nonempty value.

(ii) The multi-valued mapping $f : \mathcal{W} \longrightarrow N(\mathcal{W})$ is said to have approximative values, AV for short, if fu is approximative for each $u \in \mathcal{W}$.

(iii) f is called to have the w-approximative valued property if there exists $v \in fu$ such that $\mathcal{G}_{u,v}(i) \geq H_{fa,fu}(i)$ for all $a \in \mathcal{W}$, $u \in Ga$ and for all $i > 0$.

Definition 1.12 and Definition 1.13 are satisfied in PbM-space. Also, we need to the following definition in the sequel.

Definition 1.14. [14] Close operator is multi-valued operator $f : \mathcal{W} \to CB(\mathcal{W})$ if for two sequences $\{u_n\}$,$\{v_n\} \subset \mathcal{W}$ and u_0 ,$v_0 \in \mathcal{W}$, $u_n \to u_0$, $v_n \to v_0$ and $v_n \in f(u_n)$ imply $v_0 \in f(u_0)$.

2 Results on $(\alpha-\psi)$-E-type simulation for multi-valued mappings in PbM-space

Following the idea of Hasanvand and Khanehgir [16], Gopal et al [13] and Karapinar [2] we introduce the following definitions in the framework of Menger PbM-space.

Definition 2.1. Suppose $f:\mathcal{W}\to CB(\mathcal{W})$ is a multi-valued operator and $\gamma:\mathcal{W}\times\mathcal{W}\times(0,\infty)\to(0,\infty)$ is a function. We say that f is γ-orbital admissible if $\gamma(a,b,i)\leq 1$ for all $a\in\mathcal{W}$, $b\in f(a)$ and $i>0$, then $\gamma(u,b,i)\leq 1$ for all $u\in f(b)$.

Example 2.2. Suppose $\mathcal{W}=\mathbb{R}^+$ and $\mathcal{G}$ be as in Example 1.6. Then $(\mathcal{W},\mathcal{G},\mathcal{T})$ is a complete Menger PbM-space with $\alpha=\frac{1}{2}$. Define $f:\mathcal{W}\to CB(\mathcal{W})$ by $f(u)=[0,3u]$ for all $u\in\mathcal{W}$ and $\gamma:\mathcal{W}\times\mathcal{W}\times(0,\infty)\to(0,\infty)$ by

$$\gamma(u,v,i)=\begin{cases}\frac{1}{4} & \text{if } (u,v)=(0,0)\\ \frac{1}{2} & \text{otherwise.}\end{cases}$$

Then f is γ-orbital admissible.

Definition 2.3. Suppose that $(\mathcal{W},\mathcal{G},\mathcal{T})$ is a Menger PbM-space with coefficient α and $f:\mathcal{W}\to CB(\mathcal{W})$ is said $(\alpha-\psi)$-Ehsan type simulation if there exist $\psi\in\Psi$, $c\in Z$, $\gamma:\mathcal{W}\times\mathcal{W}\times[0,\infty)\to[0,\infty)$ and real numbers $a_1,a_2,a_3,a_4,a_5\in\mathbb{R}$ with

$$\frac{a_2-a_4}{a_1}>1,\ a_1\neq 0,\ a_2-a_5>0,$$

such that

$$c(p_{u,v}(i),l_{u,v}(i))\geq 0,$$

for all $u\in\mathcal{W}$, $v\in fu$ and $o\in fv$ where

$$\begin{aligned}p_{u,v}(i) &= -a_1(\gamma(u,v,\alpha^k i)H_{fu,fv}(\alpha^k\psi(i)))-a_2(\mathcal{G}_{u,fu}(\alpha^{k-1}\psi(\frac{i}{c}))+\mathcal{G}_{v,fv}(\alpha^{k-1}\psi(\frac{i}{c})))\\ &\quad -\min[a_3\mathcal{G}_{v,fu}(\alpha^{k-1}\psi(\frac{i}{c})),a_3\mathcal{G}_{u,fv}(\alpha^{k-1}\psi(\frac{i}{c}))],\end{aligned}$$

and

$$l_{u,v}(i)=a_4\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c}))+a_5\mathcal{G}_{u,o}(\alpha^{k-1}\psi(\frac{i}{c})),$$

where $k\in\mathbb{N}$ and $c\in(0,1)$.

