Int. J. Nonlinear Anal. Appl. In Press, 1–11
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.25243.2966

New results on fractional calculus and integral transform with extended Mittag-Leffler type function

Sunil Kumar Sharma[a], Vinod Gill[b], Dinesh Kumar[c,*], Norbert Südland[d]

[a]Department of Mathematics, Arya College of Engineering, RIICO Industrial Area, Kukas, Delhi Road, Jaipur–302028 (Raj.), India

[b]Department of Mathematics, Government College Nalwa (Hisar), Haryana-125037, India

[c]Department of Applied Sciences, College of Agriculture, Agriculture University Jodhpur, Jodhpur–342304 (Raj.), India

[d]Aage GmbH, Röntgenstraße 24, 73431 Aalen (Württemberg), Germany

(Communicated by Choonkil Park)

Abstract

In the present article, we obtain new results, based on an extended Mittag-Leffler type function $E_{u,v}^{(\{k_l\}_{l\in\mathbb{N}_0};\tau)}(\xi;\mathfrak{p})$. We also investigate some integral transforms and generalized integral formulas for this function, and established results are expressed in terms of the Wright generalized hypergeometric type function ${}_{m+1}\psi_{n+1}^{(\{k_l\}_{l\in\mathbb{N}_0})}(\xi;\mathfrak{p})$. Some presumably new and known interesting special cases are also deduced.

Keywords: Generalized Wright hypergeometric function, Extended Mittag-Leffler function, Fractional calculus operators, Integral transforms, Generalized Beta and Gamma functions
2020 MSC: Primary: 33E12, 26A33, 33B15, 33C20; Secondary: 33C90, 33C70, 44A20

1 Introduction and Preliminaries

Recently, several mathematicians and researchers established different generalizations of Gamma functions, Beta functions, Mittag-Leffler functions, Bessel functions, Wright functions, hypergeometric functions, etc. by adding some extra parameters to their series and integral repetitions. For the generalizations, properties and applications of these functions, the reader can refer to the recent work, done by many researchers (see [3, 12, 14, 16, 17, 23, 31, 33]). Throughout this article, let $\mathbb{Z},\mathbb{C},\mathbb{R},\mathbb{R}_+$, and $\mathbb{N}$ be the sets of integers, complex numbers, real numbers, positive real numbers and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

We recall, that the Wright type hypergeometric function is defined by the following series (see [34, 35]):

$$ {}_p\psi_q(\omega) = {}_p\psi_q\left[\begin{array}{c}(\alpha_i,A_i)_{1,p}\\(\beta_j,B_j)_{1,q}\end{array};\omega\right] = \sum_{r=0}^{\infty}\frac{\Gamma(\alpha_1+A_1r)\cdots\Gamma(\alpha_p+A_pr)}{\Gamma(\beta_1+B_1r)\cdots\Gamma(\beta_q+B_qr)}\,\frac{\omega^r}{r!}, \tag{1.1}$$

*Corresponding author
Email addresses: sunil94147@gmail.com (Sunil Kumar Sharma), vinod.gill08@gmail.com (Vinod Gill), dinesh_dino03@yahoo.com (Dinesh Kumar), norbert.suedland@aage-leichtbauteile.de (Norbert Südland)

Received: November 2021 *Accepted:* August 2024

where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$. Recently, Sharma and Devi [29] defined an extended Wright type function, given as:

$$
\begin{aligned}
{}_{m+1}\psi_{n+1}(\xi;\mathfrak{p}) &= {}_{m+1}\psi_{n+1}\left[\begin{array}{l}(\alpha_i, A_i)_{1,m}, (\lambda, 1)\\ (\beta_j, B_j)_{1,n}, (d, 1)\end{array}\middle|(\xi;\mathfrak{p})\right]\\
&= \sum_{r=0}^{\infty}\frac{\Gamma(\alpha_1 + A_1 r)\cdots\Gamma(\alpha_m + A_m r)}{\Gamma(\beta_1 + B_1 r)\cdots\Gamma(\beta_n + B_n r)}\,\frac{B_p(\lambda + r, d - \lambda;\mathfrak{p})}{\Gamma(d-\lambda)}\,\frac{\xi^r}{r!},
\end{aligned}
\tag{1.2}
$$

where $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$, $\Re(p) > 0$, $\Re(d) > \Re(\lambda) > 0$ and $\mathfrak{p} \geq 0$. In 2018, Agarwal et al. [2] defined a potentially new extension of the Wright hypergeometric function, given by:

$$
\begin{aligned}
{}_{m+1}\psi_{n+1}^{(\{k_l\}_{l\in\mathbb{N}_0})}(\xi;\mathfrak{p}) &= {}_{m+1}\psi_{n+1}^{(\{k_l\}_{l\in\mathbb{N}_0})}\left[\begin{array}{l}(\alpha_i, A_i)_{1,m}, (\lambda, 1)\\ (\beta_j, B_j)_{1,n}, (d, 1)\end{array}\middle|(\xi;\mathfrak{p})\right]\\
&= \frac{1}{\Gamma(d-\lambda)}\sum_{r=0}^{\infty}\frac{\prod_i^m\Gamma(\alpha_i + A_i r)}{\prod_j^n\Gamma(\beta_j + B_j r)}\,B_p^{(\{k_l\}_{l\in\mathbb{N}_0})}(\lambda + r, d - \lambda;\mathfrak{p})\,\frac{\xi^r}{r!},
\end{aligned}
\tag{1.3}
$$

where, $\xi, \lambda \in \mathbb{C}$, $\Re(d) > \Re(\lambda)$, $\mathfrak{p} \geq 0$. If we take $k_l = 1$, then (1.3) reduces to the result (1.2), due to Sharma and Devi [29].

In recent years, many researchers have investigated the importance and great consideration of the Mittag-Leffler function in the theory of special functions and fractional calculus for discovering a generalization and some applications (see [6, 7, 11, 13, 24, 28]).

Parmar [24] introduced an extended Mittag-Leffler type function, given as follows:

$$
E_{u,v}^{(\{k_l\}_{l\in\mathbb{N}_0};\tau)}(\xi;\mathfrak{p}) = \sum_{r=0}^{\infty}\frac{B^{(\{k_l\}_{l\in\mathbb{N}_0})}(\tau + r, 1 - \tau;\mathfrak{p})}{B(\tau, 1-\tau)}\,\frac{\xi^r}{\Gamma(ur + v)},
\tag{1.4}
$$

where $B^{(\{k_l\}_{l\in\mathbb{N}_0})}(\tau + r, 1 - \tau; p)$ is the extended Beta function, defined by [33], in the integral form given as:

$$
\begin{aligned}
B^{(\{k_l\}_{l\in\mathbb{N}_0})}(a, b;\mathfrak{p}) &= \int_0^1 t^{a-1}(1-t)^{b-1}\,\Theta\left(\{k_l\}_{l\in\mathbb{N}_0};\frac{-\mathfrak{p}}{t(1-t)}\right)\mathrm{d}t\\
&\quad(\min\{\Re(a), \Re(b)\} > 0;\ \mathfrak{p} \geq 0),
\end{aligned}
\tag{1.5}
$$

with $\Theta\left(\{k_l\}_{l\in\mathbb{N}_0}; y\right)$ being a function of an advisably bounded sequence $\{k_l\}_{l\in\mathbb{N}_0}$ of arbitrary real or complex numbers, given as follows:

$$
\Theta\left(\{k_l\}_{l\in\mathbb{N}_0}; y\right)\begin{cases}\sum_{\ell=0}^{\infty}\{k_\ell\}_{\ell\in\mathbb{N}_0}\frac{y^l}{\ell!} & (|y| < \mathcal{R},\ 0 < \mathcal{R} < \infty;\ k_0 = 1)\\ M_0 y^{\omega}\exp(y)\left[1 + O\left(\frac{1}{y}\right)\right], & (\Re(y) \to \infty;\ M_0 > 0;\ \omega \in \mathbb{C})\end{cases},
\tag{1.6}
$$

here, $\mathcal{R}$ is the radius of convergence for some suitable constants M_0 and ω, depending essentially on the sequence $\{k_\ell\}_{\ell\in\mathbb{N}_0}$.

