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Generalized weak contractions with control functions on metric and normed interval spaces

Marzieh Sharifi, Sayed Mohamad Sadegh Modarres Mosadegh*

Department of Mathematical Sciences, Yazd University, Yazd, Iran

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Abstract

In this paper, we discuss some near-fixed point theorems with control functions for mappings that are specifically generalized weakly contraction in metric and normed interval spaces. We prove the existence and uniqueness of near-fixed points and common near-fixed points for these mappings. Moreover, we provide some examples to demonstrate the validity of our extensions.

Keywords: Alternating distance functions, Contractive mappings, Metric interval spaces, Near fixed points, Normed interval spaces, Null set
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1 Introduction

Banach's contraction principle is a very important and fundamental tool in many branches of mathematical analysis, which has been generalized in various fields. One of the generalizations of Banach's contraction principle is about contractive mappings on metric spaces, and so researchers have studied them. Fixed point theory as an effective nonlinear analysis tool has been the research field of many mathematicians in the last fifty years [3, 7, 11]. Alber et al. first expressed the notion of weak contraction in Hilbert spaces, and Rhoades generalized this concept to metric spaces [11, 7]. Khan et al. introduced the concept of a control function called an alternating distance function [7, 11]. In 1906, Fréchet initiated the concept of metric spaces, and since then Many researchers have generalized this concept to other spaces [2]. In 2018, Wu defined the notion of near-fixed points for collecting all closed and bounded intervals $\mathcal{I}$ in $\mathbb{R}$. He also stated the notion of the null set and proposed metric interval space (MIS) and normed interval space (NIS) based on it [13]. Since the customary normed space is based on the vector space and the interval space $\mathcal{I}$ is not a vector space, we cannot consider the customary normed space $(\mathcal{I}, \|\cdot\|)$. To overcome this problem, the NIS has presented based on the null set. Due to Banach's contraction principle, a fixed point for mapping T is expressed provided that we have the metric space $(\mathcal{I}, d)$. But for some distance functions such as $d([k,l],[u,v]) = |(k+l)-(u+v)|$, we can not consider the space $(\mathcal{I}, d)$ as a metric space. To resolve this difficulty, the MIS has been introduced based on the null set, so the MIS and the NIS cannot be the customary metric and normed spaces [13]. Since there is no additive inverse member for any non-degenerated closed interval in $\mathcal{I}$, therefore $\mathcal{I}$ is not a linear space. The algebraic structure of $\mathcal{I}$ is named a quasilinear space (QLS) [5, 6, 9, 10, 15]. The QLS was first stated by Aseev in 1986 [4].

*Corresponding author
Email addresses: marzieh.sharifi@stu.yazd.ac.ir (Marzieh Sharifi), smodarres@yazd.ac.ir (Sayed Mohamad Sadegh Modarres Mosadegh)

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One of the important topics in solving practical problems is data collection. But this work is not exactly possible in some situations. For example, measuring the level of liquids due to fluctuations and recording stock prices in short time intervals due to the intensity of fluctuations in the trading market cannot be evaluated exactly. In these cases, uncertain data can be described by using bounded closed intervals. In other words, the level of liquids and the stock price can be considered to be inbounded closed intervals, which indicates uncertainty. Therefore, interval analysis can be used to cope with uncertainty in issues such as engineering, economics and social sciences [14].

The purpose of this article is to define different types of certain weak contractive mappings in the MIS and the NIS and prove near-fixed point theorems related to them. Various types of certain generalized weakly contractive mappings and related theorems have been stated in different articles, such as [1, 2, 7, 8]. We present these theorems and definitions with the metric and norm based on the null set in the MIS and the NIS. In other words, we describe these definitions and theorems in the MIS and the NIS. We investigate contractive mappings on the MIS and the NIS, which are generalizations of Banach's contraction principle, and present some examples.

In section 2, we define the interval space and the null set. In sections 3 and 4, we introduce the MIS and the NIS and their properties. In section 5, we define different types of specific generalized weakly contractive mappings in the MIS and the NIS and prove the near-fixed point theorems for these mappings. Finally, in section 6, we give a general conclusion for the article.

2 Preliminaries

Suppose that $\mathcal{I}$ is the collection of all closed and bounded intervals $[k,l]$ in $\mathbb{R}$, such that $k \leq l$. The addition and scalar multiplication operations on $\mathcal{I}$ are defined as follows:

$$[k,l] \oplus [u,v] = [k+u, l+v] \quad \text{and} \quad a\,[k,l] = \begin{cases} [ak, al] & \text{if } a \geq 0 \\ [al, ak] & \text{if } a < 0. \end{cases}$$

Note that $\mathcal{I}$ is not a customary vector space cosidering mentioned two operations, because there is no additive inverse member for any non-degenerated closed interval. Clearly we have $[0,0] \in \mathcal{I}$ as a zero member. However, the following subtraction does not give a zero element for some $[k,l] \in \mathcal{I}$,

$$[k,l] \ominus [k,l] = [k,l] \oplus [-l,-k] = [k-l, l-k]\,.$$

The null set is defined by:

$$\Omega = \big\{\, [k,l] \ominus [k,l] : \ [k,l] \in \mathcal{I} \big\}.$$

It is clear that

$$\Omega = \big\{\, [-a,a] : \ \ a \geq 0 \big\} = \big\{ a\,[-1,1] : \ \ a \geq 0 \big\}.$$

For more details, see [13]. Because some members of $\mathcal{I}$ have not a additive inverse member, so $\mathcal{I}$ is not a linear space. The algebraic structure of $\mathcal{I}$ is named a quasilinear space (QLS) [9]. The QLS was first stated by Aseev in 1986 [4]. He defined the QLS as follows.

Definition 2.1. [4] A set X is called a quasilinear space (QLS) if a partial order "$\leq$", an algebraic addition operation, and a multiplication operation on real numbers are expressed in it such that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:

(1) $x \leq x$,

(2) $x \leq z$ if $x \leq y$ and $y \leq z$,

(3) $x = y$ if $x \leq y$ and $y \leq x$,

(4) $x + y = y + x$,

(5) $x + (y + z) = (x + y) + z$,

(6) there exists an element $\theta \in X$ such that $x + \theta = x$,

(7) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$,

(8) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$,

(9) $1 \cdot x = x$,

(10) $0 \cdot x = \theta$,

(11) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$,

(12) $x + z \leq y + v$ if $x \leq y$ and $z \leq v$,

(13) $\alpha \cdot x \leq \alpha \cdot y$ if $x \leq y$.

An important example of quasilinear spaces is $\mathcal{I}$, the collection of all closed real intervals, with the inclusion relation " $\subseteq$ ", and with the addition and real-scalar multiplication operations are defined as follows:

$$A + B = \{a + b : a \in A, b \in B\} \quad \text{and} \quad \lambda \cdot A = \{\lambda a : a \in A\}.$$

This set is denoted by $\Omega_C(\mathbb{R})$. For more details, see [5, 6, 9, 10, 15].

Remark 2.2. [13] In any interval space, the following facts are valid:

- The distributive law is generally incorrect, i.e.,

$$(\alpha + \beta)[k, l] \neq \alpha [k, l] \oplus \beta [k, l], \quad \text{for any} \quad [k, l] \in \mathcal{I} \quad \text{and} \quad \alpha, \beta \in \mathbb{R}.$$

  For example, for $\alpha = -5$, $\beta = 3$ and $[k, l] = [-1, 3]$, the equality is incorrect.

- The distributive law is true provided that the scalars are both positive or both negative, i.e.,

$$(\alpha + \beta)[k, l] = \alpha [k, l] \oplus \beta [k, l], \quad \text{for any} \quad [k, l] \in \mathcal{I} \quad \text{and} \quad \alpha, \beta > 0 \;\text{ or }\; \alpha, \beta < 0.$$

- For any $[k, l], [u, v], [m, n] \in \mathcal{I}$, we have

$$[m, n] \ominus ([k, l] \oplus [u, v]) = [m, n] \oplus (-[k, l]) \oplus (-[u, v]) = [m, n] \ominus [k, l] \ominus [u, v]. \tag{2.1}$$

- We define $[k, l] \stackrel{\Omega}{=} [u, v]$ if and only if

$$\exists \omega_1, \omega_2 \in \Omega \quad \text{such that} \quad [k, l] \oplus \omega_1 = [u, v] \oplus \omega_2. \tag{2.2}$$

  Clearly, $[k, l] = [u, v]$ implies $[k, l] \stackrel{\Omega}{=} [u, v]$ by choosing $\omega_1 = \omega_2 = [0, 0]$. However, the inverse is not usually true. The following class based on the relation $\stackrel{\Omega}{=}$ for any $[k, l] \in \mathcal{I}$, is defined by:

$$\left\langle [k, l] \right\rangle = \left\{ [u, v] \in \mathcal{I} : [k, l] \stackrel{\Omega}{=} [u, v] \right\}. \tag{2.3}$$

  $\langle \mathcal{I} \rangle$ represents the family of all classes $\langle [k, l] \rangle$ for $[k, l] \in \mathcal{I}$.

Proposition 2.3. [13] The relation $\stackrel{\Omega}{=}$ is a reflexive, symmetric and transitive relation, and therefore, $\stackrel{\Omega}{=}$ will be an equivalence relation.

This proposition shows that the classes (2.3) provide the equivalence classes. Moreover, $[u, v] \in \langle [k, l] \rangle$ implies that $\langle [k, l] \rangle = \langle [u, v] \rangle$. Note that the quotient collection $\mathcal{I}$ will not necessarily be a customary vector space(see [13]).

3 Metric interval space and its properties

Definition 3.1. [13] Suppose that $\mathcal{I}$ is the collection of all bounded and closed intervals in $\mathbb{R}$ and $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ is a mapping. A pair $(\mathcal{I}, d)$ is named a metric interval space (MIS) if the following three conditions hold for d:

(i) $d([k,l], [u,v]) = 0$ if and only if $[k,l] \overset{\Omega}{=} [u,v]$, for all $[k,l], [u,v] \in \mathcal{I}$;

(ii) $d([k,l], [u,v]) = d([u,v], [k,l])$ for all $[k,l], [u,v] \in \mathcal{I}$;

(iii) $d([k,l], [u,v]) \leq d([k,l], [m,n]) + d([m,n], [u,v])$, for all $[k,l], [u,v], [m,n] \in \mathcal{I}$.

We have a pseudo-metric interval space $(\mathcal{I}, d)$ if conditions (ii) and (iii) hold for d. We say that the null equalities hold for d if it satisfies the following equalities for any $\omega_1, \omega_2 \in \Omega$ and $[k,l], [u,v] \in \mathcal{I}$:

- $d([k,l] \oplus \omega_1, [u,v] \oplus \omega_2) = d([k,l], [u,v])$;
- $d([k,l] \oplus \omega_1, [u,v]) = d([k,l], [u,v])$;
- $d([k,l], [u,v] \oplus \omega_2) = d([k,l], [u,v])$.

Definition 3.2. [13] The sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ is named a convergent sequence in pseudo-metric interval space $(\mathcal{I}, d)$ if and only if $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$ for some $[k,l] \in \mathcal{I}$. In fact, the member $[k,l]$ is the limit of sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$.

