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New results on fractional difference triple sequences of fuzzy numbers

Carlos Granados[a,*], Suman Das[b,c], Binod Chandra Tripathy[d]

[a] I.E.D. Camilo Torres Tenorio, Barranquilla, Colombia

[b] Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India

[c] Department of Education (ITEP), NIT Agartala, Jirania, 799046, Tripura, India

[d] Department of Mathematics, Tripura University, Tripura, India

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Abstract

In this paper, we introduce triple sequence spaces of fuzzy numbers defined using the fractional difference operator and Musielak-Orlicz function. Besides, some topological properties for these spaces are shown and some inclusion relations are proved. Additionally, we define and prove theorems related to η-dual space of these spaces of fuzzy numbers.

Keywords: Triple sequence spaces, fuzzy numbers, dual space, fractional difference operator
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1 Introduction

The notion of fuzzy sets was originally introduced by Zadeh and due to its far-reaching applications, fuzzy set theory has gained the attention of several researchers working in different areas of mathematics, sciences and engineering. Further, several researchers study the sequences of fuzzy numbers, and it has been proved [20] that the set of all convergent sequences of fuzzy numbers is a complete metric space. Some difference sequence spaces of fuzzy numbers are studied in [11]. On the other hand, theory of sequence spaces acts as a tool which links different branches of mathematics to functional analysis. Due to applications in numerical analysis, the study of difference sequence spaces has gained the attention of many researchers. By extending the index of difference to any real or complex number the fractional difference [3] is defined, and this leads to the theory of fractional difference sequence spaces. Extending the idea of triple sequences to triple sequences of fuzzy numbers Kumar et al. [17] proved that the set of all triple convergent sequences of fuzzy numbers is complete. For more details regarding the fractional difference equations, triple sequences and sequences of fuzzy numbers, we refer the reader to [1, 14, 16, 22] and the references therein. In the next section, we present basic and necessary facts about the fuzzy numbers, triple difference sequences and fractional difference operator, then we define and extend certain Musielak-Orlicz triple sequence space of fuzzy numbers by using fractional difference operator defined in [24]. In the last section, we prove certain topological properties of these sequence spaces and their η-dual.

*Corresponding author
Email addresses: carlosgranadosortiz@outlook.es (Carlos Granados), sumandas18842@gmail.com (Suman Das), tripathybc@gmail.com (Binod Chandra Tripathy)

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2 Preliminaries

In this section, we present basics facts about the fuzzy numbers, triple difference sequences and fractional difference operator which will be useful for the development of this paper.

Definition 2.1. A fuzzy number is a function $\tilde{x} : \mathbb{R}^n \to [0,1]$ which is normal, fuzzy convex, upper semi-continuous and the closure $\tilde{x}^o$ of $\{t \in \mathbb{R}^n : \tilde{x}(t) > 0\}$ is compact.

Remark 2.2. In view of the above definition, we have for each $0 < \alpha \leq 1$, the α-level set

$$\tilde{x}^\alpha = \{x \in \mathbb{R}^n : \tilde{x}(x) > \alpha\}$$

is a non-empty, convex and compact subset of $\mathbb{R}^n$.

Remark 2.3. We will denote $\mathcal{F}(\mathbb{R}^n)$ as the set of all fuzzy numbers.

Definition 2.4. The linear structure on $\mathcal{F}(\mathbb{R}^n)$ induces the scalar multiplication and addition in terms of α-level sets as

$$[\lambda\tilde{x}]^\alpha = \lambda[\tilde{x}]^\alpha \text{ for each } 0 \leq \alpha \leq 1 \quad \text{and} \quad [\tilde{x}+\tilde{y}]^\alpha = [\tilde{x}]^\alpha + [\tilde{y}]^\alpha$$

Definition 2.5. For each $1 \leq q < \infty$ define

$$d_q(\tilde{x},\tilde{y}) = \{\int_0^1 \delta_\infty(\tilde{x}^\alpha, \tilde{y}^\alpha)^q d\alpha\}^{1/q} \quad \text{and} \quad d_\infty(\tilde{x},\tilde{y}) = \sup_{0\leq\alpha\leq 1} \delta_\infty(\tilde{x}^\alpha, \tilde{y}^\alpha),$$

here, $\delta_\infty(\tilde{x}^\alpha, \tilde{y}^\alpha)$ denotes the Hausdorff distance between the functions $\tilde{x}$ and $\tilde{y}$. Additionally, we have

$$d_\infty(\tilde{x},\tilde{y}) = \lim_{q\to\infty} d_q(\tilde{x},\tilde{y}) \text{ with } d_q \leq d_r \ if \ q \leq r.$$

Remark 2.6. $(\mathcal{F}(\mathbb{R}^n), d_q)$ is a separable separable, complete and locally compact metric space. For more details of fuzzy metric spaces we refer the reader to [9]. For details and some recent advances regrading sequence spaces and summability methods refer [13, 15, 23].