Remark 1.1. If we set $k_\ell = \frac{(\gamma)_\ell}{(\delta)_\ell}$ $(\ell \in \mathbb{N}_0)$, then (1.4) reduces to the following extended generalized Mittag-Leffler function [24, eqn. (17), p. 1072]:

$$
\begin{aligned}
E_{u,v}^{(\gamma,\delta);\tau}(\xi;\mathfrak{p}) &= \sum_{r=0}^{\infty}\frac{B^{(\gamma,\delta)}(\tau + r, 1 - \tau;\mathfrak{p})}{B(\tau, 1-\tau)}\,\frac{\xi^r}{\Gamma(ur + v)}\\
&\quad(\xi, v, \tau \in \mathbb{C},\ \Re(\gamma) > 0, \Re(\delta) > 0, \Re(u) > 0, \Re(v) > 0, \Re(\tau) > 1;\ \mathfrak{p} \geq 0).
\end{aligned}
\tag{1.7}
$$

Remark 1.2. If we set $k_\ell = 1$ $(\ell \in \mathbb{N}_0)$ in (1.4) (see [22] with $c = 1$), then we get the following result, given by:

$$
E_{u,v}^{\tau}(\xi;\mathfrak{p}) = \sum_{r=0}^{\infty}\frac{B(\tau + r, 1 - \tau;\mathfrak{p})}{B(\tau, 1-\tau)}\,\frac{\xi^r}{\Gamma(ur + v)}\quad(\xi, v, \tau \in \mathbb{C},\ \Re(u) > 0, \Re(v) > 0, \Re(\tau) > 1;\ \mathfrak{p} \geq 0).
\tag{1.8}
$$

Remark 1.3. If we take $\mathfrak{p}=0$, then (1.8) reduces to the Prabhakar-type Mittag-Leffler function [25], given as:

$$E_{u,v}^{\tau}(\xi)=\sum_{r=0}^{\infty}\frac{(\tau)_r}{\Gamma(ur+v)}\frac{\xi^r}{r!}\qquad(\xi,v\in\mathbb{C},\ \Re(u)>0,\Re(v)>0),\tag{1.9}$$

where $(\tau)_r$ denotes the well-known Pochhamer symbol, which is defined by:

$$(\tau)_r=\begin{cases}1 & (r=0)\\ \tau(\tau+1)\cdots(\tau+r-1) & (r\in\mathbb{N},\ \tau\in\mathbb{C})\end{cases}$$

In fact, the following special cases are also satisfied:

$$E_{u,v}^{1}(\xi)=E_{u,v}(\xi);\ E_{u,1}^{1}(\xi)=E_u(\xi),\tag{1.10}$$

where the Mittag-Leffler functions $E_u(\xi)$ and $E_{u,v}(\xi)$ are defined by the following series:

$$E_u(\xi)=\sum_{r=0}^{\infty}\frac{\xi^r}{\Gamma(ur+1)}\qquad(\xi\in\mathbb{C},\ \Re(u)>0),\tag{1.11}$$

and

$$E_{u,v}(\xi)=\sum_{r=0}^{\infty}\frac{\xi^r}{\Gamma(ur+v)}\qquad(\xi,v\in\mathbb{C},\ \Re(u)>0).\tag{1.12}$$

2 Fractional calculus operators

In this section, we need to recall some fractional integral and fractional derivative operators. For $\xi>0,\ \alpha,\beta,\rho\in\mathbb{C}$ and $\Re(\alpha)>0$, the left-hand sided and right-hand sided Saigo hypergeometric fractional integral operators are defined as (see [10, 21, 26, 27]):

$$\left(I_{0,\xi}^{\alpha,\beta,\rho}f(t)\right)(\xi)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(\xi-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\rho;\alpha;1-\frac{t}{\xi}\right)f(t)\,\mathrm{d}t,\tag{2.1}$$

and

$$\left(J_{\xi,\infty}^{\alpha,\beta,\rho}f(t)\right)(\xi)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-\xi)^{\alpha-1}\,t^{\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\rho;\alpha;1-\frac{\xi}{t}\right)f(t)\,\mathrm{d}t,\tag{2.2}$$

respectively. The Riemann-Liouville $\mathbb{R}_{0,\xi}^{\alpha}(.)$ and the Erdélyi-Kober $\mathbb{E}_{0,\xi}^{\alpha,\rho}(.)$ fractional integral operators are special cases of the left-hand sided Saigo fractional integral operator by means of the following relationships:

$$\left(\mathbb{R}_{0,\xi}^{\alpha}f(t)\right)(\xi)=\left(I_{0,\xi}^{\alpha,-\alpha,\rho}f(t)\right)(\xi)=\frac{1}{\Gamma(\alpha)}\int_0^x(\xi-t)^{\alpha-1}f(t)\,\mathrm{d}t\quad\text{(Riemann-Liouville)},\tag{2.3}$$

$$\left(\mathbb{E}_{0,\xi}^{\alpha,\rho}f(t)\right)(\xi)=\left(I_{0,\xi}^{\alpha,0,\rho}f(t)\right)(\xi)=\frac{\xi^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(\xi-t)^{\alpha-1}t^{\rho}f(t)\,\mathrm{d}t\quad\text{(Erdélyi-Kober)},\tag{2.4}$$

Also, the Weyl and the Erdélyi-Kober fractional operators are particular cases of right-hand sided Saigo fractional integral operators, given as follows:

$$\left(\mathbb{W}_{\xi,\infty}^{\alpha}f(t)\right)(\xi)=\left(\mathbb{J}_{\xi,\infty}^{\alpha,-\alpha,\rho}f(t)\right)(\xi)=\frac{1}{\Gamma(\alpha)}\int_0^x(t-\xi)^{\alpha-1}f(t)\,\mathrm{d}t\quad\text{(Weyl)},\tag{2.5}$$

$$\left(\mathbb{K}_{\xi,\infty}^{\alpha,\rho}f(t)\right)(\xi)=\left(\mathbb{J}_{\xi,\infty}^{\alpha,0,\rho}f(t)\right)(\xi)=\frac{\xi^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(t-\xi)^{\alpha-1}t^{-\alpha-\rho}f(t)\,\mathrm{d}t\ \text{(Erdélyi-Kober)}.\tag{2.6}$$

For the continuation of our study, we recall the following lemmas, proven by Saigo [26], as mentioned below:

Lemma 2.1. Let $\alpha,\beta,\rho,\delta\in\mathbb{C}$, $\xi>0$ be such that $\Re(\alpha)>0$, $\Re(\delta)>\max[0,\Re(\beta-\rho)]$. Then we have:

$$\left(I_{0,\xi}^{\alpha,\beta,\rho}t^{\delta-1}\right)(\xi)=\frac{\Gamma(\delta)\Gamma(\delta-\beta+\rho)}{\Gamma(\delta-\beta)\Gamma(\delta+\alpha+\rho)}\xi^{\delta-\beta-1}.\tag{2.7}$$

Lemma 2.2. $\alpha,\beta,\rho,\delta\in\mathbb{C}$, $\xi>0$ and $\Re(\alpha)>0, \Re(\delta)<1+\max[\Re(\beta),\Re(\rho)]$, then

$$\left(J_{\xi,\infty}^{\alpha,\beta,\rho}t^{\delta-1}\right)(\xi)=\frac{\Gamma(\beta-\delta+1)\Gamma(\rho-\delta+1)}{\Gamma(1-\delta)\Gamma(\beta+\alpha-\delta+\rho+1)}\,\xi^{\delta-\beta-1}. \tag{2.8}$$

Consider $\alpha,\beta,\rho\in\mathbb{C}$ and $\xi>0$, the left-hand sided and right-hand sided Saigo fractional derivative operators are given as [26]:

$$\begin{gathered}\left(D_{0+}^{\alpha,\beta,\rho}f\right)(\xi)=\left(I_{0+}^{-\alpha,-\beta,\alpha+\rho}f\right)(\xi)=\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^m\left(I_{0,\xi}^{-\alpha+\rho,-\beta-\rho,\alpha+\rho-m}f\right)(\xi),\\ (\Re(\alpha)\geq 0,\ m=\max[\Re(\alpha)]+1),\end{gathered} \tag{2.9}$$

$$\begin{gathered}\left(D_{-}^{\alpha,\beta,\rho}f\right)(\xi)=\left(I_{-}^{-\alpha,-\beta,\alpha+\rho}f\right)(\xi)=\left(-\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^m\left(I_{-}^{-\alpha+\rho,-\beta-\rho,\alpha+\rho-m}f\right)(\xi),\\ (\Re(\alpha)\geq 0,\ m=\max[\Re(\rho)]+1).\end{gathered} \tag{2.10}$$

In particular, the Saigo fractional derivative operators, containing the Riemann-Liouville and Weyl fractional derivatives, are connected by the following relations:

$$\left(D_{0+}^{\alpha,-\alpha,\rho}f\right)(\xi)=\left(D_{0+}^{\alpha}f\right)(\xi)=\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^m\frac{1}{\Gamma(m-\alpha)}\int_0^x\frac{f(t)\,\mathrm{d}t}{(\xi-t)^{\alpha-m+1}} \tag{2.11}$$

$$\left(D_{0-}^{\alpha,-\alpha,\rho}f\right)(\xi)=\left(D_{-}^{\alpha}f\right)(\xi)=\left(-\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^m\frac{1}{\Gamma(m-\alpha)}\int_0^x\frac{f(t)\,\mathrm{d}t}{(\xi-t)^{\alpha-m+1}}, \tag{2.12}$$

for $\xi>0$, $m=[\Re(\alpha)]+1$, $\alpha\in\mathbb{C}$, $\Re(\alpha)\geq 0$. The following Saigo fractional derivative operators comprise the Erdélyi-Kobar fractional derivative operators (Kiryakova [9]), given as:

Lemma 2.3. Assume that $\alpha,\beta,\rho,\delta\in\mathbb{C}$, $\xi>0$, $\Re(\alpha)>0$ be such that $\Re(\delta)>-\min[0,\Re(\alpha+\beta+\rho)]$, then the following result holds true

$$\left(D_{0+}^{\alpha,\beta,\rho}t^{\delta-1}\right)(\xi)=\frac{\Gamma(\delta)\ \Gamma(\delta+\alpha+\beta+\rho)}{\Gamma(\delta+\beta)\ \Gamma(\delta+\rho)}\,\xi^{\delta+\beta-1}. \tag{2.13}$$

Lemma 2.4. Assume that $\alpha,\beta,\rho,\delta\in\mathbb{C}$, $\xi>0$, $\Re(\alpha)>0$ along with $\Re(\delta)<1+\min[\Re(-\beta-\rho),\Re(\alpha+\rho)]$, then the following result holds

$$\left(D_{-}^{\alpha,\beta,\rho}t^{\delta-1}\right)(\xi)=\frac{\Gamma(1-\delta-\beta)\Gamma(1-\delta+\alpha+\rho)}{\Gamma(1-\delta)\Gamma(1-\delta+\rho-\beta)}\,\xi^{\delta+\beta-1}. \tag{2.14}$$

3 Fractional integral formulas involving the extended Mittag-Leffler type function

In this section, we give fractional integral formulas for the extended Mittag-Leffler type function.

Theorem 3.1. Let $\xi>0$, $\alpha,\beta,\rho,\delta,\mu,v,\tau,\mathfrak{p}\in\mathbb{C}\,(\Re(\mu)>0,\ \Re(v)>0,\ \Re(\tau)>0,\ \mathfrak{p}\geq 0)$ be parameters such that $\Re(\delta)>\max\{0,\Re(\beta-\rho)\}$. Then the following formula holds:

$$\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}(\xi;\mathfrak{p})\right\}\right)(x)=\frac{x^{\delta-\beta-1}}{\Gamma(\tau)}\,{}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{matrix}(\delta,1),(\rho+\delta-\beta,1),(\tau,1)\\(\delta-\beta,1),(\delta+\alpha+\rho,1),(\mu,v)\end{matrix}\middle|(x;\mathfrak{p})\right]. \tag{3.1}$$

Proof . By applying the definition of the extended Mittag-Leffler type function (1.4), defined by Parmar [24], and the fractional integral formula (2.1), we get:

$$\varphi=\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}(\xi;\mathfrak{p})\right\}\right)(x)=\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\sum_{r=0}^{\infty}\frac{B^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}(\tau+r,1-\tau;\mathfrak{p})}{B(\tau,1-\tau)}\frac{\xi^r}{\Gamma(ur+v)}\right\}\right)(x).$$

Now, interchanging the order of integration and summations, finding the inner integral by applying lemma 2.1 and the well-known Beta integral formula, we yield:

$$\varphi=\frac{x^{\delta-\beta-1}}{\Gamma(\tau)}\sum_{r=0}^{\infty}\frac{B^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}(\tau+r,1-\tau;\mathfrak{p})}{\Gamma(1-\tau)}\frac{\xi^r}{\Gamma(ur+v)}\frac{\Gamma(\delta+r)\ \Gamma(\delta+r-\beta+\rho)}{\Gamma(\delta+r-\beta)\ \Gamma(\delta+r+\alpha+\rho)}.$$

Next, by the virtue of definition (1.2), defined by Sharma and Devi [29], we receive the required result:

$$\varphi = \frac{x^{\delta-\beta-1}}{\Gamma(\tau)}\, {}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (\delta,1), (\rho+\delta-\beta,1), (\tau,1) \\ (\delta-\beta,1), (\delta+\alpha+\rho,1), (\mu,v) \end{matrix} \middle| (x;\mathfrak{p}) \right].$$