Proposition 3.3. [13] If the sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ exists in $\mathcal{I}$ such that $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$ and the null equality holds for d, then $\lim_{n\to\infty} d([k_n, l_n], [u,v]) = 0$ for any $[u,v] \in \langle [k,l] \rangle$.

Definition 3.4. [13] Let the sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ exists in $\mathcal{I}$ such that $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$ for some $[k,l] \in \mathcal{I}$. Then the class $\langle [k,l] \rangle$ is named the class limit of $\{ [k_n, l_n] \}_{n=1}^{\infty}$. In addition, the uniqueness of the limit class in a MIS is easily obtained (see [13]).

Definition 3.5. [13] Let $\{ [k_n, l_n] \}_{n=1}^{\infty}$ be a sequence in the MIS $(\mathcal{I}, d)$ such that

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \ \text{such that} \ d([k_n, l_n], [k_m, l_m]) < \varepsilon, \ \ \forall n, m > N.$$

Then $\{ [k_n, l_n] \}_{n=1}^{\infty}$ is called a Cauchy sequence in $\mathcal{I}$. Moreover, a subset $\mathcal{W}$ of $\mathcal{I}$ in the MIS $(\mathcal{I}, d)$ is complete if and only if every Cauchy sequence $\{ [k_n, l_n] \}_{n=1}^{\infty}$ in $\mathcal{W}$ converges to an element $[k,l] \in \mathcal{W}$.

Example 3.6. [13] Suppose that $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ is defined by $d([k,l], [u,v]) = |(k+l) - (u+v)|$. Then, it is obvious that $(\mathcal{I}, d)$ is a complete metric interval space (CMIS) and the null equality holds for d.

4 Normed interval space and its properties

Definition 4.1. [13] For a mapping $\|\cdot\| : \mathcal{I} \to \mathbb{R}^+$ that $\mathbb{R}^+$ is nonnegative real numbers, we present the following features:

(i) $\|\lambda [k,l]\| = |\lambda| \cdot \|[k,l]\|$, for all $[k,l] \in \mathcal{I}$ and $\lambda \in \mathbb{F}$;

(i°) $\|\lambda [k,l]\| = |\lambda| \cdot \|[k,l]\|$, for all $[k,l] \in \mathcal{I}$ and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$;

(ii) $\|[k,l] \oplus [u,v]\| \leq \|[k,l]\| + \|[u,v]\|$, for all $[k,l], [u,v] \in \mathcal{I}$;

(iii) $\|[k,l]\| = 0$ implies $[k,l] \in \Omega$.

- It is said that $(\mathcal{I}, \|\cdot\|)$ is a pseudo-seminormed interval space if it satisfies cases (i°) and (ii).
- It is said that $(\mathcal{I}, \|\cdot\|)$ is a normed interval space (NIS) if it satisfies cases (i), (ii) and (iii).

- It is said that the null condition holds for $\|\cdot\|$ if item (iii) is changed to $\|[k,l]\| = 0 \Leftrightarrow [k,l] \in \Omega$.
- It is said that the null super-inequality holds for $\|\cdot\|$ if $\|[k,l] \oplus \omega\| \geq \|[k,l]\|$, for all $[k,l] \in \mathcal{I}$ and $\omega \in \Omega$.
- It is said that the null equality holds for $\|\cdot\|$ if $\|[k,l] \oplus \omega\| = \|[k,l]\|$, for all $[k,l] \in \mathcal{I}$ and $\omega \in \Omega$.

A complete normed interval space is named a Banach interval space (BIS).

Example 4.2. [13] Assume that the mapping $\|\cdot\| : \mathcal{I} \to \mathbb{R}^+$ is defined by $\|[k,l]\| = |k+l|$. Then, it is obvious that $(\mathcal{I}, \|\cdot\|)$ is a BIS and the null equality holds for $\|\cdot\|$.

For more details about normed interval spaces, see [13].

5 Near fixed point results

Definition 5.1. [13] A point $[k,l] \in \mathcal{I}$ is named a near fixed point for a self-mapping $T : \mathcal{I} \to \mathcal{I}$ if $T[k,l] \overset{\Omega}{=} [k,l]$.

According to the definition, we have $T[k,l] \overset{\Omega}{=} [k,l]$ if and only if there exist $[-b_1, b_1], [-b_2, b_2] \in \Omega$ where $b_1, b_2 \in \mathbb{R}^+$ so that at least one of the following equalities holds:

- $T[k,l] \oplus [-b_1, b_1] = [k,l]$;
- $T[k,l] = [k,l] \oplus [-b_1, b_1]$;
- $T[k,l] \oplus [-b_1, b_1] = [k,l] \oplus [-b_2, b_2]$.

Definition 5.2. [12] A point $[k,l] \in \mathcal{I}$ is a common near fixed point for functions $T, S : \mathcal{I} \to \mathcal{I}$ if $T[k,l] \overset{\Omega}{=} S[k,l] \overset{\Omega}{=} [k,l]$.

Example 5.3. Assume that $T, S : \mathcal{I} \to \mathcal{I}$ are defined by

$$T[k,l] = [k-3, l+3] \quad \text{and} \quad S[k,l] = [k-4, l+4].$$

We indicate that $[k,l]$ is a common near fixed point of T and S. For $\omega_1 = [0,0]$ and $\omega_2 = [-3,3] \in \Omega$, we have $T[k,l] \overset{\Omega}{=} [k,l]$, i.e.,

$$[k-3, l+3] \overset{\Omega}{=} [k,l] \Longleftrightarrow [k-3, l+3] \oplus [0,0] = [k,l] \oplus [-3,3] \Longleftrightarrow [k-3, l+3] = [k-3, l+3].$$

Similarly, for $\omega_1 = [0,0]$ and $\omega_2 = [-4,4] \in \Omega$, we have $S[k,l] \overset{\Omega}{=} [k,l]$. According to proposition 2.3, because $\overset{\Omega}{=}$ is an equivalence relation, then $T[k,l] \overset{\Omega}{=} S[k,l]$. Hence $T[k,l] \overset{\Omega}{=} S[k,l] \overset{\Omega}{=} [k,l]$.

Definition 5.4. [8] A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a strongly monotone increasing function if for $x, y \in [0,\infty)$, $x \leq y \Longleftrightarrow \psi(x) \leq \psi(y)$.

Definition 5.5. [8] Suppose that $\psi : [0,\infty) \to [0,\infty)$ is a strongly monotone increasing and continuous function and $\psi(t) = 0 \Longleftrightarrow t = 0$. Therefore, ψ is named an alternating distance function.

Lemma 5.6. Suppose that $\lim_{n\to\infty} d([k_n, l_n], [k,l]) = 0$ in the MIS $(\mathcal{I}, d)$. Then for each $[u,v] \in \mathcal{I}$, we have

$$\lim_{n\to\infty} d([k_n, l_n], [u,v]) = d([k,l], [u,v]).$$

Proof . Due to the triangle inequality, we obtain

$$d([k_n, l_n], [u, v]) \leq d([k_n, l_n], [k, l]) + d([k, l], [u, v])$$

and

$$d([k, l], [u, v]) \leq d([k, l], [k_n, l_n]) + d([k_n, l_n], [u, v]).$$

So, we obtain

$$0 \leq |d([k_n, l_n], [u, v]) - d([k, l], [u, v])| \leq d([k_n, l_n], [k, l]).$$

Letting $n \to \infty$ and using the assumption of Lemma, we have

$$\lim_{n\to\infty} d([k_n, l_n], [u, v]) = d([k, l], [u, v]).$$

□

Theorem 5.7. Suppose that $(\mathcal{I}, d)$ is a CMIS and the null equality holds for d. Assume that $T, S : (\mathcal{I}, d) \to (\mathcal{I}, d)$ are self-mappings of $\mathcal{I}$ such that for all $[k, l], [u, v] \in \mathcal{I}$,

$$\psi\big(d(S[k, l], T[u, v])\big) \leq \psi\big(M([k, l], [u, v])\big) - \phi\big(\max\{d([k, l], [u, v]), d([k, l], S[k, l]), d([u, v], T[u, v])\}\big), \tag{5.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function with the property $\phi(t) = 0 \Leftrightarrow t = 0$, $\psi : [0, \infty) \to [0, \infty)$ is an alternating distance function and

$$M([k, l], [u, v]) = \alpha d([k, l], [u, v]) + \beta\big[d([k, l], S[k, l]) + d([u, v], T[u, v])\big] + \gamma\big[d([k, l], T[u, v]) + d([u, v], S[k, l])\big],$$

with $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + 2\gamma \leq 1$. Then a common near fixed point $[k, l] \in \mathcal{I}$ exists for T and S satisfying $T[k, l] \stackrel{\Omega}{=} [k, l] \stackrel{\Omega}{=} S[k, l]$.
Moreover, a unique equivalence class $\langle [k, l] \rangle$ exists for $\mathcal{I}$ such that if $[\bar{k}, \bar{l}]$ is another common near fixed point for T and S, then $\langle [k, l] \rangle = \langle [\bar{k}, \bar{l}] \rangle$ and $[k, l] \stackrel{\Omega}{=} [\bar{k}, \bar{l}]$. Every point $[\bar{k}, \bar{l}] \in \langle [k, l] \rangle$ is also a common near fixed point for T and S. In addition, any near fixed point of T is a near fixed point of S and conversely.

Proof . Assume that T and S have two common near fixed points $[k, l]$ and $[\bar{k}, \bar{l}]$ such that $[k, l] \stackrel{\Omega}{\neq} [\bar{k}, \bar{l}]$, i.e., $[\bar{k}, \bar{l}] \stackrel{\Omega}{=} T[\bar{k}, \bar{l}] \stackrel{\Omega}{=} S[\bar{k}, \bar{l}]$, $[k, l] \stackrel{\Omega}{=} T[k, l] \stackrel{\Omega}{=} S[k, l]$ and $[\bar{k}, \bar{l}] \notin \langle [k, l] \rangle$. Then

$$[\bar{k}, \bar{l}] \oplus \omega_1 = T[\bar{k}, \bar{l}] \oplus \omega_2 \quad \text{and} \quad [k, l] \oplus \omega_5 = T[k, l] \oplus \omega_6 \tag{5.2}$$

$$[k, l] \oplus \omega_3 = S[k, l] \oplus \omega_4 \quad \text{and} \quad [\bar{k}, \bar{l}] \oplus \omega_7 = S[\bar{k}, \bar{l}] \oplus \omega_8 \tag{5.3}$$

for some $\omega_i \in \Omega$, $i = 1, \ldots, 8$. Note that $d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]) = 0$ and $d([k, l], S[k, l]) = 0$. Then using (5.1), (5.2) and (5.3) and making use of the null equality and the triangle inequality, we have

$$\begin{aligned}
\psi\big(d([k, l], [\bar{k}, \bar{l}])\big) &= \psi\big(d([k, l] \oplus \omega_3, [\bar{k}, \bar{l}] \oplus \omega_1)\big) = \psi\big(d(S[k, l] \oplus \omega_4, T[\bar{k}, \bar{l}] \oplus \omega_2)\big) \\
&= \psi\big(d(S[k, l], T[\bar{k}, \bar{l}])\big) \\
&\leq \psi\big(M\big([k, l], [\bar{k}, \bar{l}]\big)\big) - \phi\big(\max\{d([k, l], [\bar{k}, \bar{l}]), d([k, l], S[k, l]), d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\}\big),
\end{aligned}$$

where

$$\begin{aligned}
M([k, l], [\bar{k}, \bar{l}]) &= \alpha d([k, l], [\bar{k}, \bar{l}]) + \beta\big[d([k, l], S[k, l]) + d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big] + \gamma\big[d([k, l], T[\bar{k}, \bar{l}]) + d([\bar{k}, \bar{l}], S[k, l])\big] \\
&\leq \alpha d([k, l], [\bar{k}, \bar{l}]) + \beta\big[d([k, l], S[k, l]) + d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big] \\
&\quad + \gamma\big[d([k, l], [\bar{k}, \bar{l}]) + d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]) + d([k, l], [\bar{k}, \bar{l}]) + d([k, l], S[k, l])\big] \\
&= (\alpha + 2\gamma) d([k, l], [\bar{k}, \bar{l}]).
\end{aligned}$$