The following is the main result of this section. We present a fixed point theorem for $(\alpha-\psi)$-E-type simulation multi-valued mappings in Menger PbM-spaces.

Theorem 2.4. Suppose $(\mathcal{W},\mathcal{G},\mathcal{T})$ is a complete Menger PbM-space with coefficient α, which satisfies $\mathcal{T}(a,a)\geq a$ with $a\in[0,1]$. Also, let $f:\mathcal{W}\to CB(\mathcal{W})$ has the w-approximative value property and be satisfying the following conditions:

(i) f is γ-orbital admissible;

(ii) for some $u_0\in\mathcal{W}$ there exists $u_1\in f(u_0)$ such that $\gamma(u_0,u_1,i)\leq 1$ for all $i>0$;

(iii) f is $(\alpha-\psi)$-E-type simulation;

(iv) f is closed operator.

Then f has a fixed point.

Proof . If $u_1=u_0$, then the proof is complete. Suppose that $u_1\neq u_0$, i.e., $u_0\notin fu_0$. Since f has w-approximative value property, there exists $u_2\in fu_1$ such that $\mathcal{G}_{u_1,u_2}(i)\geq H_{fu_0,fu_1}(i)$ for all $i>0$. For $u_2\in fu_1$, from (i) and (ii) we have $\gamma(u_0,u_1,i)\leq 1$ and $\gamma(u_1,u_2,i)\leq 1$ for all $i>0$. If $u_1\in fu_1$, then u_1 is a fixed point of f. Suppose that $u_2\neq u_1$. Again, by the assumptions, there exists $u_3\in fu_2$ such that $\mathcal{G}_{u_3,u_2}(i)\geq H_{fu_2,fu_1}(i)$, $\gamma(u_2,u_1,i)\leq 1$ and

$\gamma(u_2, u_3, i) \leq 1$ for all $i > 0$. By continuing this process, we obtain a sequence $\{u_n\}$ in $\mathcal{W}$ such that $u_n \in fu_{n-1}$ with $u_n \neq u_{n-1}$, where $\mathcal{G}_{u_n,u_{n+1}}(i) \geq H_{fu_{n-1},fu_n}(i)$, $\gamma(u_n, u_{n-1}, i) \leq 1$ and $\gamma(u_n, u_{n+1}, i) \leq 1$ for all $i > 0$. By (iii) we have,

$$0 \leq c(p_{u_n,u_{n+1}}(i), l_{u_n,u_{n+1}}(i)) < l_{u_n,u_{n+1}}(i) - p_{u_n,u_{n+1}}(i),$$

where

$$\begin{aligned} p_{u_n,u_{n+1}}(i) = &-a_1(\gamma(u_n, u_{n+1}, \alpha^k i)H_{fu_n,fu_{n+1}}(\alpha^k\psi(i))) - a_2(\mathcal{G}_{u_n,fu_n}(\alpha^{k-1}\psi(\frac{i}{c})) \\ &+ \mathcal{G}_{u_{n+1},fu_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c}))) - \min[a_3\mathcal{G}_{u_{n+1},fu_n}(\alpha^{k-1}\psi(\frac{i}{c})), a_3\mathcal{G}_{u_n,fu_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c}))], \end{aligned}$$

and

$$l_{u_n,u_{n+1}}(i) = a_4\mathcal{G}_{u_n,u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})) + a_5\mathcal{G}_{u_n,u_{n+2}}(\alpha^{k-1}\psi(\frac{i}{c})).$$