□

Theorem 3.2. Let $\xi > 0, \alpha, \beta, \rho, \delta, \mu, v, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \mathfrak{p} \geq 0)$ be parameters such that $\Re(\delta) > 1 + \min\{\Re(\beta), \Re(\rho)\}$. Then, we get the following fractional integral formula:

$$\left(J_{x,\infty}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1} E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta-\beta-1}}{\Gamma(\tau)}\, {}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (\beta-\delta+1,1), (\rho-\delta+1,1), (\tau,1) \\ (1-\delta,1), (\alpha+\beta-\delta+\rho+1,1), (\mu,v) \end{matrix} \middle| \left(\frac{1}{x};\mathfrak{p}\right) \right]. \tag{3.2}$$

Proof . By using suitable facts and formulas, given in this paper, the proof of this theorem would run analogously to the one of theorem 3.1. □ Setting $\beta = 0$ in theorems 3.1 and 3.2, we get interesting results, asserted by the following corollaries:

Corollary 3.3. Let $\xi > 0$, $\alpha, \rho, \delta, \mu, v, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \mathfrak{p} \geq 0)$ be parameters such that $\Re(\delta) > \Re(-\rho)$. Then the following Erdélyi-Kober fractional integral formula holds true:

$$\left(\mathbb{E}_{0,x}^{\alpha,\rho}\left\{\xi^{\delta-1} E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}(\xi;\mathfrak{p})\right\}\right)(x) = \frac{x^{\delta-1}}{\Gamma(\tau)}\, {}_2\psi_2^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (\rho+\delta,1), (\tau,1) \\ (\delta+\alpha+\rho,1), (\mu,v) \end{matrix} \middle| (x;\mathfrak{p}) \right]. \tag{3.3}$$

Corollary 3.4. Suppose $\xi > 0$, $\alpha, \rho, \delta, \mu, v, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \mathfrak{p} \geq 0)$ such that $\Re(\delta) < \Re(\rho)$, then we have the following Erdélyi-Kober fractional integral formula, given as follows:

$$\left(\mathbb{K}_{x,\infty}^{\alpha,\rho}\left\{\xi^{\delta-1} E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta-1}}{\Gamma(\tau)}\, {}_2\psi_2^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (\rho-\delta+1,\ 1), (\tau,1) \\ (\alpha+\rho-\delta+1,\ 1), (\mu,v) \end{matrix} \middle| \left(\frac{1}{x};\mathfrak{p}\right) \right]. \tag{3.4}$$

Further, by replacing β by $-\alpha$ in theorems 3.1 and 3.2, we get the Riemann-Liouville fractional integrals of the extended Mittag-Leffler type functions, given in the next following two corollaries.

Corollary 3.5. Assume that $\xi > 0$, $\{\alpha, \rho, \delta, \mu, v, \tau, \mathfrak{p}\} \subset \mathbb{C}$ with $\Re(\mu) > 0$, $\Re(v) > 0$, $\Re(\tau) > 0$, and $\mathfrak{p} \geq 0$, such that $\min\{\Re(\delta), \Re(\alpha)\} > 0$, then the following Riemann-Liouville fractional integral formula holds true:

$$\left(\mathbb{R}_{0,x}^{\alpha}\left\{\xi^{\delta-1} E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}(\xi;\mathfrak{p})\right\}\right)(x) = \frac{x^{\delta+\alpha-1}}{\Gamma(\tau)}\, {}_2\psi_2^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (1-\alpha-\delta,1), (\tau,1) \\ (1-\delta,1), (\mu,v) \end{matrix} \middle| (x;\mathfrak{p}) \right]. \tag{3.5}$$

Corollary 3.6. Let $\xi > 0, \alpha, \rho, \delta, \mu, v, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \mathfrak{p} \geq 0)$ such that $\min\{\Re(\delta), \Re(\rho)\}$, then we get the following Weyl fractional integral formula, given as:

$$\left(\mathbb{W}_{x,\infty}^{\alpha,\rho}\left\{\xi^{\delta-1} E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta+\alpha-1}}{\Gamma(\tau)}\, {}_2\psi_2^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (1-\alpha-\delta,1), (\tau,1) \\ (1-\delta,1), (\mu,v) \end{matrix} \middle| \left(\frac{1}{x};\mathfrak{p}\right) \right]. \tag{3.6}$$

4 Fractional derivative formulas involving the extended Mittag-Leffler type function

In this section, we establish the fractional derivative formulas for the extended Mittag-Leffler type function.

Theorem 4.1. Suppose that $\alpha, \beta, \rho, \delta, \mu, v, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \mathfrak{p} \geq 0)$ and $\xi > 0, \Re(\alpha) \geq 0$ be parameters such that $\Re(\delta) > -\min\{0, \Re(\alpha+\beta+\rho)\}$ and $\min\{\Re(\delta), \Re(\beta)\} > 0$. Then the following formula holds true:

$$\left(D_{0,+}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1} E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}(\xi;\mathfrak{p})\right\}\right)(x) = \frac{x^{\delta+\beta-1}}{\Gamma(\tau)}\, {}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)} \left[\begin{matrix} (\delta,1), (\rho+\delta+\beta+\delta,1), (\tau,1) \\ (\beta+\delta,1), (\rho+\delta,1), (\mu,v) \end{matrix} \middle| (x;\mathfrak{p}) \right]. \tag{4.1}$$

Proof . As suitable, we denote the left-hand side of (4.1) by Ω. Using the definition (1.4) and the fractional derivative formula (2.9), the left-hand side of (4.1) is reduced to:

$$\Omega = \left(D_{0,+}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\xi;\mathfrak{p}\right)\right\}\right)(x)$$

or:

$$\Omega = \sum_{r=0}^{\infty}\frac{B^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left(\tau+r,\ 1-\tau;\mathfrak{p}\right)}{B\left(\tau,\ 1-\tau\right)}\frac{1}{\Gamma\left(ur+v\right)}\left(D_{0,+}^{\alpha,\beta,\rho}\xi^{\delta+r-1}\right)(x),$$

by applying the result (2.13) and after simplification, the above equation reduces to:

$$\Omega = \frac{x^{\delta+\beta-1}}{\Gamma\left(\tau\right)}\sum_{r=0}^{\infty}\frac{B^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left(\tau+r,1-\tau;\mathfrak{p}\right)}{\Gamma\left(1-\tau\right)}\frac{\xi^r}{\Gamma\left(ur+v\right)}\frac{\Gamma\left(\delta+r\right)\Gamma\left(\delta+r+\alpha+\beta+\rho\right)}{\Gamma\left(\delta+r+\beta\right)\Gamma\left(\delta+r+\rho\right)}.$$

Next, by the virtue of definition (1.2) we get:

$$\Omega = \frac{x^{\delta+\beta-1}}{\Gamma\left(\tau\right)}\,{}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}\left(\delta,1\right),\left(\rho+\delta+\beta+\delta,1\right),\left(\tau,1\right)\\ \left(\beta+\delta,1\right),\left(\rho+\delta,1\right),\left(\mu,v\right)\end{array}\middle|\left(x;\mathfrak{p}\right)\right].$$