Therefore

$$\psi\big(d([k, l], [\bar{k}, \bar{l}])\big) \leq \psi\big((\alpha + 2\gamma) d([k, l], [\bar{k}, \bar{l}])\big) - \phi\big(d([k, l], [\bar{k}, \bar{l}])\big),$$

Due to $\alpha + 2\gamma \leq 1$ and the strongly monotone property of ψ, it follows that

$$\begin{aligned}\psi\big(d([k,l]\,,[\bar{k},\bar{l}])\big) &\leq \psi\big((\alpha+2\gamma)d([k,l]\,,[\bar{k},\bar{l}])\big) - \phi\big(d([k,l]\,,[\bar{k},\bar{l}])\big)\\ &\leq \psi\big(d([k,l]\,,[\bar{k},\bar{l}])\big) - \phi\big(d([k,l]\,,[\bar{k},\bar{l}])\big),\end{aligned}$$

which results that $\phi\big(d([k,l]\,,[\bar{k},\bar{l}])\big) \leq 0$. Hence $\phi\big(d([k,l]\,,[\bar{k},\bar{l}])\big) = 0$. Thus $d([k,l]\,,[\bar{k},\bar{l}]) = 0$, which contradicts $[\bar{k},\bar{l}] \notin \langle\, [k,l]\,\rangle$. Therefore, any $[\bar{k},\bar{l}] \notin \langle\, [k,l]\,\rangle$ cannot be a common near fixed point for T and S. In fact, if $[\bar{k},\bar{l}]$ is another common near fixed point for T and S, then $[\bar{k},\bar{l}] \in \langle\, [k,l]\,\rangle$ i.e., $[k,l] \stackrel{\Omega}{=} [\bar{k},\bar{l}]$. Now assume that $[k,l]$ is a near fixed point of S and $[k,l] \stackrel{\Omega}{\neq} T\,[k,l]$. Then by using (5.1), (5.3) and the null equality, we have

$$\begin{aligned}\psi\big(d([k,l]\,,T\,[k,l])\big) =&\psi\big(d([k,l]\oplus\omega_3,T\,[k,l])\big) = \psi\big(d(S\,[k,l]\oplus\omega_4,T\,[k,l])\big)\\ =&\psi\big(d(S\,[k,l]\,,T\,[k,l])\big)\\ \leq&\psi\big(M\big(\,[k,l]\,,[k,l]\,\big)\big) - \phi\big(\max\big\{d([k,l]\,,[k,l]),d([k,l]\,,S\,[k,l]),d([k,l]\,,T\,[k,l])\big\}\big),\end{aligned}$$

where

$$\begin{aligned}M([k,l]\,,[k,l]) &= \alpha d([k,l]\,,[k,l]) + \beta\big[d([k,l]\,,S\,[k,l]) + d([k,l]\,,T\,[k,l])\big] + \gamma\big[d([k,l]\,,T\,[k,l]) + d([k,l]\,,S\,[k,l])\big]\\ &= (\beta+\gamma)d([k,l]\,,T\,[k,l]).\end{aligned}$$

Thus

$$\psi\big(d([k,l]\,,T\,[k,l])\big) \leq \psi\big((\beta+\gamma)d([k,l]\,,T\,[k,l])\big) - \phi\big(d([k,l]\,,T\,[k,l])\big),$$

since $\beta + \gamma \leq 1$ and ψ is strongly monotone increasing, it results that

$$\begin{aligned}\psi\big(d([k,l]\,,T\,[k,l])\big) &\leq \psi\big((\beta+\gamma)d([k,l]\,,T\,[k,l])\big) - \phi\big(d([k,l]\,,T\,[k,l])\big)\\ &\leq \psi\big(d([k,l]\,,T\,[k,l])\big) - \phi\big(d([k,l]\,,T\,[k,l])\big),\end{aligned}$$

which results that $\phi\big(d([k,l]\,,T\,[k,l])\big) \leq 0$. Hence, $\phi\big(d([k,l]\,,T\,[k,l])\big) = 0$. Thus $d([k,l]\,,T\,[k,l]) = 0$, which contradicts $[k,l] \stackrel{\Omega}{\neq} T\,[k,l]$. Similarly, any near fixed point for T is also a near fixed point for S.

Let $[k_0,l_0] \in \mathcal{I}$ be an arbitrary element. Consider a sequence $\big\{\,[k_n,l_n]\,\big\}_{n=1}^{\infty}$ in $\mathcal{I}$ such that $[k_{2n+1},l_{2n+1}] = S\,[k_{2n},l_{2n}]$ and $[k_{2n+2},l_{2n+2}] = T\,[k_{2n+1},l_{2n+1}]$ for $n \geq 0$. If there is a positive integer $2N$ such that $[k_{2N},l_{2N}] = [k_{2N+1},l_{2N+1}]$, then $[k_{2N},l_{2N}]$ is a near fixed point for S and hence it is also a near fixed point for T. We have a similar result if $[k_{2N+1},l_{2N+1}] = [k_{2N+2},l_{2N+2}]$ for some N. Therefore, suppose that $d([k_{n+1},l_{n+1}]\,,[k_n,l_n]) \neq 0$ for all $n \geq 0$. Then from (5.1), we obtain

$$\begin{aligned}&\psi\big(d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n},l_{2n}]\,\big)\big) = \psi\big(d(S\,[k_{2n},l_{2n}]\,,T\,[k_{2n-1},l_{2n-1}])\big)\\ &\leq \psi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n-1},l_{2n-1}]\,\big)\big) - \phi\Big(\max\Big\{d\big(\,[k_{2n},l_{2n}]\,,[k_{2n-1},l_{2n-1}]\,\big),\\ &\quad d\big(\,[k_{2n},l_{2n}]\,,S\,[k_{2n},l_{2n}]\,\big),d\big(\,[k_{2n-1},l_{2n-1}]\,,T\,[k_{2n-1},l_{2n-1}]\,\big)\Big\}\Big)\\ &= \psi\big(\alpha d\big(\,[k_{2n},l_{2n}]\,,[k_{2n-1},l_{2n-1}]\,\big) + \beta\big[d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) + d\big(\,[k_{2n-1},l_{2n-1}]\,,[k_{2n},l_{2n}]\,\big)\big]\\ &\quad + \gamma\big[d\big(\,[k_{2n},l_{2n}]\,,[k_{2n},l_{2n}]\,\big) + d\big(\,[k_{2n-1},l_{2n-1}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big]\big)\\ &\quad - \phi\Big(\max\Big\{d\big(\,[k_{2n},l_{2n}]\,,[k_{2n-1},l_{2n-1}]\,\big),d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big),d\big(\,[k_{2n-1},l_{2n-1}]\,,[k_{2n},l_{2n}]\,\big)\Big\}\Big).\end{aligned}$$

Since

$$d([k_{2n-1},l_{2n-1}]\,,[k_{2n+1},l_{2n+1}]) \leq d([k_{2n-1},l_{2n-1}]\,,[k_{2n},l_{2n}]) + d([k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]),$$

and ψ is a strongly monotone increasing function, it results that

$$\begin{aligned}\psi\big(d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n},l_{2n}]\,\big)\big) \leq&\psi\big(\alpha d\big(\,[k_{2n},l_{2n}]\,,[k_{2n-1},l_{2n-1}]\,\big)\\ &+ \beta\big[d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) + d\big(\,[k_{2n-1},l_{2n-1}]\,,[k_{2n},l_{2n}]\,\big)\big]\\ &+ \gamma\big[d\big(\,[k_{2n-1},l_{2n-1}]\,,[k_{2n},l_{2n}]\,\big) + d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big]\big)\\ &- \phi\big(\max\big\{d\big(\,[k_{2n},l_{2n}]\,,[k_{2n-1},l_{2n-1}]\,\big),d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big\}\big).\end{aligned} \tag{5.4}$$

So using the non-negativity of ϕ, we have

$$\begin{aligned}\psi\big(d\big([k_{2n+1},l_{2n+1}],[k_{2n},l_{2n}]\big)\big) \leq & \psi\big(\alpha d\big([k_{2n},l_{2n}],[k_{2n-1},l_{2n-1}]\big) \\ & + \beta\big[d\big([k_{2n},l_{2n}],[k_{2n+1},l_{2n+1}]\big)+d\big([k_{2n-1},l_{2n-1}],[k_{2n},l_{2n}]\big)\big] \\ & + \gamma\big[d\big([k_{2n-1},l_{2n-1}],[k_{2n},l_{2n}]\big)+d\big([k_{2n},l_{2n}],[k_{2n+1},l_{2n+1}]\big)\big]\big).\end{aligned}$$

Thus due to the strongly monotone property of ψ, we obtain

$$\begin{aligned}d\big([k_{2n+1},l_{2n+1}],[k_{2n},l_{2n}]\big) \leq & \alpha d\big([k_{2n},l_{2n}],[k_{2n-1},l_{2n-1}]\big) \\ & + \beta\big[d\big([k_{2n},l_{2n}],[k_{2n+1},l_{2n+1}]\big)+d\big([k_{2n-1},l_{2n-1}],[k_{2n},l_{2n}]\big)\big] \\ & + \gamma\big[d\big([k_{2n-1},l_{2n-1}],[k_{2n},l_{2n}]\big)+d\big([k_{2n},l_{2n}],[k_{2n+1},l_{2n+1}]\big)\big].\end{aligned}$$

Therefore

$$(1-\beta-\gamma)d\big([k_{2n+1},l_{2n+1}],[k_{2n},l_{2n}]\big) \leq (\alpha+\beta+\gamma)d\big([k_{2n},l_{2n}],[k_{2n-1},l_{2n-1}]\big).$$

This implies that

$$d\big([k_{2n+1},l_{2n+1}],[k_{2n},l_{2n}]\big) \leq \frac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma)}d\big([k_{2n},l_{2n}],[k_{2n-1},l_{2n-1}]\big).$$

Since $\dfrac{(\alpha+\beta+\gamma)}{(1-\beta-\gamma)} \leq 1$, it follows that

$$d\big([k_{2n+1},l_{2n+1}],[k_{2n},l_{2n}]\big) \leq d\big([k_{2n},l_{2n}],[k_{2n-1},l_{2n-1}]\big).$$

Similarly, we can obtain

$$d\big([k_{2n+2},l_{2n+2}],[k_{2n+1},l_{2n+1}]\big) \leq d\big([k_{2n+1},l_{2n+1}],[k_{2n},l_{2n}]\big).$$