Since f has w-approximative value property we have,

$$\begin{aligned} a_1\mathcal{G}_{u_n,u_{n+1}}(i) &\geq a_1 H_{fu_{n-1},fu_n}(\alpha^k\psi(i)) \\ &\geq a_1\gamma(u_{n-1}, u_n, \alpha^k i)H_{fu_{n-1},fu_n}(\alpha^k\psi(i)) \\ &\geq a_2(\mathcal{G}_{u_{n-1},fu_{n-1}}(\alpha^{k-1}\psi(\frac{i}{c})) + \mathcal{G}_{u_n,fu_n}(\alpha^{k-1}\psi(\frac{i}{c}))) \\ &\quad + \min[a_3\mathcal{G}_{u_n,fu_{n-1}}(\alpha^{k-1}\psi(\frac{i}{c})), a_3\mathcal{G}_{u_{n-1},fu_n}(\alpha^{k-1}\psi(\frac{i}{c}))] \\ &\quad - a_4\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})) - a_5\mathcal{G}_{u_{n-1},u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})) \\ &\geq a_2(\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})) + \mathcal{G}_{u_n,u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c}))) \\ &\quad + \min[a_3\mathcal{G}_{u_n,u_n}(\alpha^{k-1}\psi(\frac{i}{c})), a_3\mathcal{G}_{u_{n-1},u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c}))] \\ &\quad - a_4\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})) - a_5\mathcal{G}_{u_{n-1},u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})). \end{aligned}$$

So we have,

$$\begin{aligned} a_1\mathcal{G}_{u_n,u_{n+1}}(i) &\geq a_1\mathcal{G}_{u_n,u_{n+1}}(i) - a_2\mathcal{G}_{u_n,u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})) \\ &\geq (a_2 - a_4)(\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c}))) + (a_3 - a_5)\mathcal{G}_{u_{n-1},u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})), \end{aligned}$$

which implies,

$$\mathcal{G}_{u_n,u_{n+1}}(i) \geq \frac{(a_2 - a_4)}{(a_1)}\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})) \geq \mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})).$$

So we have,

$$\mathcal{G}_{u_n,u_{n+1}}(i) \geq \mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})) \geq \cdots \geq \mathcal{G}_{u_0,u_1}(\alpha^{k-n}\psi(\frac{i}{c^n})),$$

that is, $\mathcal{G}_{u_n,u_{n+1}}(i) \geq \mathcal{G}_{u_0,u_1}(\alpha^{k-n}\psi(\frac{r}{c^n})$ for arbitrary $n \in \mathbb{N}$. Next, let $m, n \in \mathbb{N}$ with $m > n$. Then, by $(PbM3)$ and strictly increasing of ψ, we have

$$\begin{aligned} \mathcal{G}_{u_n,u_m}((m-n)i) &\geq \min\{\mathcal{G}_{u_n,u_{n+1}}(\alpha i), \cdots, \mathcal{G}_{u_{m-1},u_{m-2}}(\alpha^{m-n-1}i), \mathcal{G}_{u_{m-1},u_m}(\alpha^{m-n-1}i)\} \\ &\geq \min\{\mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^n})), \cdots, \mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^{m-2}})), \mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^{m-1}}))\} \\ &= \mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^n})). \end{aligned}$$

Since $\alpha^{1-n}\psi(\frac{i}{c^n}) \to \infty$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^n})) > 1-\epsilon$ for fixed $\epsilon \in (0,1)$, whenever $n \geq n_0$. This implies that $\mathcal{G}_{u_n,u_m}((m-n)i) > 1-\epsilon$ for every $m > n \geq n_0$. Since $i > 0$ and $\epsilon \in (0,1)$ are arbitrary, we deduce that $\{u_n\}$ is a Cauchy sequence in the complete Menger PbM-space $(\mathcal{W}, \mathcal{G}, \mathcal{T})$. Then there exists $u \in \mathcal{W}$ such that $u_n \to u$ as $n \to \infty$. We are going to show that u is a fixed point of f. Now, by condition (iv), since $u_n \to u$, $u_{n+1} \to u$ and $u_{n+1} \in fu_n$ we conclude that $u \in fu$. This complete the proof. □