This completes the proof of the theorem. □

Theorem 4.2. Assume, that $\alpha,\beta,\rho,\delta,\mu,v,\tau,\mathfrak{p}\in\mathbb{C}$, $\xi>0$, $(\Re(\mu)>0,\Re(v)>0,\Re(\tau)>0,\ \ \mathfrak{p}\geq 0,\Re(\alpha)\geq 0)$ be parameters along with the conditions $\Re(\delta)<1+\min\{\Re(-\beta-\rho),\ \Re(\alpha+\beta)\}$ and $\min\{\Re(\delta),\Re(\beta)\}>0$. Then the following fractional derivative formula holds true:

$$\left(D_{0-}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta+\beta-1}}{\Gamma\left(\tau\right)}\,{}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}\left(1-\beta-\delta,1\right),\left(1+\alpha+\rho-\delta,1\right),\left(\tau,1\right)\\ \left(1-\delta,1\right),\left(1+\rho-\beta-\delta,1\right),\left(\mu,v\right)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right]. \tag{4.2}$$

Proof . By making use of the operator (2.10) and the result (2.14) into account, we can readily prove the desired result (4.2). □ By setting $\beta=0$ in theorems 4.1 and 4.2, we obtain certain results, asserted by the following corollaries:

Corollary 4.3. Let $\alpha,\rho,\delta,\mu,v,\tau,\mathfrak{p}\in\mathbb{C}$, $\xi>0$, $(\Re(\mu)>0,\Re(v)>0,\Re(\tau)>0,\mathfrak{p}\geq 0)$, such that $\Re(\delta)>-\min\{0,\Re(\alpha+\rho)\}$. Then the right-hand sided Erdélyi-Kober fractional derivative of the extended Mittag-Leffler type functions is given by:

$$\left(D_{0+}^{\alpha,0,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\xi;\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta-1}}{\Gamma\left(\tau\right)}\,{}_2\psi_2^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}\left(\alpha+\rho+\delta,1\right),\left(\tau,1\right)\\ \left(\rho+\delta,1\right),\left(\mu,v\right)\end{array}\middle|\left(x;\mathfrak{p}\right)\right]. \tag{4.3}$$

Corollary 4.4. Let $\alpha,\rho,\delta,\mu,v,\tau,\mathfrak{p}\in\mathbb{C}$, $\xi>0$, $(\Re(\mu)>0,\Re(v)>0,\Re(\tau)>0,\mathfrak{p}\geq 0)$, such that

$$\Re(\delta)<1+\min\{\Re(\delta),\Re(\alpha+\rho)\}.$$

Then the left-hand sided Erdélyi-Kober fractional derivative of the extended Mittag-Leffler type functions is given as follows:

$$\left(D_{0-}^{\alpha,0,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta-1}}{\Gamma\left(\tau\right)}\,{}_2\psi_2^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}\left(1+\alpha+\rho-\delta,1\right),\left(\tau,1\right)\\ \left(1+\rho-\delta,1\right),\left(\mu,v\right)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right]. \tag{4.4}$$

Further, if we replace β by $-\alpha$ in (4.1) and (4.2), then we get the following results:

Corollary 4.5. Let $\alpha,\rho,\delta,\mu,v,\tau,\mathfrak{p}\in\mathbb{C},\xi>0$, $(\Re(\mu)>0,\Re(v)>0,\Re(\tau)>0,\ \mathfrak{p}\geq 0)$, such that $\min\{\Re(\delta),\Re(\alpha)\}$. Then, we get the following right-hand sided Riemann-Liouville fractional derivative results, given as:

$$\left(D_{0+}^{\alpha,-\alpha,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\xi;\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta-\alpha-1}}{\Gamma\left(\tau\right)}\,{}_2\psi_2^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}\left(\delta,1\right),\left(\tau,1\right)\\ \left(\delta-\alpha,1\right),\left(\mu,v\right)\end{array}\middle|\left(x;\mathfrak{p}\right)\right]. \tag{4.5}$$

Corollary 4.6. Let $\xi>0$, $\alpha,\rho,\delta,\mu,v,\tau,\mathfrak{p}\in\mathbb{C}$, $(\Re(\mu)>0,\Re(v)>0,\Re(\tau)>0,\mathfrak{p}\geq 0)$, such that

$$\Re(\delta)<1+\min\{\Re(\alpha-\rho),\Re(\alpha+\rho)\},\min\{\Re(\delta),\Re(\alpha)\}.$$

Then we obtain the following left-hand sided Riemann-Liouville fractional derivative formula, given by:

$$\left(D_{0-}^{\alpha,-\alpha,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in\mathbb{N}_0};\tau\right)}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta-\alpha-1}}{\Gamma\left(\tau\right)}\,{}_2\psi_2^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}\left(1+\alpha-\delta,1\right),\left(\tau,1\right)\\ \left(1-\delta,1\right),\left(\mu,v\right)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right]. \tag{4.6}$$

5 Special cases and applications

The results in theorems 3.1, 3.2, 4.1 and 4.2 can be readily specialized to develop the corresponding formulas, involving Mittag-Leffler type function, extended confluent hypergeometric function, etc. Choosing $k_l = \frac{(\eta)_l}{(\lambda)_l}$ in theorems 3.1 and 3.2, we get the following examples:

Example 5.1. Let $\xi > 0, \alpha, \beta, \rho, \delta, \mu, v, \tau, \lambda, \eta, \mathfrak{p} \in \mathbb{C}$ $(\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \quad \mathfrak{p} \geq 0)$ be such that $\Re(\delta) > \max\{0, \Re(\beta - \rho)\}$. Then the following formula holds true:

$$\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\, E_{\mu,v}^{(\eta,\lambda);\tau}(\xi;\mathfrak{p})\right\}\right)(x) = \frac{x^{\delta-\beta-1}\Gamma(\lambda)}{\Gamma(\tau)\Gamma(\eta)}\, {}_4\psi_4\left[\begin{array}{c}(\delta,1),(\rho+\delta-\beta,1),(\tau,1),(\eta,1)\\(\delta-\beta,1),(\delta+\alpha+\rho,1),(\mu,v),(\lambda,1)\end{array}\middle|(x;\mathfrak{p})\right]. \tag{5.1}$$

Example 5.2. Let $\xi > 0, \alpha, \beta, \rho, \delta, \mu, v, \tau, \lambda, \eta, \mathfrak{p} \in \mathbb{C}$ $(\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \quad \mathfrak{p} \geq 0)$ be parameters such that $\Re(\delta) > 1 + \min\{\Re(\beta), \Re(\rho)\}$. Then we obtain:

$$\left(J_{x,\infty}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\, E_{\mu,v}^{(\eta,\lambda);\tau}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta+\beta-1}\Gamma(\lambda)}{\Gamma(\tau)\Gamma(\eta)}\, {}_4\psi_4\left[\begin{array}{c}(\beta-\delta+1,1),(\rho-\delta+1,1),(\tau,1),(\eta,1)\\(1-\delta,1),(\alpha+\beta-\delta+\rho+1,1),(\mu,v),(\lambda,1)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right]. \tag{5.2}$$