Then according to the above facts, $\big\{d([k_n,l_n],[k_{n+1},l_{n+1}])\big\}$ is a monotone non-increasing sequence of non-negative real numbers. Therefore there is an $r \geq 0$ such that

$$\lim_{n\to\infty} d([k_n,l_n],[k_{n+1},l_{n+1}]) = r. \tag{5.5}$$

Assume that $r > 0$. Taking $n \to \infty$ in (5.4) and using (5.5) and the continuity of ψ and ϕ, we obtain

$$\psi(r) \leq \psi((\alpha+2\beta+2\gamma)r) - \phi(r),$$

since $\alpha+2\beta+2\gamma \leq 1$ and ψ is strongly monotone increasing, it follows that

$$\psi(r) \leq \psi((\alpha+2\beta+2\gamma)r) - \phi(r) \leq \psi(r) - \phi(r) \Longrightarrow \psi(r) \leq \psi(r) - \phi(r),$$

which results that $\phi(r) = 0$. Thus $r = 0$. Therefore,

$$\lim_{n\to\infty} d([k_n,l_n],[k_{n+1},l_{n+1}]) = 0. \tag{5.6}$$

Now, we indicate that $\big\{[k_n,l_n]\big\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{I},d)$. Due to (5.6), it is sufficient to show that $\big\{[k_{2n},l_{2n}]\big\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{I},d)$. Assume that $\big\{[k_{2n},l_{2n}]\big\}$ is not a Cauchy sequence. So there is an $\varepsilon > 0$ that we can detect two sequences of positive integers $\{2m(h)\}$ and $\{2n(h)\}$ with $2n(h) > 2m(h) > h$ such that

$$d\big(\big[k_{2m(h)},l_{2m(h)}\big],\big[k_{2n(h)},l_{2n(h)}\big]\big) \geq \varepsilon \quad \text{and} \quad d\big(\big[k_{2m(h)},l_{2m(h)}\big],\big[k_{2n(h)-2},l_{2n(h)-2}\big]\big) < \varepsilon,$$

for all positive integers h. Therefore we have

$$\begin{aligned}\varepsilon \leq & d\big(\big[k_{2m(h)},l_{2m(h)}\big],\big[k_{2n(h)},l_{2n(h)}\big]\big) \leq d\big(\big[k_{2m(h)},l_{2m(h)}\big],\big[k_{2n(h)-2},l_{2n(h)-2}\big]\big) \\ & + d\big(\big[k_{2n(h)-2},l_{2n(h)-2}\big],\big[k_{2n(h)-1},l_{2n(h)-1}\big]\big) + d\big(\big[k_{2n(h)-1},l_{2n(h)-1}\big],\big[k_{2n(h)},l_{2n(h)}\big]\big) \\ < & \varepsilon + d\big(\big[k_{2n(h)-2},l_{2n(h)-2}\big],\big[k_{2n(h)-1},l_{2n(h)-1}\big]\big) + d\big(\big[k_{2n(h)-1},l_{2n(h)-1}\big],\big[k_{2n(h)},l_{2n(h)}\big]\big).\end{aligned}$$

Taking $h \to \infty$ in the above inequality and making use of (5.6), we have

$$\lim_{h\to\infty} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big) = \varepsilon. \tag{5.7}$$

Again,

$$\begin{aligned} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big) \leq & d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big) \\ & + d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \\ & + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big), \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \leq & d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2m(h)}, l_{2m(h)}\right] \big) \\ & + d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big) \\ & + d\big(\left[k_{2n(h)}, l_{2n(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big). \end{aligned} \tag{5.9}$$

Taking $h \to \infty$ in the inequalities (5.8) and (5.9) and using (5.6) and (5.7), we obtain

$$\lim_{h\to\infty} d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) = \varepsilon. \tag{5.10}$$

Again,

$$\begin{aligned} d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big) \leq & d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \\ & + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big), \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big) \leq & d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right] \big) \\ & + d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big). \end{aligned} \tag{5.12}$$

Again,

$$\begin{aligned} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right] \big) \leq & d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big) \\ & + d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \\ & + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right] \big), \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \leq & d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2m(h)}, l_{2m(h)}\right] \big) \\ & + d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right] \big) \\ & + d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big). \end{aligned} \tag{5.14}$$

Further,

$$d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) \leq d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big) + d\big(\left[k_{2n(h)}, l_{2n(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big), \tag{5.15}$$

and

$$d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big) \leq d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right] \big) + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)}, l_{2n(h)}\right] \big). \tag{5.16}$$

Letting $h \to \infty$ in the inequalities (5.11)-(5.16) and using (5.6), (5.7) and (5.10), we have

$$\lim_{h\to\infty} d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right] \big) = \varepsilon, \tag{5.17}$$

$$\lim_{h\to\infty} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right] \big) = \varepsilon, \tag{5.18}$$

and

$$\lim_{h\to\infty} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) = \varepsilon. \tag{5.19}$$

By putting $[k,l] = \left[k_{2m(h)}, l_{2m(h)}\right]$ and $[u,v] = \left[k_{2n(h)+1}, l_{2n(h)+1}\right]$ in (5.1), we obtain

$$\begin{aligned}
&\psi\big(d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big)\big) = \psi\big(d\big(S\left[k_{2m(h)}, l_{2m(h)}\right], T\left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big)\big)\\
&\quad\le \psi\Big(\alpha d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) + \beta\Big[d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big)\\
&\quad + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big)\Big] + \gamma\Big[d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big)\\
&\quad + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big)\Big]\Big)\\
&\quad - \phi\Big(\max\Big\{d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big), d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big)\\
&\qquad d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big)\Big\}\Big).
\end{aligned}$$

Now taking $h\to\infty$ in the above inequality, using (5.6), (5.10) and (5.17)-(5.19) and making use of the continuity of ψ and ϕ, we have

$$\psi(\varepsilon) \le \psi((\alpha+2\gamma)\varepsilon) - \phi(\varepsilon),$$

since $\alpha + 2\gamma \le 1$ and ψ is strongly monotone increasing, we have

$$\psi(\varepsilon) \le \psi((\alpha+2\gamma)\varepsilon) - \phi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon) \Longrightarrow \psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon),$$

which is a contradiction as $\varepsilon > 0$ and $\phi(t) = 0$ for $t = 0$. Thus $\left\{ [k_{2n}, l_{2n}] \right\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{I}, d)$. Hence according to (5.6), $\left\{ [k_n, l_n] \right\}_{n=1}^{\infty}$ is also a Cauchy sequence in $(\mathcal{I}, d)$. Due to the completeness of the MIS $(\mathcal{I}, d)$, there is $[k,l] \in \mathcal{I}$ such that

$$d([k_n, l_n], [k,l]) \to 0. \tag{5.20}$$

Now we prove that any point $[\bar{k}, \bar{l}] \in \langle [k,l] \rangle$ is a common near fixed point for T and S. Due to $[\bar{k}, \bar{l}] \in \langle [k,l] \rangle$, we have

$$[\bar{k}, \bar{l}] \oplus \omega_1 = [k,l] \oplus \omega_2 \quad \text{for some } \ \omega_1, \omega_2 \in \Omega. \tag{5.21}$$

Since the null equality holds for d, according to proposition 3.3 and (5.20) we have

$$\lim_{n\to\infty} d([k_n, l_n], [\bar{k}, \bar{l}]) = 0 \ \text{ for any } \ [\bar{k}, \bar{l}] \in \langle [k,l] \rangle. \tag{5.22}$$

Therefore by setting $[k,l] = [k_{2n}, l_{2n}]$ and $[u,v] = [\bar{k}, \bar{l}]$ in (5.1), using the monotone increasing property of ψ and the triangle inequality, we obtain

$$\begin{aligned}
&\psi\big(d\big([k_{n+1}, l_{n+1}], T[\bar{k}, \bar{l}]\big)\big) = \psi\big(d\big(S[k_n, l_n], T[\bar{k}, \bar{l}]\big)\big)\\
&\quad\le \psi\Big(\alpha d\big([k_n, l_n], [\bar{k}, \bar{l}]\big) + \beta\big[d\big([k_n, l_n], [k_{n+1}, l_{n+1}]\big) + d\big([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]\big)\big]\\
&\qquad + \gamma\big[d\big([k_n, l_n], T[\bar{k}, \bar{l}]\big) + d\big([\bar{k}, \bar{l}], [k_{n+1}, l_{n+1}]\big)\big]\Big)\\
&\qquad - \phi\Big(\max\Big\{d\big([k_n, l_n], [\bar{k}, \bar{l}]\big), d\big([k_n, l_n], [k_{n+1}, l_{n+1}]\big), d\big([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]\big)\Big\}\Big)\\
&\quad\le \psi\Big(\alpha d\big([k_n, l_n], [\bar{k}, \bar{l}]\big) + \beta\big[d\big([k_n, l_n], [k_{n+1}, l_{n+1}]\big) + d\big([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]\big)\big]\\
&\qquad + \gamma\big[d\big([k_n, l_n], [\bar{k}, \bar{l}]\big) + d\big([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]\big) + d\big([\bar{k}, \bar{l}], [k_{n+1}, l_{n+1}]\big)\big]\Big)\\
&\qquad - \phi\Big(\max\Big\{d\big([k_n, l_n], [\bar{k}, \bar{l}]\big), d\big([k_n, l_n], [k_{n+1}, l_{n+1}]\big), d\big([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]\big)\Big\}\Big).
\end{aligned} \tag{5.23}$$

Note that due to Lemma 5.6 and (5.22), we have $\lim_{n\to\infty} d\big([k_{n+1}, l_{n+1}], T[\bar{k}, \bar{l}]\big) = d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])$. Thus taking $n\to\infty$ in (5.23), using Lemma 5.6 and the continuity of ϕ and ψ and making use of (5.6) and (5.22), we have

$$\psi\big(d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big) \le \psi\big((\beta+\gamma) d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big) - \phi\big(d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big).$$

Due to $\beta+\gamma \le 1$ and the strongly monotone property of ψ, it follows that

$$\begin{aligned}\psi\big(d(\left[\bar{k},\bar{l}\right],T\left[\bar{k},\bar{l}\right])\big) &\le \psi\big((\beta+\gamma)d(\left[\bar{k},\bar{l}\right],T\left[\bar{k},\bar{l}\right])\big)-\phi\big(d(\left[\bar{k},\bar{l}\right],T\left[\bar{k},\bar{l}\right])\big)\\ &\le \psi\big(d(\left[\bar{k},\bar{l}\right],T\left[\bar{k},\bar{l}\right])\big)-\phi\big(d(\left[\bar{k},\bar{l}\right],T\left[\bar{k},\bar{l}\right])\big),\end{aligned}$$

which implies that $\phi\big(d(\left[\bar{k},\bar{l}\right],T\left[\bar{k},\bar{l}\right])\big)\le 0$. Hence according to nonnegativity of ϕ, we have $\phi\big(d(\left[\bar{k},\bar{l}\right],T\left[\bar{k},\bar{l}\right])\big)=0$. Thus $d(\left[\bar{k},\bar{l}\right],T\left[\bar{k},\bar{l}\right])=0$, i.e., $T\left[\bar{k},\bar{l}\right]\stackrel{\Omega}{=}\left[\bar{k},\bar{l}\right]$ for any point $\left[\bar{k},\bar{l}\right]\in\left\langle\,[k,l]\,\right\rangle$. Since any near fixed point for T is a near fixed point for S, then $S\left[\bar{k},\bar{l}\right]\stackrel{\Omega}{=}\left[\bar{k},\bar{l}\right]$ for any $\left[\bar{k},\bar{l}\right]\in\left\langle\,[k,l]\,\right\rangle$. Hence according to proposition 2.3, we have $T\left[\bar{k},\bar{l}\right]\stackrel{\Omega}{=}\left[\bar{k},\bar{l}\right]\stackrel{\Omega}{=}S\left[\bar{k},\bar{l}\right]$ for any point $\left[\bar{k},\bar{l}\right]\in\left\langle\,[k,l]\,\right\rangle$. □