Example 2.5. Suppose $\mathcal{W} = \mathbb{R}^+$, $\mathcal{G}$ and $\mathcal{T}$ are as in Example 1.6. Then $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ is a complete Menger PbM-space with $\alpha = \frac{1}{2}$. Define the mapping $f : \mathcal{W} \to CB(\mathcal{W})$ by $fu = [0, \frac{u}{4}]$, $c : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $c(j,i) = \frac{i-j}{2}$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(i) = i$. Also, let $\gamma(u,v,i) = 1$ for all $u, v \in \mathcal{W}$. To prove that f is a $(\alpha-\psi)$-E-type simulation, it suffices to check the following condition:

$$c(p_{u,v}(i), l_{u,v}(i)) \geq 0,$$

for all $u \in \mathcal{W}$, $v \in fu$ and $o \in fv$ where,

$$\begin{aligned} p_{u,v}(i) \;&=\; -a_1(\gamma(u,v,\alpha^k i)H_{fu,fv}(\alpha^k\psi(i))) - a_2(\mathcal{G}_{u,fu}(\alpha^{k-1}\psi(\frac{i}{c}) + \mathcal{G}_{v,fv}(\alpha^{k-1}\psi(\frac{i}{c})) \\ &\quad - \min[a_3\mathcal{G}_{v,fu}(\alpha^{k-1}\psi(\frac{i}{c}), a_3\mathcal{G}_{u,f}(\alpha^{k-1}\psi(\frac{i}{c})], \end{aligned}$$

and

$$l_{u,v}(i) = a_4\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c})) + a_5\mathcal{G}_{u,o}(\alpha^{k-1}\psi(\frac{i}{c})),$$

for all $u \in fu$ and $v \in fv$. Note that $\gamma(u,v,\alpha^k i)H_{fu,fv}(\alpha^k\psi(i)) = 1.H_{fu,fv}(\alpha^k\psi(i))$. Using the definition of the probabilistic Hausdorff metric in Definition 1.12, we have

$$\begin{aligned} a_1H_{fu,fv}(\alpha^k\psi(i)) &= a_1\frac{\frac{1}{2^k}i}{\frac{1}{2^k}i + |\frac{u}{4} - \frac{v}{4}|^2} = a_1\frac{i}{i + 2^{k-4}|u-v|^2} \\ &\geq a_1\frac{i}{i + 2^{k-2}|u-v|^2} \\ &= a_1\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c})) \\ &\geq -a_4\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c})) - a_5\mathcal{G}_{u,o}(\alpha^{k-1}\psi(\frac{i}{c})), \end{aligned}$$

where, $c = \frac{1}{2}, a_1 = 1, a_4 = -2, a_5 = -1$ and $a_2 = a_3 = 0$. This means that f is $(\alpha-\psi)$-E-type simulation. Since f has the compact values it has w-approximative valued property and it is closed mapping. On the other hands, by definitions of f and γ, all of the conditions of Theorem 3.2 (i)-(iv) are satisfies. Therefore, Theorem 3.2 implies that f has a fixed point.

Corollary 2.6. Suppose $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ is a complete Menger PbM-space with coefficient α, which satisfies $\mathcal{T}(a,a) \geq a$ with $a \in [0,1]$. Also, let $f : \mathcal{W} \to \mathcal{W}$ is a continuous mapping and be satisfying the following conditions:

(i) f is γ-orbital admissible,;

(ii) for some $u_0 \in \mathcal{W}$ there exists $u_1 = f(u_0)$ such that $\gamma(u_0, u_1, i) \leq 1$ for all $i > 0$;

(iii) f is $(\alpha-\psi)$-E-type simulation.

Then f has a fixed point.