Example 5.3. Suppose that $\alpha, \beta, \rho, \delta, \mu, v, \tau, \lambda, \eta, \mathfrak{p} \in \mathbb{C}$ $(\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \quad \mathfrak{p} \geq 0)$, $\xi > 0$, $\Re(\alpha) \geq 0$ be parameters such that $Re(\delta) > -\min\{0, \Re(\alpha+\beta+\rho)\}$ and $\min\{\Re(\delta), \Re(\beta)\} > 0$. Then the following formula holds:

$$\left(D_{0+}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\, E_{\mu,v}^{(\eta,\lambda);\tau}(\xi;\mathfrak{p})\right\}\right)(x) = \frac{x^{\delta+\beta-1}\Gamma(\lambda)}{\Gamma(\tau)\Gamma(\eta)}\, {}_4\psi_4\left[\begin{array}{c}(\delta,1),(\rho+\delta+\beta+\delta,1),(\tau,1),(\eta,1)\\(\beta+\delta,1),(\rho+\delta,1),(\mu,v),(\lambda,1)\end{array}\middle|(x;\mathfrak{p})\right]. \tag{5.3}$$

Example 5.4. Consider that $\alpha, \beta, \rho, \delta, \mu, v, \tau, \lambda, \eta, \mathfrak{p}, \xi > 0, (\Re(\mu) > 0, \Re(v) > 0, \Re(\tau) > 0, \quad \mathfrak{p} \geq 0, \Re(\alpha) \geq 0)$ be parameters along with the conditions $\Re(\delta) < 1 + \min\{\Re(-\beta-\rho), \quad \Re(\alpha+\beta)\}$ and $\min\{\Re(\delta), \Re(\beta)\} > 0$. Then we receive the following fractional derivative formula:

$$\begin{aligned}&\left(D_{0-}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\, E_{\mu,v}^{(\eta,\lambda);\tau}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x)\\&= \frac{x^{\delta+\beta-1}\Gamma(\lambda)}{\Gamma(\tau)\Gamma(\eta)}\, {}_4\psi_4\left[\begin{array}{c}(1-\beta-\delta,1),(1+\alpha+\rho-\delta,1),(\tau,1),(\eta,1)\\(1-\delta,1),(1+\rho-\beta-\delta,1),(\mu,v),(\lambda,1)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right].\end{aligned} \tag{5.4}$$

Next, if we take $k_l = 1$ with $\Re(\mu) > 0$ and $\Re(\rho) > 0$, then the theorems 3.1, 3.2, 4.1 and 4.2 reduce as follows:

Example 5.5. With all assumptions and conditions on parameters, as stated in theorem 3.1, the following result holds true:

$$\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\, E_{\mu,v}^{\tau}(\xi;\mathfrak{p})\right\}\right)(x) = \frac{x^{\delta-\beta-1}}{\Gamma(\tau)}\, {}_3\psi_3\left[\begin{array}{c}(\delta,1),(\rho+\delta-\beta,1),(\tau,1)\\(\delta-\beta,1),(\delta+\alpha+\rho,1),(\mu,v)\end{array}\middle|(x;\mathfrak{p})\right]. \tag{5.5}$$

Example 5.6. With all assumptions and conditions on parameters, as given in theorem 3.2. the following fractional integral formula holds true:

$$\left(J_{x,\infty}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\, E_{\mu,v}^{\tau}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta-\beta-1}}{\Gamma(\tau)}\, {}_3\psi_3\left[\begin{array}{c}(\beta-\delta+1,1),(\rho-\delta+1,1),(\tau,1)\\(1-\delta,1),(\alpha+\beta-\delta+\rho+1,1),(\mu,v)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right]. \tag{5.6}$$

Example 5.7. With all assumptions and conditions on parameters, as discussed in theorem 4.1, we yield:

$$\left(D_{0+}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\, E_{\mu,v}^{\tau}(\xi;\mathfrak{p})\right\}\right)(x) = \frac{x^{\delta+\beta-1}}{\Gamma(\tau)}\, {}_3\psi_3\left[\begin{array}{c}(\delta,1),(\rho+\delta+\beta+\delta,1),(\tau,1)\\(\beta+\delta,1),(\rho+\delta,1),(\mu,v)\end{array}\middle|(x;\mathfrak{p})\right]. \tag{5.7}$$

Example 5.8. As stated in theorem 4.2, keeping in mind all assumption and conditions on parameters, we get the following fractional derivative formula, given as:

$$\left(D_{0-}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\, E_{\mu,v}^{\tau}\left(\frac{1}{\xi};\mathfrak{p}\right)\right\}\right)(x) = \frac{x^{\delta+\beta-1}}{\Gamma(\tau)}\, {}_3\psi_3\left[\begin{array}{c}(1-\beta-\delta,1),(1+\alpha+\rho-\delta,1),(\tau,1)\\(1-\delta,1),(1+\rho-\beta-\delta,1),(\mu,v)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right]. \tag{5.8}$$

Remark 5.9. If we set $\mathfrak{p} = 0$ in the examples 5.5, 5.6, 5.7, and 5.8, then we can obtain the fractional integral and derivative formulas, involving the Prabhakar type Mittag-Leffler function [25]. We omit the details here.

In a similar way, choosing $\mu = \upsilon = 1$ in theorems 3.1, 3.2, 4.1, and 4.2, we consequently obtain the following interesting results:

Example 5.10. Let $\xi > 0, \alpha, \beta, \rho, \delta, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\tau) > 0, \mathfrak{p} \geq 0)$ be parameters such that $\Re(\delta) > \max\{0, \Re(\beta - \rho)\}$. Then we have:

$$\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\Phi_{\mathfrak{p}}^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}(\tau;1;\xi)\right\}\right)(x) = \frac{x^{\delta-\beta-1}}{\Gamma(\tau)}\,{}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}(\delta,1),\,(\rho+\delta-\beta,1),\,(\tau,1)\\(\delta-\beta,1),\,(\delta+\alpha+\rho,1),\,(1,1)\end{array}\middle|(x;\mathfrak{p})\right]. \tag{5.9}$$

Example 5.11. Let $\xi > 0, \alpha, \beta, \rho, \delta, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\tau) > 0, \mathfrak{p} \geq 0)$ be parameters such that $\Re(\delta) > 1 + \min\{\Re(\beta), \Re(\rho)\}$. Then we get following fractional integral formula:

$$\begin{aligned}&\left(J_{x,\infty}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,\Phi_{\mathfrak{p}}^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left(\tau;1;\frac{1}{\xi}\right)\right\}\right)(x)\\&=\frac{x^{\delta-\beta-1}}{\Gamma(\tau)}\,{}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}(\beta-\delta+1,1),\,(\rho-\delta+1,1),\,(\tau,1)\\(1-\delta,1),\,(\alpha+\beta-\delta+\rho+1,1),\,(1,1)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right].\end{aligned} \tag{5.10}$$

Example 5.12. Suppose that $\alpha, \beta, \rho, \delta, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\tau) > 0, \mathfrak{p} \geq 0)$, $\xi > 0, \Re(\alpha) \geq 0$ be parameters such that $\Re(\delta) > -\min\{0, \Re(\alpha + \beta + \rho)\}$ and $\min\{\Re(\delta), \Re(\beta)\} > 0$. Then the following formula holds true:

$$\left(D_{0+}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,\Phi_{\mathfrak{p}}^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}(\tau;1;\xi)\right\}\right)(x) = \frac{x^{\delta+\beta-1}}{\Gamma(\tau)}\,{}_3\psi_3^{\left(\{k_l\}_{l\in\mathbb{N}_0}\right)}\left[\begin{array}{c}(\delta,1),\,(\rho+\delta+\beta+\delta,1),\,(\tau,1)\\(\beta+\delta,1),\,(\rho+\delta,1),\,(1,1)\end{array}\middle|(x;\mathfrak{p})\right]. \tag{5.11}$$