Example 5.8. Let $\phi,\psi:[0,\infty)\to[0,\infty)$ and $S,T:\mathcal{I}\to\mathcal{I}$ in the MIS $(\mathcal{I},d)$ be defined by $\phi(t)=\dfrac{t}{8}$, $\psi(t)=t$, $S[k,l]=\left[-1+\frac{k}{5},1+\frac{l}{5}\right]$ and $T[k,l]=\left[-2+\frac{k}{5},2+\frac{l}{5}\right]$, respectively. Define a mapping $d:\mathcal{I}\times\mathcal{I}\to\mathbb{R}^+$ by $d([k,l],[u,v])=|(k+l)-(u+v)|$ for all $[k,l],[u,v]\in\mathcal{I}$. Suppose that $\alpha=\frac{1}{2}$, $\beta=\frac{1}{8}$ and $\gamma=\frac{1}{8}$. Then by (5.1), we have

$$\begin{aligned}&\psi\bigg(d\Big(\Big[-1+\frac{k}{5},1+\frac{l}{5}\Big],\Big[-2+\frac{u}{5},2+\frac{v}{5}\Big]\Big)\bigg)\le\psi\bigg(\alpha d([k,l],[u,v])+\beta\Big[d\Big([k,l],\Big[-1+\frac{k}{5},1+\frac{l}{5}\Big]\Big)\\ &+d\Big([u,v],\Big[-2+\frac{u}{5},2+\frac{v}{5}\Big]\Big)\Big]+\gamma\Big[d([k,l],\Big[-2+\frac{u}{5},2+\frac{v}{5}\Big])+d([u,v],\Big[-1+\frac{k}{5},1+\frac{l}{5}\Big])\Big]\\ &-\phi\bigg(\max\Big\{d([k,l],[u,v]),d\Big([k,l],\Big[-1+\frac{k}{5},1+\frac{l}{5}\Big]\Big),d\Big([u,v],\Big[-2+\frac{u}{5},2+\frac{v}{5}\Big]\Big)\Big\}\bigg).\end{aligned}$$

Then,

$$\begin{aligned}&\psi\Big(\frac{1}{5}\big|(k+l)-(u+v)\big|\Big)\le\psi\bigg(\alpha\big|(k+l)-(u+v)\big|+\beta\Big[\frac{4}{5}|k+l|+\frac{4}{5}|u+v|\Big]\\ &+\gamma\Big[\big|(k+l)-\frac{1}{5}(u+v)\big|+\big|(u+v)-\frac{1}{5}(k+l)\big|\Big]\bigg)-\phi\bigg(\max\Big\{\big|(k+l)-(u+v)\big|,\frac{4}{5}\big|k+l\big|,\frac{4}{5}\big|u+v\big|\Big\}\bigg).\end{aligned}$$

So,

$$\begin{aligned}&\frac{1}{5}\big|(k+l)-(u+v)\big|\le\alpha\big|(k+l)-(u+v)\big|+\frac{4}{5}\beta\Big[\big|k+l\big|+\big|u+v\big|\Big]\\ &+\gamma\Big[\big|(k+l)-\frac{1}{5}(u+v)\big|+\big|(u+v)-\frac{1}{5}(k+l)\big|\Big]-\phi\bigg(\max\Big\{\big|(k+l)-(u+v)\big|,\frac{4}{5}\big|k+l\big|,\frac{4}{5}\big|u+v\big|\Big\}\bigg).\end{aligned}$$

Thus, all conditions of Theorem 5.7 hold for this example. Hence, T and S have a unique equivalence class of common near fixed points $\left\langle\,[-1,1]\,\right\rangle$ in $\mathcal{I}$.

Corollary 5.9. *Suppose that $(\mathcal{I},d)$ is a CMIS and the null equality holds for d. Assume that $T:(\mathcal{I},d)\to(\mathcal{I},d)$ is a self-mapping satisfying the following inequality for all $[k,l],[u,v]\in\mathcal{I}$,*

$$\psi\big(d(T[k,l],T[u,v])\big)\le\psi\big(M([k,l],[u,v])\big)-\phi\big(\max\{d([k,l],[u,v]),d([u,v],T[u,v])\}\big),$$

where $\phi:[0,\infty)\to[0,\infty)$ is a continuous, nondecreasing function with the property $\phi(t)=0\Leftrightarrow t=0$, $\psi:[0,\infty)\to[0,\infty)$ is an alternating distance function and

$$M([k,l],[u,v])=\alpha d([k,l],[u,v])+\beta\big[d([k,l],T[k,l])+d([u,v],T[u,v])\big]+\gamma\big[d([k,l],T[u,v])+d([u,v],T[k,l])\big],$$

with $\alpha,\beta,\gamma\ge 0$, $\alpha+2\beta+2\gamma\le 1$. Then a near fixed point $[k,l]\in\mathcal{I}$ exists for T where $T[k,l]\stackrel{\Omega}{=}[k,l]$.
Moreover, a unique equivalence class $\left\langle\,[k,l]\,\right\rangle$ exists for $\mathcal{I}$ such that if $\left[\bar{k},\bar{l}\right]$ is another near fixed point for T, then $\left\langle\,[k,l]\,\right\rangle=\left\langle\,\left[\bar{k},\bar{l}\right]\,\right\rangle$. In addition, every point $\left[\bar{k},\bar{l}\right]\in\left\langle\,[k,l]\,\right\rangle$ is a near fixed point for T.

Corollary 5.10. *Suppose that $(\mathcal{I},\|\cdot\|)$ is a BIS such that the null equality and null condition hold for $\|\cdot\|$. Assume that $S,T:(\mathcal{I},\|\cdot\|)\to(\mathcal{I},\|\cdot\|)$ are self-mappings of $\mathcal{I}$ such that for all $[k,l],[u,v]\in\mathcal{I}$,*

$$\psi\big(\|S[k,l]\ominus T[u,v]\|\big)\le\psi\big(M([k,l],[u,v])\big)-\phi\big(\max\{\|[k,l]\ominus[u,v]\|,\|[k,l]\ominus S[k,l]\|,\|[u,v]\ominus T[u,v]\|\}\big),\quad(5.24)$$

where $\phi:[0,\infty)\to[0,\infty)$ is a continuous, nondecreasing function with the property $\phi(t)=0\Leftrightarrow t=0$, $\psi:[0,\infty)\to[0,\infty)$ is an alternating distance function and

$$M([k,l],[u,v])=\alpha\|[k,l]\ominus[u,v]\|+\beta\big[\|[k,l]\ominus S[k,l]\|+\|[u,v]\ominus T[u,v]\|\big]\gamma\big[\|[k,l]\ominus T[u,v]\|+\|[u,v]\ominus S[k,l]\|\big],$$

with $\alpha,\beta,\gamma\ge0$, $\alpha+2\beta+2\gamma\le1$. Then a common near fixed point $[k,l]\in\mathcal{I}$ exists for T and S satisfying $T[k,l]\overset{\Omega}{=}[k,l]\overset{\Omega}{=}S[k,l]$. Moreover, a unique equivalence class $\langle\,[k,l]\,\rangle$ exists for $\mathcal{I}$ such that if $[\bar{k},\bar{l}]$ is another common near fixed point for T and S, then $\langle\,[k,l]\,\rangle=\langle\,[\bar{k},\bar{l}]\,\rangle$. In addition, any near fixed point of T is a near fixed point of S and conversely.

Example 5.11. Let $\phi,\psi:[0,\infty)\to[0,\infty)$ and $S,T:\mathcal{I}\to\mathcal{I}$ in the BIS $(\mathcal{I},\|\cdot\|)$ be defined by $\phi(t)=\dfrac{t}{9}$, $\psi(t)=t$, $S[k,l]=\big[-1+\frac{k}{3},1+\frac{l}{3}\big]$ and $T[k,l]=\big[-2+\frac{k}{3},2+\frac{l}{3}\big]$, respectively. Define $\|\cdot\|:\mathcal{I}\to\mathbb{R}^+$ by $\|[k,l]\|=|k+l|$. Suppose that $\alpha=\frac{1}{2}$, $\beta=\frac{1}{8}$ and $\gamma=\frac{1}{8}$. Note that for all $[k,l],[u,v]\in\mathcal{I}$, we have

$$\|[k,l]\ominus[u,v]\|=\|[k,l]\oplus[-v,-u]\|=\|[k-v,l-u]\|=|(k-v)+(l-u)|=|(k+l)-(u+v)|.$$

Then by (5.24), we have

$$\begin{aligned}\psi\bigg(\bigg\|\Big[-1+\frac{k}{3},1+\frac{l}{3}\Big]\ominus\Big[-2+\frac{u}{3},2+\frac{v}{3}\Big]\bigg\|\bigg)&\le\psi\bigg(\alpha\|[k,l]\ominus[u,v]\|+\beta\bigg[\bigg\|[k,l]\ominus\Big[-1+\frac{k}{3},1+\frac{l}{3}\Big]\bigg\|\\&+\bigg\|[u,v]\ominus\Big[-2+\frac{u}{3},2+\frac{v}{3}\Big]\bigg\|\bigg]+\gamma\bigg[\bigg\|[k,l]\ominus\Big[-2+\frac{u}{3},2+\frac{v}{3}\Big]\bigg\|+\bigg\|[u,v]\ominus\Big[-1+\frac{k}{3},1+\frac{l}{3}\Big]\bigg\|\bigg]\\&-\phi\bigg(\max\bigg\{\|[k,l]\ominus[u,v]\|,\bigg\|[k,l]\ominus\Big[-1+\frac{k}{3},1+\frac{l}{3}\Big]\bigg\|,\bigg\|[u,v]\ominus\Big[-2+\frac{u}{3},2+\frac{v}{3}\Big]\bigg\|\bigg\}\bigg).\end{aligned}$$

This implies that

$$\begin{aligned}\psi\Big(\frac{1}{3}\big|(k+l)-(u+v)\big|\Big)&\le\psi\bigg(\alpha\big|(k+l)-(u+v)\big|+\beta\Big[\frac{2}{3}|k+l|+\frac{2}{3}|u+v|\Big]\\&+\gamma\Big[\big|(k+l)-\frac{1}{3}(u+v)\big|+\big|(u+v)-\frac{1}{3}(k+l)\big|\Big]\bigg)-\phi\bigg(\max\Big\{\big|(k+l)-(u+v)\big|,\frac{2}{3}\big|k+l\big|,\frac{2}{3}\big|u+v\big|\Big\}\bigg).\end{aligned}$$

So,

$$\begin{aligned}\frac{1}{3}\big|(k+l)-(u+v)\big|&\le\alpha\big|(k+l)-(u+v)\big|+\frac{2}{3}\beta\Big[\big|k+l\big|+\big|u+v\big|\Big]\\&+\gamma\Big[\big|(k+l)-\frac{1}{3}(u+v)\big|+\big|(u+v)-\frac{1}{3}(k+l)\big|\Big]-\phi\bigg(\max\Big\{\big|(k+l)-(u+v)\big|,\frac{2}{3}\big|k+l\big|,\frac{2}{3}\big|u+v\big|\Big\}\bigg).\end{aligned}$$

Thus, this example satisfies all conditions of Corollary 5.10. Hence, T and S have a unique equivalence class of common near fixed points $\langle\,[-2,2]\,\rangle$ in $\mathcal{I}$.