Next, we will show the notion of generalized fractional difference operator.

Definition 2.7. [2] The fractional difference operator for a real (or complex) sequence $x = (x_n)$ and $\beta \in \mathbb{R}$ is defined as

$$(\Delta^\beta x)_n = \sum_{j=0}^{\infty} (-1)^j \left(\frac{\Gamma(\beta+1)}{j!\Gamma(\beta-j+1)}\right) x_{n+j},$$

here, Γ denotes the gamma function, and the series is assumed to be convergent throughout this paper.

For more details about the fractional difference operators and their applications to sequence spaces we refer reader to [5, 6, 7, 8, 24].

Definition 2.8. A triple infinite array of fuzzy numbers is called a fuzzy triple sequence.

Remark 2.9. We will denote by $(\tilde{x}_{mne})_{m,n,e\in\mathbb{N}}$ a fuzzy triple sequence, here $\tilde{x}_{mne}$ is a fuzzy number and by $\mathbb{S}(F)$ the of all triple sequences of fuzzy numbers.

Taking into account definition of fractional difference operator [4, 25, 24] we introduce the following definition.

Definition 2.10 (*Fractional triple difference fuzzy sequence*). The fractional difference Δ^β_{ijk} of order $\beta \in \mathbb{R}$ for a triple fuzzy sequence is

$$(\Delta^\beta_{ijk}\tilde{x}_{mne}) = \sum_{p=0}^{\infty}\sum_{a=0}^{i}\sum_{s=0}^{j}\sum_{d=0}^{k} \binom{i}{a}\binom{j}{s}\binom{k}{d}(-1)^p \frac{\Gamma(\beta+1)}{p!\Gamma(\beta-p+1)} \tilde{x}_{p+ia+m,p+js+n,p+kd+e}.$$

Definition 2.11 (*Pringsheim limit* [21]). A triple sequence $(\tilde{x}_{mne})$ of fuzzy numbers (considered as a metric space with metric $d_q, q \geq 1$) is P-convergent (Pringsheim convergent) to P-limit $\tilde{x}$ if for all $\varepsilon < 0$, there exists $N_\varepsilon \in \mathbb{N} : m, n, e > N_\varepsilon$, then $d_q(\tilde{x}_{mne} - \tilde{x}) < \varepsilon$.

Definition 2.12. A six dimensional matrix with fuzzy entries, which transforms P-convergent bounded sequences of fuzzy numbers to P-convergent sequences of fuzzy numbers with the same P-limit is called a RH-regular fuzzy matrix.

Definition 2.13. A continuous, convex, non-decreasing function φ with $\varphi(x) > 0$ for $x > 0$; $\varphi(0) = 0$ and $\varphi(x) \to \infty$ whenever $x \to \infty$ is called and Orlicz function.

Remark 2.14. Orlicz sequence space, denoted by ℓ_φ is space of the sequences $x = (x_n)$ which satisfy

$$\sum_{i=1}^{\infty} \varphi(\frac{|x_i|}{t}) < \infty, \ t > 0.$$

Theorem 2.15. [18] The space $(\ell_\varphi, \|x\| = \inf\{t > 0 : \sum_{i=1}^{\infty} \varphi(\frac{|x_i|}{t}) \leq 1\})$ is a Banach space.

Definition 2.16. A Musielak-Orlicz function denoted Φ is a sequence of Orlicz functions ϕ_i.

Definition 2.17. [18] Orlicz function ϕ is said to satisfy the Δ_2-condition if for every $\lambda > 1$, there exists a constant K depending upon λ and a positive number x, such that $\phi(\lambda x) \leq K(\lambda)\phi(x)$ for $0 \leq x \leq x(\lambda)$.

Using Musielak–Orlicz function we define the following triple sequence spaces of fuzzy numbers:

$$^3\tilde{l}(\Delta^\beta_{ijk}, \Phi) = \{x = (\tilde{x}_{mne}) \in^3 \mathbb{S}(F) : \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta^\beta_{ijk}\tilde{x}_{mne}, \tilde{x}_0)}{t}) < \infty\}. \tag{2.1}$$

$$^3\tilde{l}^\infty(\Delta^\beta_{ijk}, \Phi) = \{x = (\tilde{x}_{mne}) \in^3 \mathbb{S}(F) : \sup_{k,j,i,e,n,m} \Phi(\frac{d_q(\Delta^\beta_{ijk}\tilde{x}_{mne}, \tilde{x}_0)}{t}) < \infty\}, \tag{2.2}$$

where, $i, j, k, m, n, e \in \mathbb{N}$.