3 Results on $(\beta-\psi)$-E-type simulation for multi-valued mappings in PbM-space

Definition 3.1. Suppose $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ is a Menger PbM-space with coefficient α and $f : \mathcal{W} \to CB(\mathcal{W})$ is said $(\beta-\psi)$-E-type simulation if there exist $p, q \geq 0$, $\psi \in \Psi$, $c \in Z$, $\beta : \mathcal{W} \times \mathcal{W} \times [0,\infty) \to [0,\infty)$ such that

$$c(b_{u,v}(i), e_{u,v}(i)) \geq 0,$$

for all $u \in \mathcal{W}$ and $v \in fu$ where

$$b_{u,v}(i) = min[p\mathcal{G}_{u,fv}(\alpha^{k-1}\psi(\frac{i}{c})), q\mathcal{G}_{v,fu}(\alpha^{k-1}\psi(\frac{i}{c}))]$$

and

$$e_{u,v}(i) = \beta(u,v,i)H_{fu,fv}(i) - \min[\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u,fu}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{v,fv}(\alpha^{k-1}\psi(\frac{i}{c}))]$$

where, $k \in \mathbb{N}$ and $c \in (0,1)$.

The following is the main result of this section.

Theorem 3.2. Let $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ be a complete Menger PbM-space with coefficient α, which satisfies $\mathcal{T}(a,a) \geq a$ with $a \in [0,1]$. Also, let $f : \mathcal{W} \to CB(\mathcal{W})$ has the w-approximative value property and be satisfying the following conditions:

(i) f is γ-orbital admissible;

(ii) for some $u_0 \in \mathcal{W}$ there exists $u_1 \in f(u_0)$ such that $\beta(u_0, u_1, i) \leq 1$ for all $i > 0$;

(iii) f is $(\beta - \psi)$-Ehsan type simulation;

(iv) f is closed operator.

Then f has a fixed point.

Proof . If $u_1 = u_0$, then the proof is complete. Let $u_1 \neq u_0$, i.e., $u_0 \notin fu_0$. Since f has w-approximative value property, there exists $u_2 \in fu_1$ such that $\mathcal{G}_{u_1,u_2}(i) \geq H_{fu_0,fu_1}(i)$ for all $i > 0$. For $u_2 \in fu_1$, from (i) and (ii) we have $\beta(u_0, u_1, i) \leq 1$ and $\beta(u_1, u_2, i) \leq 1$ for all $i > 0$. If $u_1 \in fu_1$, then u_1 is a fixed point of f. Suppose that $u_2 \neq u_1$. Again, by the assumptions, there exists $u_3 \in fu_2$ such that $\mathcal{G}_{u_3,u_2}(i) \geq H_{fu_2,fu_1}(i)$, $\beta(u_2, u_1, i) \leq 1$ and $\beta(u_2, u_3, i) \leq 1$ for all $i > 0$. By continuing this process, we obtain a sequence $\{u_n\}$ in $\mathcal{W}$ such that $u_n \in fu_{n-1}$ with $u_n \neq u_{n-1}$, where $\mathcal{G}_{u_n,u_{n+1}}(i) \geq H_{fu_{n-1},fu_n}(i)$, $\beta(u_n, u_{n-1}, i) \leq 1$ and $\beta(u_n, u_{n+1}, i) \leq 1$ for all $i > 0$. Since ψ is continuous at zero. By (iii) we have

$$0 \leq c(b_{u_n,u_{n+1}}(i), e_{u_n,u_{n+1}}(i)) < e_{u_n,u_{n+1}}(i) - b_{u_n,u_{n+1}}(i),$$

where,

$$b_{u_n,u_{n+1}}(i) = \min[p\mathcal{G}_{u_n,fu_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})), q\mathcal{G}_{u_{n+1},fu_n}(\alpha^{k-1}\psi(\frac{i}{c}))]$$

and

$$e_{u_n,u_{n+1}}(i) = \beta(u_n, u_{n+1}, i)H_{fu_n,fu_{n+1}}(i) - \min[\mathcal{G}_{u_n,u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u_n,fu_n}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u_{n+1},fu_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c}))].$$