Theorem 5.12. *Suppose that $(\mathcal{I},d)$ is a CMIS and the null equality holds for d. Assume that $T,S:(\mathcal{I},d)\to(\mathcal{I},d)$ are self-mappings of $\mathcal{I}$ such that for all $[k,l],[u,v]\in\mathcal{I}$,*

$$\psi\big(d(S[k,l],T[u,v])\big)\le\psi\big(M([k,l],[u,v])\big)-\phi\big(M([k,l],[u,v])\big),\quad(5.25)$$

where $\phi:[0,\infty)\to[0,\infty)$ is a continuous, nondecreasing function with the property $\phi(t)=0\Leftrightarrow t=0$, $\psi:[0,\infty)\to[0,\infty)$ is an alternating distance function and

$$M([k,l],[u,v])=\alpha d([k,l],[u,v])+\beta\big[d([k,l],S[k,l])+d([u,v],T[u,v])\big]\gamma\big[d([k,l],T[u,v])+d([u,v],S[k,l])\big],\quad(5.26)$$

with $\alpha, \beta > 0, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma \leq 1$. Then a common near fixed point $[k,l] \in \mathcal{I}$ exists for T and S such that $T\,[k,l] \overset{\Omega}{=} [k,l] \overset{\Omega}{=} S\,[k,l]$. Moreover, a unique equivalence class $\langle\, [k,l]\,\rangle$ exists for $\mathcal{I}$ such that if $[\bar{k},\bar{l}]$ is another common near fixed point for T and S, then $\langle\, [k,l]\,\rangle = \langle\, [\bar{k},\bar{l}]\,\rangle$ and $[k,l] \overset{\Omega}{=} [\bar{k},\bar{l}]$. Every point $[\bar{k},\bar{l}] \in \langle\, [k,l]\,\rangle$ is also a common near fixed point for T and S. In addition, any near fixed point of T is a near fixed point of S and conversely.

Proof . Assume that T and S have two common near fixed points $[k,l]$ and $\left[\tilde{k},\tilde{l}\right]$ such that $[k,l] \overset{\Omega}{\neq} \left[\tilde{k},\tilde{l}\right]$, i.e., $\left[\tilde{k},\tilde{l}\right] \overset{\Omega}{=} T\left[\tilde{k},\tilde{l}\right] \overset{\Omega}{=} S\left[\tilde{k},\tilde{l}\right]$, $[k,l] \overset{\Omega}{=} T\,[k,l] \overset{\Omega}{=} S\,[k,l]$ and $\left[\tilde{k},\tilde{l}\right] \notin \langle\, [k,l]\,\rangle$. Then

$$\left[\tilde{k},\tilde{l}\right] \oplus \omega_1 = T\left[\tilde{k},\tilde{l}\right] \oplus \omega_2 \quad \text{and} \quad [k,l] \oplus \omega_5 = T\,[k,l] \oplus \omega_6 \tag{5.27}$$

$$[k,l] \oplus \omega_3 = S\,[k,l] \oplus \omega_4 \quad \text{and} \quad \left[\tilde{k},\tilde{l}\right] \oplus \omega_7 = S\left[\tilde{k},\tilde{l}\right] \oplus \omega_8 \tag{5.28}$$

for some $\omega_i \in \Omega$, $i = 1, \ldots, 8$. Note that $d(\left[\tilde{k},\tilde{l}\right], T\left[\tilde{k},\tilde{l}\right]) = 0$ and $d([k,l]\,, S\,[k,l]) = 0$. Then using (5.25), (5.27) and (5.28) and making use of the null equality and the triangle inequality, we have

$$\begin{aligned}
\psi\big(d([k,l]\,,\left[\tilde{k},\tilde{l}\right])\big) &= \psi\big(d([k,l] \oplus \omega_3, \left[\tilde{k},\tilde{l}\right] \oplus \omega_1)\big) = \psi\big(d(S\,[k,l] \oplus \omega_4, T\left[\tilde{k},\tilde{l}\right] \oplus \omega_2)\big) \\
&= \psi\big(d(S\,[k,l]\,, T\left[\tilde{k},\tilde{l}\right])\big) \\
&\leq \psi\big(M\big(\,[k,l]\,,\left[\tilde{k},\tilde{l}\right]\,\big)\big) - \phi\big(M\big(\,[k,l]\,,\left[\tilde{k},\tilde{l}\right]\,\big)\big),
\end{aligned}$$

where

$$\begin{aligned}
M([k,l]\,,\left[\tilde{k},\tilde{l}\right]) &= \alpha d([k,l]\,,\left[\tilde{k},\tilde{l}\right]) + \beta\big[d([k,l]\,, S\,[k,l]) + d(\left[\tilde{k},\tilde{l}\right], T\left[\tilde{k},\tilde{l}\right])\big] + \gamma\big[d([k,l]\,, T\left[\tilde{k},\tilde{l}\right]) + d(\left[\tilde{k},\tilde{l}\right], S\,[k,l])\big] \\
&\leq \alpha d([k,l]\,,\left[\tilde{k},\tilde{l}\right]) + \beta\big[d([k,l]\,, S\,[k,l]) + d(\left[\tilde{k},\tilde{l}\right], T\left[\tilde{k},\tilde{l}\right])\big] \\
&\quad + \gamma\big[d([k,l]\,,\left[\tilde{k},\tilde{l}\right]) + d(\left[\tilde{k},\tilde{l}\right], T\left[\tilde{k},\tilde{l}\right]) + d([k,l]\,,\left[\tilde{k},\tilde{l}\right]) + d([k,l]\,, S\,[k,l])\big] \\
&= (\alpha + 2\gamma) d([k,l]\,,\left[\tilde{k},\tilde{l}\right]).
\end{aligned} \tag{5.29}$$

Therefore, according to the monotone increasing property of ψ, we have

$$\psi\big(d([k,l]\,,\left[\tilde{k},\tilde{l}\right])\big) \leq \psi\big((\alpha + 2\gamma) d([k,l]\,,\left[\tilde{k},\tilde{l}\right])\big) - \phi\big(M([k,l]\,,\left[\tilde{k},\tilde{l}\right])\big).$$

Due to $\alpha + 2\gamma \leq 1$ and the monotone increasing property of ψ, it results that

$$\psi\big(d([k,l]\,,\left[\tilde{k},\tilde{l}\right])\big) \leq \psi\big(d([k,l]\,,\left[\tilde{k},\tilde{l}\right])\big) - \phi\big(M([k,l]\,,\left[\tilde{k},\tilde{l}\right])\big),$$

which implies that $\phi\big(M([k,l]\,,\left[\tilde{k},\tilde{l}\right])\big) \leq 0$. So we have $M([k,l]\,,\left[\tilde{k},\tilde{l}\right]) = 0$. Which is a contradiction since $d([k,l]\,,\left[\tilde{k},\tilde{l}\right]) > 0$ and $\alpha > 0$. Hence $d([k,l]\,,\left[\tilde{k},\tilde{l}\right]) = 0$, i.e., $[k,l] \overset{\Omega}{=} \left[\tilde{k},\tilde{l}\right]$. Thus, any $\left[\tilde{k},\tilde{l}\right] \notin \langle\, [k,l]\,\rangle$ cannot be a common near fixed point for S and T. In fact, if $\left[\tilde{k},\tilde{l}\right]$ is another common near fixed point of T and S, then $\left[\tilde{k},\tilde{l}\right] \in \langle\, [k,l]\,\rangle$ i.e., $[k,l] \overset{\Omega}{=} \left[\tilde{k},\tilde{l}\right]$. Now assume that $[k,l]$ is a near fixed point of S and $[k,l] \overset{\Omega}{\neq} T\,[k,l]$. Then by using (5.25), (5.28) and the null equality, we have

$$\begin{aligned}
\psi\big(d([k,l]\,, T\,[k,l])\big) &= \psi\big(d([k,l] \oplus \omega_3, T\,[k,l])\big) = \psi\big(d(S\,[k,l] \oplus \omega_4, T\,[k,l])\big) \\
&= \psi\big(d(S\,[k,l]\,, T\,[k,l])\big) \\
&\leq \psi\big(M\big(\,[k,l]\,,[k,l]\,\big)\big) - \phi\big(M\big(\,[k,l]\,,[k,l]\,\big)\big),
\end{aligned}$$

where

$$\begin{aligned}
M([k,l]\,,[k,l]) &= \alpha d([k,l]\,,[k,l]) + \beta\big[d([k,l]\,, S\,[k,l]) + d([k,l]\,, T\,[k,l])\big] + \gamma\big[d([k,l]\,, T\,[k,l]) + d([k,l]\,, S\,[k,l])\big] \\
&= (\beta + \gamma) d([k,l]\,, T\,[k,l]).
\end{aligned} \tag{5.30}$$

Thus

$$\psi\big(d([k,l]\,,T\,[k,l])\big) \leq \psi\big((\beta+\gamma)d([k,l]\,,T\,[k,l])\big) - \phi\big((\beta+\gamma)M([k,l]\,,[k,l])\big),$$

since $\beta+\gamma \leq 1$ and ψ is the monotone increasing, it implies that

$$\psi\big(d([k,l]\,,T\,[k,l])\big) \leq \psi\big(d([k,l]\,,T\,[k,l])\big) - \phi\big((\beta+\gamma)d([k,l]\,,T\,[k,l])\big),$$

which is a contradiction as $d([k,l]\,,T\,[k,l]) > 0$ and $\beta > 0$. Hence $d([k,l]\,,T\,[k,l]) = 0$, i.e., $[k,l] \overset{\Omega}{=} T\,[k,l]$. Similarly, any near fixed point for T is also a near fixed point for S.