Example 5.13. Assume that $\xi > 0$, $\alpha, \beta, \rho, \delta, \mu, \upsilon, \tau, \mathfrak{p} \in \mathbb{C}$ $(\Re(\tau) > 0, \mathfrak{p} \geq 0, \Re(\alpha) \geq 0)$ be parameters along with the conditions $\Re(\delta) < 1 + \min\{\Re(-\beta - \rho), \Re(\alpha + \beta)\}$ and $\min\{\Re(\delta), \Re(\beta)\} > 0$. Then the following fractional derivative formula holds true:

$$\begin{aligned}&\left(D_{0-}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,\Phi_{\mathfrak{p}}^{\left(\{k_l\}_{l\in N_0}\right)}\left(\tau;1;\frac{1}{\xi}\right)\right\}\right)(x)\\&=\frac{x^{\delta+\beta-1}}{\Gamma(\tau)}\,{}_3\psi_3^{\left(\{k_l\}_{l\in N_0}\right)}\left[\begin{array}{c}(1-\beta-\delta,1),(1+\alpha+\rho-\delta,1),(\tau,1)\\(1-\delta,1),(1+\rho-\beta-\delta,1),(1,1)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right].\end{aligned} \tag{5.12}$$

If we set $\mu = \upsilon = 1$ into the examples 5.1, 5.2, 5.3 and 5.4, then we obtain the following results:

Example 5.14. Let $\xi > 0, \alpha, \beta, \rho, \delta, \tau, \lambda, \eta, \mathfrak{p} \in \mathbb{C}$ $(\Re(\tau) > 0, \mathfrak{p} \geq 0)$ be such that $\Re(\delta) > \max\{0, \Re(\beta - \rho)\}$. We get the following formula:

$$\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,\Phi_{\mathfrak{p}}^{(\eta,\lambda)}(\tau;1;\xi)\right\}\right)(x) = \frac{x^{\delta-\beta-1}\Gamma(\lambda)}{\Gamma(\tau)\Gamma(\eta)}\,{}_4\psi_4\left[\begin{array}{c}(\delta,1),(\rho+\delta-\beta,1),(\tau,1),(\eta,1)\\(\delta-\beta,1),(\delta+\alpha+\rho,1),(1,1),(\lambda,1)\end{array}\middle|(x;\mathfrak{p})\right]. \tag{5.13}$$

Example 5.15. Let $\xi > 0$, $\alpha, \beta, \rho, \delta, \lambda, \eta, \mathfrak{p} \in \mathbb{C}$ $(\Re(\tau) > 0, \mathfrak{p} \geq 0)$ be parameters such that $\Re(\delta) > 1 + \min\{\Re(\beta), \Re(\rho)\}$. Then we have:

$$\left(J_{x,\infty}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,\Phi_{\mathfrak{p}}^{(\eta,\lambda)}\left(\tau;1;\frac{1}{\xi}\right)\right\}\right)(x) = \frac{x^{\delta-\beta-1}\Gamma(\lambda)}{\Gamma(\tau)\Gamma(\eta)}\,{}_4\psi_4\left[\begin{array}{c}(\beta-\delta+1,1),(\rho-\delta+1,1),(\tau,1),(\eta,1)\\(1-\delta,1),(\alpha+\beta-\delta+\rho+1,1),(1,1),(\lambda,1)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right]. \tag{5.14}$$

Example 5.16. Suppose that $\xi > 0$, $\alpha, \beta, \rho, \delta, \tau, \lambda, \eta, \mathfrak{p} \in \mathbb{C}$ $(\Re(\tau) > 0, \Re(\alpha) \geq 0, \mathfrak{p} \geq 0)$ be parameters such that $\Re(\delta) > -\min\{0, \Re(\alpha + \beta + \rho)\}$ and $\min\{\Re(\delta), \Re(\beta)\} > 0$. Then the following formula holds true:

$$\left(D_{0+}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,\Phi_{\mathfrak{p}}^{(\eta,\lambda)}(\tau;1;\xi)\right\}\right)(x) = \frac{x^{\delta+\beta-1}\Gamma(\lambda)}{\Gamma(\tau)\Gamma(\eta)}\,{}_4\psi_4\left[\begin{array}{c}(\delta,1),(\rho+\delta+\beta+\delta,1),(\tau,1),(\eta,1)\\(\beta+\delta,1),(\rho+\delta,1),(1,1),(\lambda,1)\end{array}\middle|(x;\mathfrak{p})\right]. \tag{5.15}$$

Example 5.17. Assume that $\xi > 0$, $\alpha, \beta, \rho, \delta, \tau, \lambda, \eta, \mathfrak{p} \in \mathbb{C}$ $(\Re(\tau) > 0, \mathfrak{p} \geq 0, \Re(\alpha) \geq 0)$ be parameters along with the conditions $\Re(\delta) < 1 + \min\{\Re(-\beta - \rho), \Re(\alpha + \beta)\}$ and $\min\{\Re(\delta), \Re(\beta)\} > 0$. Then we establish the following fractional derivative formula, given as:

$$\left(D_{0-}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,\Phi_{\mathfrak{p}}^{(\eta,\lambda)}\left(\tau;1;\frac{1}{\xi}\right)\right\}\right)(x) = \frac{x^{\delta+\beta-1}\Gamma(\lambda)}{\Gamma(\tau)\Gamma(\eta)}\,{}_4\psi_4\left[\begin{array}{c}(1-\beta-\delta,1),(1+\alpha+\rho-\delta,1),(\tau,1),(\eta,1)\\(1-\delta,1),(1+\rho-\beta-\delta,1),(1,1),(\lambda,1)\end{array}\middle|\left(\frac{1}{x};\mathfrak{p}\right)\right]. \tag{5.16}$$

6 Integral transform associated with extended Mittag-Leffler type functions

A large number of fractional calculus and integral transform formulas have been developed by several researchers (for example, see [1, 4, 5, 8, 15, 18, 19]). This section deals with integral transform formulas, associated with the results being obtained in the previous section. We start with recalling the following integral transform:

Beta Transform: The Beta transform of a function $f(y)$ is defined by Sneddon [32], given as:

$$\mathfrak{B}\left\{f\left(y\right);a,b\right\}=\int_0^1 y^{a-1}(1-y)^{b-1}f\left(y\right). \tag{6.1}$$

Verma Transform: The Verma transform of $f\left(y\right)$ is given by the following integral equation [20]:

$$\mathbb{V}\left\{f,\ell,m;s\right\}=\int_0^1 (sy)^{m-\frac{1}{2}}\exp\left(-\frac{1}{2}sy\right)\mathcal{W}_{\ell,m}\left(sy\right)f\left(y\right)\,\mathrm{d}y \qquad \left(\Re\left(s\right)>0\right), \tag{6.2}$$

where $\mathcal{W}_{l,m}$ is the Whittaker function, given as:

$$\mathcal{W}_{\ell,m}\left(y\right)=\sum_{m,-m}\frac{\Gamma\left(-2m\right)}{\Gamma\left(\frac{1}{2}-\ell-m\right)}\mathcal{M}_{\ell,m}, \tag{6.3}$$

here,

$$\mathcal{M}_{\ell,m}\left(y\right)=y^{m+\frac{1}{2}}\exp\left(-\frac{y}{2}\right)\,{}_1F_1\left(\frac{1}{2}-\ell-m;2m+1;y\right). \tag{6.4}$$

By setting $\ell=-v+\frac{1}{2}$, the Verma transform converts to the Laplace transform [32], given as:

$$\mathcal{L}\left\{f\left(y\right);s\right\}=\int_0^{-\infty} e^{-sy}f\left(y\right)\,\mathrm{d}y. \tag{6.5}$$

Next, we give the Beta, Verma and Laplace transforms, concerning an extended Mittag-Leffler type function and fractional calculus operators, the proofs are omitted.