Since f has w-approximative value property we have,

$$\begin{aligned}
\mathcal{G}_{u_n,u_{n+1}}(i) &\geq \beta(u_{n-1}, u_n, i)H_{fu_{n-1},fu_n}(i) \\
&\geq \min[\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u_{n-1},fu_{n-1}}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u_n,fu_n}(\alpha^{k-1}\psi(\frac{i}{c}))] \\
&\quad + \min[p\mathcal{G}_{u_{n-1},fu_n}(\alpha^{k-1}\psi(\frac{i}{c})), q\mathcal{G}_{u_n,fu_{n-1}}(\alpha^{k-1}\psi(\frac{i}{c}))] \\
&\geq \min[\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u_n,u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c}))] \\
&\quad + \min[p\mathcal{G}_{u_{n-1},u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})), q\mathcal{G}_{u_n,u_n}(\alpha^{k-1}\psi(\frac{i}{c}))] \\
&= \min[\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u_n,u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c}))] + \min[p\mathcal{G}_{u_{n-1},u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c})), q] \\
&\geq \min[\mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u_n,u_{n+1}}(\alpha^{k-1}\psi(\frac{i}{c}))] \\
&\geq \mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})).
\end{aligned}$$

So we have,

$$\mathcal{G}_{u_n,u_{n+1}}(i) \geq \mathcal{G}_{u_{n-1},u_n}(\alpha^{k-1}\psi(\frac{i}{c})) \geq \cdots \geq \mathcal{G}_{u_0,u_1}(\alpha^{k-n}\psi(\frac{i}{c^n})),$$

that is, $\mathcal{G}_{u_n,u_{n+1}}(i) \geq \mathcal{G}_{u_0,u_1}(\alpha^{k-n}\psi(\frac{r}{c^n})$ for arbitrary $n \in \mathbb{N}$. Next, let $m, n \in \mathbb{N}$ with $m > n$. Then, by $(PbM3)$ and strictly increasing of ψ, we have

$$\begin{aligned}\mathcal{G}_{u_n,u_m}((m-n)i) &\geq \min\{\mathcal{G}_{u_n,u_{n+1}}(\alpha i), \cdots, \mathcal{G}_{u_{m-1},u_{m-2}}(\alpha^{m-n-1}i), \mathcal{G}_{u_{m-1},u_m}(\alpha^{m-n-1}i)\}\\ &\geq \min\{\mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^n})), \cdots, \mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^{m-2}})), \mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^{m-1}}))\}\\ &= \mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^n})).\end{aligned}$$

Since $\alpha^{1-n}\psi(\frac{i}{c^n}) \to \infty$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{G}_{u_0,u_1}(\alpha^{1-n}\psi(\frac{i}{c^n})) > 1-\epsilon$ for fixed $\epsilon \in (0,1)$, whenever $n \geq n_0$. This implies that $\mathcal{G}_{u_n,u_m}((m-n)i) > 1-\epsilon$ for every $m > n \geq n_0$. Since $i > 0$ and $\epsilon \in (0,1)$ are arbitrary, we deduce that $\{u_n\}$ is a Cauchy sequence in the complete Menger PbM-space $(\mathcal{W}, \mathcal{G}, \mathcal{T})$. Then there exists $u \in \mathcal{W}$ such that $u_n \to u$ as $n \to \infty$. We are going to show that u is a fixed point of f. Now, by condition (iv), since $u_n \to u$, $u_{n+1} \to u$ and $u_{n+1} \in fu_n$ we conclude that $u \in fu$. This complete the proof. □

Example 3.3. Suppose $\mathcal{W} = \mathbb{R}^+$, $\mathcal{G}$ and $\mathcal{T}$ are as in Example 1.6. Then $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ is a complete Menger PbM-space with $\alpha = \frac{1}{2}$. Define the mapping $f : \mathcal{W} \to CB(\mathcal{W})$ by $fu = [0, \frac{u}{4}]$, $c : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $c(j,i) = \frac{i-j}{2}$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(i) = i$. Also, let $\beta(u,v,i) = 1$ for all $u, v \in \mathcal{W}$. To prove that f is a $(\beta-\psi)$-E-type simulation, it suffices to check the following condition:

$$c(b_{u,v}(i), e_{u,v}(i)) \geq 0,$$

for all $u \in \mathcal{W}$ and $v \in fu$. Note that $\beta(u,v,i)H_{fu,fv}(i) = 1.H_{fu,fv}(i)$. Using the definition of the probabilistic Hausdorff metric in Definition 1.12, we have

$$\begin{aligned}H_{fu,fv}(i) =& \frac{i}{i+|\frac{u}{4}-\frac{v}{4}|^2} = \frac{i}{i+|u-v|^2}\\ \geq& \frac{i}{i+|u-v|^2}\\ =& \mathcal{G}_{u,v}(i)\\ \geq& \min[\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u,fu}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{v,fv}(\alpha^{k-1}\psi(\frac{i}{c}))]\\ \geq& \min[p\mathcal{G}_{u,fv}(\alpha^{k-1}\psi(\frac{i}{c})), q\mathcal{G}_{v,fu}(\alpha^{k-1}\psi(\frac{i}{c}))] + \min[\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u,fu}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{v,fv}(\alpha^{k-1}\psi(\frac{i}{c}))],\end{aligned}$$

where $p, q = 0$. This means that f is $(\beta-\psi)$-E-type simulation. Since f has the compact values it has the w-approximative valued property and it is closed mapping. On the others hands, by definitions of f and γ, all of the conditions of Theorem 3.2 (i)-(iv) are satisfied. Therefore, Theorem 3.2 implies that f has a fixed point.

Corollary 3.4. Suppose $(\mathcal{W}, \mathcal{G}, \mathcal{T})$ is a complete Menger PbM-space with coefficient α, which satisfies $\mathcal{T}(a,a) \geq a$ with $a \in [0,1]$. Also, let $f : \mathcal{W} \to \mathcal{W}$ is a continuous mapping and be satisfying the following conditions:

(i) f is γ-orbital admissible;

(ii) for some $u_0 \in \mathcal{W}$ there exists $u_1 = f(u_0)$ such that $\beta(u_0, u_1, i) \leq 1$, for all $i > 0$;

(iii) f is $(\beta-\psi)$-E-type simulation.

Then f has a fixed point.

4 An application to Volterra integral equation

As an application of our results, we consider the following Volterra integral equation:

$$u(i) = \int_0^i K(i,j,u(j))ds + v(i), \tag{4.1}$$

where $i \in I = [0,1]$, $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $v \in C(I, \mathbb{R})$.

Suppose that $C(I, \mathbb{R})$ is the Banach space of all real continuous functions defined on I with norm $||u||_\infty = \max_{i \in I} |u(i)|$ for all $u \in C(I, \mathbb{R})$ and $C(I \times I \times C(I, \mathbb{R}), \mathbb{R})$ is the space of all continuous functions defined on $I \times I \times C(I, \mathbb{R})$. Alternatively, the Banach space $C(I, \mathbb{R})$ can be endowed with Bielecki norm $||u||_B = \sup_{i \in I}\{|u(i)|e^{-Li}\}$ for all $u \in C(I, \mathbb{R})$ and $L > 0$, and induced b-metric $d(u,v) = ||u - v||_B^2$ for all $u, v \in C(I, \mathbb{R})$. Note that d is complete b-metric with $j = 2$. Define the mapping $\mathcal{G} : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to D^+$ by $\mathcal{G}_{u,v}(i) = \chi(i - d(u,v))$ for $u, v \in C(I, \mathbb{R})$ and $i > 0$, where

$$\chi(i) = \begin{cases} 0 & \text{if} \quad i \leq 0 \\ 1 & \text{if} \quad i > 0. \end{cases}$$