Now, let $[k_0,l_0] \in \mathcal{I}$ an arbitrary element. Consider a sequence $\big\{\,[k_n,l_n]\,\big\}_{n=1}^{\infty}$ in $\mathcal{I}$ such that $[k_{2n+1},l_{2n+1}] = S\,[k_{2n},l_{2n}]$ and $[k_{2n+2},l_{2n+2}] = T\,[k_{2n+1},l_{2n+1}]$ for $n \geq 0$. If there is a positive integer $2N$ such that $[k_{2N},l_{2N}] = [k_{2N+1},l_{2N+1}]$, then $[k_{2N},l_{2N}]$ is a near fixed point for S and hence it is also a near fixed point for T. We have a similar result if $[k_{2N+1},l_{2N+1}] = [k_{2N+2},l_{2N+2}]$ for some N. Therefore, suppose that $d([k_{n+1},l_{n+1}]\,,[k_n,l_n]) \neq 0$ for all $n \geq 0$. Then from (5.26), we have

$$M\big(\,[k_n,l_n]\,,[k_{n+1},l_{n+1}]\,\big) > 0, \tag{5.31}$$

for all $n \in \mathbb{N}$. Hence from (5.25), we have

$$\begin{aligned}\psi\big(d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)\big) =&\psi\big(d(S\,[k_{2n},l_{2n}]\,,T\,[k_{2n+1},l_{2n+1}])\big)\\ \leq&\psi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big) - \phi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big),\end{aligned} \tag{5.32}$$

where

$$\begin{aligned}M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) =&\alpha d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\\ &+\beta\big[d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) + d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)\big]\\ &+\gamma\big[d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+2},l_{2n+2}]\,\big) + d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big].\end{aligned}$$

Since

$$d([k_{2n},l_{2n}]\,,[k_{2n+2},l_{2n+2}]) \leq d([k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]) + d([k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]),$$

it results that

$$\begin{aligned}M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) \leq&\alpha d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\\ &+\beta\big[d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) + d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)\big]\\ &+\gamma\big[d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) + d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)\big]\\ =&(\alpha+\beta+\gamma)d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) + (\beta+\gamma)d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big).\end{aligned} \tag{5.33}$$

Now, we show that $d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big) \leq d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)$ for all $n \in \mathbb{N}$. Suppose that $d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big) > d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)$ for some $k \in \mathbb{N}$. Then from (5.32) and (5.33), we obtain

$$\begin{aligned}&\psi\big(d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)\big) \leq \psi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big) - \phi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big)\\ &= \psi\Big((\alpha+\beta+\gamma)d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) + (\beta+\gamma)d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)\Big)\\ &\quad - \phi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big)\\ &\leq \psi\big((\alpha+2\beta+2\gamma)d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)\big) - \phi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big)\\ &\leq \psi\big(d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)\big) - \phi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big)\ (\text{since } (\alpha+2\beta+2\gamma) \leq 1),\end{aligned}$$

which results that $\phi\big(M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)\big) \leq 0$. Thus we have $M\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big) = 0$, which is a contradiction according to (5.31). Therefore $d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big) \leq d\big(\,[k_{2n},l_{2n}]\,,[k_{2n+1},l_{2n+1}]\,\big)$ for all $n \in \mathbb{N}$. Similarly, we also obtain $d\big(\,[k_{2n+2},l_{2n+2}]\,,[k_{2n+3},l_{2n+3}]\,\big) \leq d\big(\,[k_{2n+1},l_{2n+1}]\,,[k_{2n+2},l_{2n+2}]\,\big)$ for all $n \in \mathbb{N}$. So $\big\{d([k_n,l_n]\,,[k_{n+1},l_{n+1}])\big\}$ is a monotone non-increasing sequence of non-negative real numbers. Therefore, there is an $r \geq 0$ such that

$$\lim_{n\to\infty} d([k_n,l_n]\,,[k_{n+1},l_{n+1}]) = r. \tag{5.34}$$

Assume that $r > 0$. Then taking $n \to \infty$ in (5.33), we have

$$\lim_{n\to\infty} M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big) \le (\alpha + 2\beta + 2\gamma)r. \tag{5.35}$$

Therefore according to the above facts, from (5.32) and (5.33), we obtain

$$\begin{aligned}
&\psi\big(d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)\big) \le \psi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big) - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big)\\
&= \psi\Big((\alpha+\beta+\gamma)d\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big) + (\beta+\gamma)d\big([k_{2n+1}, l_{2n+1}] , [k_{2n+2}, l_{2n+2}] \big)\Big)\\
&\quad - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big)\\
&\le \psi\big((\alpha+2\beta+2\gamma)d\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big) - \phi\big(M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big).
\end{aligned} \tag{5.36}$$

Letting $n \to \infty$ in (5.36) and making use of (5.34) and the continuity of ψ and ϕ, we have

$$\psi(r) \le \psi((\alpha + 2\beta + 2\gamma)r) - \phi\big(\lim_{n\to\infty} M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big),$$

since $\alpha + 2\beta + 2\gamma \le 1$ and ψ is monotone increasing, it implies that

$$\psi(r) \le \psi(r) - \phi\big(\lim_{n\to\infty} M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big),$$

which implies that $\phi\big(\lim_{n\to\infty} M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big)\big) \le 0$. Hence, we have $\lim_{n\to\infty} M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big) = 0$. Which is a contradiction since $r, \alpha > 0$ and from (5.31) and (5.35), we have $0 < \lim_{n\to\infty} M\big([k_{2n}, l_{2n}] , [k_{2n+1}, l_{2n+1}] \big) \le (\alpha + 2\beta + 2\gamma)r$. Thus

$$\lim_{n\to\infty} d([k_n, l_n] , [k_{n+1}, l_{n+1}]) = 0. \tag{5.37}$$

Now we indicate that $\{ [k_n, l_n] \}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{I}, d)$. Due to (5.37), it is sufficient to show that $\{ [k_{2n}, l_{2n}] \}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{I}, d)$. Assume that $\{ [k_{2n}, l_{2n}] \}$ is not a Cauchy sequence. So there is an $\varepsilon > 0$ that we can detect two sequences of positive integers $\{2m(h)\}$ and $\{2n(h)\}$ with $2n(h) > 2m(h) > h$ such that

$$d\big(\big[k_{2m(h)}, l_{2m(h)}\big] , \big[k_{2n(h)}, l_{2n(h)}\big] \big) \ge \varepsilon \quad \text{and} \quad d\big(\big[k_{2m(h)}, l_{2m(h)}\big] , \big[k_{2n(h)-2}, l_{2n(h)-2}\big] \big) < \varepsilon,$$

for all positive integers h. Therefore we have

$$\begin{aligned}
\varepsilon \le& d\big(\big[k_{2m(h)}, l_{2m(h)}\big] , \big[k_{2n(h)}, l_{2n(h)}\big] \big) \le d\big(\big[k_{2m(h)}, l_{2m(h)}\big] , \big[k_{2n(h)-2}, l_{2n(h)-2}\big] \big)\\
&+ d\big(\big[k_{2n(h)-2}, l_{2n(h)-2}\big] , \big[k_{2n(h)-1}, l_{2n(h)-1}\big] \big) + d\big(\big[k_{2n(h)-1}, l_{2n(h)-1}\big] , \big[k_{2n(h)}, l_{2n(h)}\big] \big)\\
<& \varepsilon + d\big(\big[k_{2n(h)-2}, l_{2n(h)-2}\big] , \big[k_{2n(h)-1}, l_{2n(h)-1}\big] \big) + d\big(\big[k_{2n(h)-1}, l_{2n(h)-1}\big] , \big[k_{2n(h)}, l_{2n(h)}\big] \big).
\end{aligned}$$

Taking $h \to \infty$ in the above inequality and making use of (5.37), we have

$$\lim_{h\to\infty} d\big(\big[k_{2m(h)}, l_{2m(h)}\big] , \big[k_{2n(h)}, l_{2n(h)}\big] \big) = \varepsilon. \tag{5.38}$$

Again,

$$\begin{aligned}
d\big(\big[k_{2m(h)}, l_{2m(h)}\big] , \big[k_{2n(h)}, l_{2n(h)}\big] \big) \le& d\big(\big[k_{2m(h)}, l_{2m(h)}\big] , \big[k_{2m(h)+1}, l_{2m(h)+1}\big] \big)\\
&+ d\big(\big[k_{2m(h)+1}, l_{2m(h)+1}\big] , \big[k_{2n(h)+1}, l_{2n(h)+1}\big] \big)\\
&+ d\big(\big[k_{2n(h)+1}, l_{2n(h)+1}\big] , \big[k_{2n(h)}, l_{2n(h)}\big] \big),
\end{aligned} \tag{5.39}$$

and

$$\begin{aligned}
d\big(\big[k_{2m(h)+1}, l_{2m(h)+1}\big] , \big[k_{2n(h)+1}, l_{2n(h)+1}\big] \big) \le& d\big(\big[k_{2m(h)+1}, l_{2m(h)+1}\big] , \big[k_{2m(h)}, l_{2m(h)}\big] \big)\\
&+ d\big(\big[k_{2m(h)}, l_{2m(h)}\big] , \big[k_{2n(h)}, l_{2n(h)}\big] \big)\\
&+ d\big(\big[k_{2n(h)}, l_{2n(h)}\big] , \big[k_{2n(h)+1}, l_{2n(h)+1}\big] \big).
\end{aligned} \tag{5.40}$$

Taking $h \to \infty$ in the inequalities (5.39) and (5.40) and using (5.37) and (5.38), we obtain

$$\lim_{h\to\infty} d\big(\big[k_{2m(h)+1}, l_{2m(h)+1}\big] , \big[k_{2n(h)+1}, l_{2n(h)+1}\big] \big) = \varepsilon. \tag{5.41}$$

Again,

$$\begin{aligned} d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) \leq & d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) \\ & + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big), \end{aligned} \tag{5.42}$$

and

$$\begin{aligned} d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) \leq & d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big) \\ & + d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big). \end{aligned} \tag{5.43}$$

Moreover,

$$\begin{aligned} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big) \leq & d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) \\ & + d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) \\ & + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big), \end{aligned} \tag{5.44}$$

and

$$\begin{aligned} d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) \leq & d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2m(h)}, l_{2m(h)}\right]\big) \\ & + d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big) \\ & + d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big). \end{aligned} \tag{5.45}$$

Furthermore,

$$d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) \leq d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right]\big) + d\big(\left[k_{2n(h)}, l_{2n(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big), \tag{5.46}$$

and

$$d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)}, l_{2n(h)}\right]\big) \leq d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)}, l_{2n(h)}\right]\big). \tag{5.47}$$

Letting $h \to \infty$ in the inequalities (5.42)-(5.47) and using (5.37), (5.38) and (5.41), we have

$$\lim_{h\to\infty} d\big(\left[k_{2n(h)+2}, l_{2n(h)+2}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) = \varepsilon, \tag{5.48}$$

$$\lim_{h\to\infty} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big) = \varepsilon, \tag{5.49}$$

and

$$\lim_{h\to\infty} d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) = \varepsilon. \tag{5.50}$$

By putting $[k, l] = \left[k_{2m(h)}, l_{2m(h)}\right]$ and $[u, v] = \left[k_{2n(h)+1}, l_{2n(h)+1}\right]$ in (5.25), we obtain

$$\begin{aligned} \psi(d\big(\left[k_{2m(h)+1}, l_{2m(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big)) &= \psi(d(S\left[k_{2m(h)}, l_{2m(h)}\right], T\left[k_{2n(h)+1}, l_{2n(h)+1}\right])) \\ &\leq \psi(M\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big)) - \phi(M\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big)), \end{aligned} \tag{5.51}$$

where

$$\begin{aligned} & M\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) \\ & \quad = \alpha d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+1}, l_{2n(h)+1}\right]\big) + \beta\Big[d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big) \\ & \quad + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big)\Big] + \gamma\Big[d\big(\left[k_{2m(h)}, l_{2m(h)}\right], \left[k_{2n(h)+2}, l_{2n(h)+2}\right]\big) \\ & \quad + d\big(\left[k_{2n(h)+1}, l_{2n(h)+1}\right], \left[k_{2m(h)+1}, l_{2m(h)+1}\right]\big)\Big]. \end{aligned} \tag{5.52}$$