Theorem 6.1. (Beta Transform)

$$\begin{aligned}&\mathfrak{B}\left[\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in N_0};\tau\right)}\left(z\xi;\mathfrak{p}\right)\right\}\right)\left(x\right);a,b\right]\\&=\frac{\Gamma\left(b\right)}{\Gamma\left(\tau\right)}\,x^{\delta-\beta-1}\,{}_4\psi_4^{\left(\{k_l\}_{l\in N_0}\right)}\left[\begin{array}{c}\left(\delta,1\right),\left(\rho+\delta-\beta,1\right),\left(\tau,1\right),\left(a,1\right)\\\left(\delta-\beta,1\right),\left(\delta+\alpha+\rho,1\right),\left(\mu,v\right),\left(a+b,1\right)\end{array}\middle|\left(x;\mathfrak{p}\right)\right].\end{aligned} \tag{6.6}$$

Theorem 6.2. (Beta Transform)

$$\begin{aligned}&\mathfrak{B}\left[\left(J_{x,\infty}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in N_0};\tau\right)}\left(\frac{z}{\xi};\mathfrak{p}\right)\right\}\right)\left(x\right);a,b\right]=\frac{\Gamma\left(b\right)}{\Gamma\left(\tau\right)}x^{\delta-\beta-1}\\&\times\,{}_4\psi_4^{\left(\{k_l\}_{l\in N_0}\right)}\left[\begin{array}{c}\left(\beta-\delta+1,1\right),\left(\rho-\delta+1,1\right),\left(\tau,1\right),\left(a,1\right)\\\left(1-\delta,1\right),\left(\alpha+\beta+\rho-\delta+1,1\right),\left(\mu,v\right),\left(a+b,1\right)\end{array}\middle|\left(x^{-1};\mathfrak{p}\right)\right].\end{aligned} \tag{6.7}$$

Theorem 6.3. (Verma Transform)

$$\begin{aligned}&\mathbb{V}\left[t^a\left(\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in N_0};\tau\right)}\left(t\xi;\mathfrak{p}\right)\right\}\right)\left(x\right);s\right)\right]\\&=\frac{x^{\delta-\beta-1}}{\Gamma\left(\tau\right)s^a}\,{}_5\psi_4^{\left(\{k_l\}_{l\in N_0}\right)}\left[\begin{array}{c}\left(\delta,1\right),\left(\rho+\delta-\beta,1\right),\left(\tau,1\right),\left(2m+a,1\right),\left(a,1\right)\\\left(\delta-\beta,1\right),\left(\alpha+\beta+\rho,1\right),\left(\mu,v\right),\left(m+a-q,1\right)\end{array}\middle|\left(\frac{x}{s};\mathfrak{p}\right)\right].\end{aligned} \tag{6.8}$$

Theorem 6.4. (Verma Transform)

$$\begin{aligned}&\mathbb{V}\left[t^a\left(\left(J_{x,\infty}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in N_0};\tau\right)}\left(\frac{t}{\xi};\mathfrak{p}\right)\right\}\right)\left(x\right);s\right)\right]=\frac{x^{\delta-\beta-1}}{\Gamma\left(\tau\right)s^a}\\&\times\,{}_5\psi_4^{\left(\{k_l\}_{l\in N_0}\right)}\left[\begin{array}{c}\left(\beta-\delta+1,1\right),\left(\rho-\delta,1\right),\left(\tau,1\right),\left(2m+a,1\right),\left(a,1\right)\\\left(1-\delta,1\right),\left(\alpha+\beta+\rho-\delta+1,1\right),\left(\mu,v\right),\left(m+a-q,1\right)\end{array}\middle|\left(\frac{x}{s};\mathfrak{p}\right)\right].\end{aligned} \tag{6.9}$$

Laplace Transform: The Laplace transform is a particular case of the Verma transform. So, we establish an interesting Laplace transform, asserted by the following results in from of corollaries:

Corollary 6.5. Assume that the conditions of theorem 3.1 are satisfied. Then the following integral transforms hold true:

$$\begin{aligned}&\mathcal{L}\left[t^{a-1}\left(\left(I_{0,x}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in N_0};\tau\right)}\left(t\xi;\mathfrak{p}\right)\right\}\right)(x)\,;s\right)\right]\\&=\frac{x^{\delta-\beta-1}}{\Gamma\left(\tau\right)s^{a}}\;{}_4\psi_3^{\left(\{k_l\}_{l\in N_0}\right)}\left[\begin{array}{c}\left(\delta,1\right),\left(\rho+\delta-\beta,1\right),\left(\tau,1\right),\left(a,1\right)\\\left(\delta-\beta,1\right),\left(\alpha+\beta+\rho,1\right),\left(\mu,v\right)\end{array}\middle|\left(\frac{x}{s};\mathfrak{p}\right)\right].\end{aligned}\tag{6.10}$$

Corollary 6.6. Assume that the conditions of theorem 3.2 are satisfied. Then the following integral transforms hold true:

$$\begin{aligned}&\mathcal{L}\left[t^{a-1}\left(\left(J_{x,\infty}^{\alpha,\beta,\rho}\left\{\xi^{\delta-1}\,E_{\mu,v}^{\left(\{k_l\}_{l\in N_0};\tau\right)}\left(\frac{t}{\xi};\mathfrak{p}\right)\right\}\right)(x)\,;s\right)\right]\\&=\frac{x^{\delta-\beta-1}}{\Gamma\left(\tau\right)s^{a}}\;{}_4\psi_3^{\left(\{k_l\}_{l\in N_0}\right)}\left[\begin{array}{c}\left(\beta-\delta+1,1\right),\left(\rho-\delta,1\right),\left(\tau,1\right),\left(a,1\right)\\\left(1-\delta,1\right),\left(\alpha+\beta+\rho-\delta+1,1\right),\left(\mu,v\right)\end{array}\middle|\left(\frac{1}{xs};\mathfrak{p}\right)\right].\end{aligned}\tag{6.11}$$

7 Concluding remark

In this study, we obtained some results on fractional calculus, involving extended Mittag-Leffler type function. The results are expressed in terms of the newly defined generalized hypergeometric function. We may also emphasize, that due to practical importance of the Mittag-Leffler functions, the results deduced are of the general in character and significant. Hence, we can lead to yield a large number of fractional calculus and integral transform formulas involving various special functions. Discussing also initial values will be another task to be done.

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