Then the space $(C(I, \mathbb{R}), \mathcal{G}, \mathcal{T})$ with $\mathcal{T}(a,b) = \min\{a,b\}$ is a complete Menger PbM space with coefficient $\alpha = \frac{1}{2}$. ([16]) Also, define $f : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ by

$$fu(i) = \int_0^i K(i,j,u(j))ds + v(i), \quad v \in C(I, \mathbb{R}).$$

Theorem 4.1. Suppose $(C(I, \mathbb{R}), \mathcal{G}, \mathcal{T})$ is the complete PbM-space, $G : C(I, \mathbb{R}) \to CB(C(I, \mathbb{R}))$ be a multi-valued operator such that $G(u) = \{fu(i)\}$ and $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$ be an operator. Suppose that the following conditions hold:

(i) $||K||_\infty = \sup_{i,j \in I,\, u \in C(I,\mathbb{R})} |K(i,j,u(j))| < \infty$;

(ii) for all $u, v \in C(I, \mathbb{R})$ and all $i, j \in I$, there exists $L > 0$ such that

$$||K(i,j,fu(j)) - K(i,j,fv(j))|| \leq L \max\{|u(j) - v(j)|, |u(j) - fu(j)|, |v(j) - fv(j)|\}.$$

Then, the Volterra-type integral equation (4.1) has a solution in $C(I, \mathbb{R})$.

Proof . Suppose $d(u,v) = \max_{i \in I}(|u(i) - v(i)|^2 e^{-Lt})$, for $u, v \in C(I, \mathbb{R})$, where L satisfies condition (ii). Also, by definition of G, for all $u \in Gx$ and $v \in Gy$ we have $u = fu$ and $v = fv$. Thus, we obtain that

$$\begin{aligned} d_B(fu, fv) &\leq \max_{i \in I} \int_0^i |K(i,j,u(j)) - K(i,j,v(j))| e^{L(j-i)} e^{-Ls} ds \\ &\leq L \max\{d_B(u,v), d_B(u,fu), d_B(v,fv)\} \max_{i \in I} \int_0^i e^{L(j-i)} ds \\ &\leq (1 - e^{-aL}) \max\{d_B(u,v), d_B(u,fu), d_B(v,fv)\} \\ &\leq \max\{d_B(u,v), d_B(u,fu), d_B(v,fv)\}, \end{aligned}$$

for all $u, v \in C(I, \mathbb{R})$. Then we have,

$$\begin{aligned} \mathcal{G}_{u,v}(i) = \mathcal{G}_{fu,fv}(i) &= \chi(i - d_B(fu,fv)) \\ &\geq \chi(i - \max\{d_B(u,v), d_B(u,fu), d_B(v,fv)\}) \\ &= \min\{\chi(i - d_B(u,v)), \chi(i - d_B(u,fu)), \chi(i - d_B(v,fv))\} \\ &= \min\{\mathcal{G}_{u,v}(i), \mathcal{G}_{u,fu}(i), \mathcal{G}_{v,fv}(i)\} \\ &\geq \min\{\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u,fu}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{v,fv}(\alpha^{k-1}\psi(\frac{i}{c}))\} \\ &\geq \min\{\mathcal{G}_{u,v}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{u,fu}(\alpha^{k-1}\psi(\frac{i}{c})), \mathcal{G}_{v,fv}(\alpha^{k-1}\psi(\frac{i}{c}))\} + \min\{p\mathcal{G}_{u,fv}(\alpha^{k-1}\psi(\frac{i}{c})), q\mathcal{G}_{v,fu}(\alpha^{k-1}\psi(\frac{i}{c}))\}, \end{aligned}$$

by letting $p, q = 0$ all conditions of Theorem 3.2 are satisfied Therefore, Theorem 3.2 ensures the existence of a fixed point of G such that this fixed point is the solution of the integral equation. □

5 Conclusion

In this paper, motivated by [2, 16], we considered some nonlinear contractions for multi-valued mappings and obtained several fixed point theorems in PbM-spaces. In section 2, by applying the definitions of $(\alpha - \psi)$-E-type simulation function in PbM-space, we established some new fixed point theorems. In section 3, by applying the definitions of $(\beta - \psi)$-E-type simulation function in PbM-space, we established some new fixed point theorems. Finally, we presented an application to the existence of a solution for the Volterra type integral equation.

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