Now taking $h \to \infty$ in (5.51) and (5.52), using (5.37), (5.41) and (5.48)-(5.50) and making use of the continuity of ψ and ϕ, we have

$$\psi(\varepsilon) \leq \psi((\alpha+2\gamma)\varepsilon) - \phi((\alpha+2\gamma)\varepsilon),$$

since $\alpha + 2\gamma \leq 1$ and ψ is monotone increasing, it results that

$$\psi(\varepsilon) \leq \psi((\alpha+2\gamma)\varepsilon) - \phi((\alpha+2\gamma)\varepsilon) \leq \psi(\varepsilon) - \phi((\alpha+2\gamma)\varepsilon) \Longrightarrow \psi(\varepsilon) \leq \psi(\varepsilon) - \phi((\alpha+2\gamma)\varepsilon).$$

Which is a contradiction as $\varepsilon > 0$ and $\alpha > 0$. So $\{ [k_{2n}, l_{2n}] \}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{I}, d)$. Hence according to (5.37), $\{ [k_n, l_n] \}_{n=1}^{\infty}$ is also a Cauchy sequence in $(\mathcal{I}, d)$. Due to the completeness of the MIS $(\mathcal{I}, d)$, there is $[k,l] \in \mathcal{I}$ such that

$$d([k_n, l_n], [k,l]) \to 0. \tag{5.53}$$

Now, we indicate that any point $[\bar{k}, \bar{l}] \in \langle [k,l] \rangle$ is a common near fixed point for T and S. for this purpose, suppose that $d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]) > 0$ for any $[\bar{k}, \bar{l}] \in \langle [k,l] \rangle$. Due to $[\bar{k}, \bar{l}] \in \langle [k,l] \rangle$, we have

$$[\bar{k}, \bar{l}] \oplus \omega_1 = [k,l] \oplus \omega_2, \quad \text{for some} \quad \omega_1, \omega_2 \in \Omega. \tag{5.54}$$

Since the null equality holds for d, according to proposition 3.3 and (5.53), we have

$$\lim_{n\to\infty} d([k_n, l_n], [\bar{k}, \bar{l}]) = 0 \text{ for any } [\bar{k}, \bar{l}] \in \langle [k,l] \rangle. \tag{5.55}$$

Therefore by setting $[k,l] = [k_{2n}, l_{2n}]$ and $[u,v] = [\bar{k}, \bar{l}]$ in (5.25) and making use of the monotone increasing property of ψ and the triangle inequality, we obtain

$$\begin{aligned} \psi\big(d\big([k_{n+1}, l_{n+1}], T[\bar{k}, \bar{l}]\big)\big) =& \psi\big(d\big(S[k_n, l_n], T[\bar{k}, \bar{l}]\big)\big) \\ \leq& \psi\big(M\big([k_n, l_n], [\bar{k}, \bar{l}]\big)\big) - \phi\big(M\big([k_n, l_n], [\bar{k}, \bar{l}]\big)\big), \end{aligned} \tag{5.56}$$

where

$$\begin{aligned} M\big([k_n, l_n], [\bar{k}, \bar{l}]\big) =& \alpha d\big([k_n, l_n], [\bar{k}, \bar{l}]\big) + \beta\big[d\big([k_n, l_n], [k_{n+1}, l_{n+1}]\big) \\ &+ d\big([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]\big)\big] + \gamma\big[d\big([k_n, l_n], T[\bar{k}, \bar{l}]\big) + d\big([\bar{k}, \bar{l}], [k_{n+1}, l_{n+1}]\big)\big] \\ \leq& \alpha d\big([k_n, l_n], [\bar{k}, \bar{l}]\big) + \beta\big[d\big([k_n, l_n], [k_{n+1}, l_{n+1}]\big) + d\big([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]\big)\big] \\ &+ \gamma\big[d\big([k_n, l_n], [\bar{k}, \bar{l}]\big) + d\big([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]\big) + d\big([\bar{k}, \bar{l}], [k_{n+1}, l_{n+1}]\big)\big]. \end{aligned} \tag{5.57}$$

Note that due to Lemma 5.6 and (5.55), we have $\lim_{n\to\infty} d\big([k_{n+1}, l_{n+1}], T[\bar{k}, \bar{l}]\big) = d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])$. Thus taking $n \to \infty$ in (5.56), using (5.37), Lemma 5.6 and the continuity of ϕ and ψ and making use of (5.55) and (5.57), we have

$$\psi\big(d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big) \leq \psi\big((\beta+\gamma) d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big) - \phi\big(\lim_{n\to\infty} M\big([k_n, l_n], [\bar{k}, \bar{l}]\big)\big),$$

Due to $\beta + \gamma \leq 1$ and the monotone increasing property of ψ, it results that

$$\psi\big(d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big) \leq \psi\big(d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}])\big) - \phi\big(\lim_{n\to\infty} M\big([k_n, l_n], [\bar{k}, \bar{l}]\big)\big),$$

which implies that $\phi\big(\lim_{n\to\infty} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big)\big) \leq 0$. Thus, we have $\lim_{n\to\infty} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) = 0$, which is a contradiction since $\beta > 0$, $d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]) > 0$ and from (5.31) and (5.57), we have

$$0 < \lim_{n\to\infty} M\big([k_{2n}, l_{2n}], [k_{2n+1}, l_{2n+1}]\big) \leq (\beta+\gamma) d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]).$$

So, $d([\bar{k}, \bar{l}], T[\bar{k}, \bar{l}]) = 0$. Thus, $T[\bar{k}, \bar{l}] \stackrel{\Omega}{=} [\bar{k}, \bar{l}]$ for any point $[\bar{k}, \bar{l}] \in \langle [k,l] \rangle$. Since any near fixed point for T is a near fixed point for S, then $S[\bar{k}, \bar{l}] \stackrel{\Omega}{=} [\bar{k}, \bar{l}]$ for any $[\bar{k}, \bar{l}] \in \langle [k,l] \rangle$. Hence according to proposition 2.3, we have $T[\bar{k}, \bar{l}] \stackrel{\Omega}{=} [\bar{k}, \bar{l}] \stackrel{\Omega}{=} S[\bar{k}, \bar{l}]$ for any point $[\bar{k}, \bar{l}] \in \langle [k,l] \rangle$. □

Example 5.13. Let $\phi, \psi : [0, \infty) \to [0, \infty)$ and $S, T : \mathcal{I} \to \mathcal{I}$ in the MIS $(\mathcal{I}, d)$ be defined by $\phi(t) = \frac{t}{6}$, $\psi(t) = t$, $S[k,l] = \left[-1 + \frac{k}{6}, 1 + \frac{l}{6}\right]$ and $T[k,l] = \left[-2 + \frac{k}{6}, 2 + \frac{l}{6}\right]$, respectively. Define a mapping $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ by $d([k,l], [u,v]) = |(k+l) - (u+v)|$ for all $[k,l], [u,v] \in \mathcal{I}$. Suppose that $\alpha = \frac{1}{2}$, $\beta = \frac{1}{8}$ and $\gamma = \frac{1}{8}$. Then by (5.25), we have

$$\begin{aligned}
&\psi\big(d(S[k,l], T[u,v])\big) = \psi\bigg(d\Big(\Big[-1 + \frac{k}{6}, 1 + \frac{l}{6}\Big], \Big[-2 + \frac{u}{6}, 2 + \frac{v}{6}\Big]\Big)\bigg) \\
&= \psi\bigg(\frac{1}{6}\big|(k+l) - (u+v)\big|\bigg) = \frac{1}{6}\big|(k+l) - (u+v)\big| \\
&\leq \frac{5}{6}\bigg(\alpha\big|(k+l) - (u+v)\big| + \beta\Big[\frac{5}{6}\big|k+l\big| + \frac{5}{6}\big|u+v\big|\Big] + \gamma\Big[\big|(k+l) - \frac{1}{5}(u+v)\big| + \big|(u+v) - \frac{1}{5}(k+l)\big|\Big]\bigg) \\
&= \frac{5}{6}M\big([k,l], [u,v]\big) = M\big([k,l], [u,v]\big) - \frac{1}{6}M\big([k,l], [u,v]\big) \\
&= \psi\big(M\big([k,l], [u,v]\big)\big) - \phi\big(M\big([k,l], [u,v]\big)\big).
\end{aligned}$$

Thus, this example satisfies all conditions of Theorem 5.12. Hence, T and S have a unique equivalence class of common near fixed points $\big\langle [-2, 2] \big\rangle$ in $\mathcal{I}$.

Corollary 5.14. *Suppose that $(\mathcal{I}, \|\cdot\|)$ is a BIS such that the null equality and null condition hold for $\|\cdot\|$. Assume that $S, T : (\mathcal{I}, \|\cdot\|) \to (\mathcal{I}, \|\cdot\|)$ are self-mappings of $\mathcal{I}$ such that for all $[k,l], [u,v] \in \mathcal{I}$,*

$$\psi\big(\|S[k,l] \ominus T[u,v]\|\big) \leq \psi\big(M([k,l], [u,v])\big) - \phi\big(M([k,l], [u,v])\big), \tag{5.58}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function with the property $\phi(t) = 0 \Leftrightarrow t = 0$, $\psi : [0, \infty) \to [0, \infty)$ is an alternating distance function and

$$\begin{aligned}
M([k,l], [u,v]) = &\alpha\|[k,l] \ominus [u,v]\| + \beta\big[\|[k,l] \ominus S[k,l]\| + \|[u,v] \ominus T[u,v]\|\big] \\
&+ \gamma\big[\|[k,l] \ominus T[u,v]\| + \|[u,v] \ominus S[k,l]\|\big],
\end{aligned}$$

with $\alpha, \beta > 0, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma \leq 1$. Then a common near fixed point $[k,l] \in \mathcal{I}$ exists for T and S such that $T[k,l] \stackrel{\Omega}{=} [k,l] \stackrel{\Omega}{=} S[k,l]$. Moreover, a unique equivalence class $\big\langle [k,l] \big\rangle$ exists for $\mathcal{I}$ such that if $\big[\bar{k}, \bar{l}\big]$ is another common near fixed point for T and S, then $\big\langle [k,l] \big\rangle = \big\langle \big[\bar{k}, \bar{l}\big] \big\rangle$ and $[k,l] \stackrel{\Omega}{=} \big[\bar{k}, \bar{l}\big]$. Every point $\big[\bar{k}, \bar{l}\big] \in \big\langle [k,l] \big\rangle$ is also a common near fixed point for T and S. In addition, any near fixed point of T is a near fixed point of S and conversely.

6 Conclusion

In the last few decades, fixed point theory has become one of the most amazing topics that many researchers have been inclined towards. Recently, Wu defined the near fixed point, the MIS and the NIS based on the the notion of the null set. He proved some near fixed point theorems in these spaces [13]. Building upon this research, we have presented some near fixed point theorems for mappings with the certain weak contraction property in the MIS and the NIS. To demonstrate the validity of the results, we have provided examples.

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