3 Main Results

We will begin proving some topological properties and inclusion relations.

Theorem 3.1. For $\beta \in [0, 1)$ the space $^3\tilde{l}_{(\Delta^\beta_{ijk}, \Phi)}$ is a linear space.

Proof . For $(\tilde{x}_{mne}), (\tilde{y}_{mne}) \in^3 \tilde{l}_{(\Delta^\beta_{ijk}, \Phi)}$ we have,

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta^\beta_{ijk}\tilde{x}_{mne}, \tilde{x}_0)}{t}) < \infty \text{ and } \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta^\beta_{ijk}\tilde{y}_{mne}, \tilde{x}_0)}{t}) < \infty.$$

Now,

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta^\beta_{ijk}(\tilde{x}_{mne} - \tilde{y}_{mne}), \tilde{x}_0)}{t}) = \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta^\beta_{ijk}\tilde{x}_{mne} - \Delta^\beta_{ijk}\tilde{y}_{mne}, \tilde{x}_0)}{t}).$$

As the Orlicz function is non decreasing and convex so we obtain,

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta^\beta_{ijk}\tilde{x}_{mne} - \Delta^\beta_{ijk}\tilde{y}_{mne}, \tilde{x}_0)}{t}) \leq \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta^\beta_{ijk}\tilde{x}_{mne}, \tilde{x}_0)}{t}) + \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta^\beta_{ijk}\tilde{y}_{mne}, \tilde{x}_0)}{t})$$
$$< \infty.$$

Thus, $(\tilde{x}_{mne}) - (\tilde{y}_{mne}) \in^3 \tilde{l}_{(\Delta^\beta_{ijk}, \Phi)}$ and therefore $^3\tilde{l}_{(\Delta^\beta_{ijk}, \Phi)}$ is a linear space. □

Theorem 3.2. $\|(\tilde{x}_{mne})\| = \inf\{t > 0 : \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t}) \leq 1\}$ is a norm on the space ${}^3\tilde{l}_{(\Delta_{ijk}^{\beta},\Phi)}$.

Proof . First at all, we have $\|(\tilde{x}_{mne})\| \geq 0$. If for some $\tilde{x}_{mne} \neq \tilde{0}$, $\inf\{t > 0 : \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t}) \leq 1\} = 0$ then for a given $\varepsilon > 0$, there exists $t_\varepsilon \in (0, \varepsilon)$ such that

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t_\varepsilon}) \leq 1.$$

Therefore, we have

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t}) \leq \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t_\varepsilon}) \leq 1.$$

Taking $\varepsilon \to 0$, we get

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t}) \to \infty \Rightarrow \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t_\varepsilon}) \to \infty.$$

This is a contradiction. Therefore, $\|(\tilde{x}_{mne})\| = 0$ if and only if $\tilde{x}_{mne} = \tilde{0}$. For a scalar μ, we consider,

$$\|\mu(\tilde{x}_{mne})\| = \inf\{t > 0 : \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\mu\tilde{x}_{mne}, \tilde{x}_0)}{t}) \leq 1\}.$$

For $t = \mu\rho$, we obtain,

$$\|\mu(\tilde{x}_{mne})\| = \inf\{|\mu|\rho > 0 : \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{\rho}) \leq 1\} = |\mu|\|(\tilde{x}_{mne})\|.$$

Now, we will prove triangular inequality. Let's consider $(\tilde{x}_{mne}), (\tilde{y}_{mne}) \in^3 \tilde{l}_{(\Delta_{ijk}^{\beta},\Phi)}$ with $t_1, t_2 \in (0, \infty)$ s.t.

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t_1}) \leq 1 \text{ and } \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{y}_{mne}, \tilde{x}_0)}{t_2}) \leq 1.$$

For $t = t_1 + t_2$ and using Minkowski's inequality, we have

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}(\tilde{x}_{mne} + \tilde{y}_{mne}), \tilde{x}_0)}{t}) \leq \sum_{k,j,i,e,n,m}^{\infty} [\frac{t_1}{t}\Phi(\frac{d_q(\Delta_{ijk}^{\beta}(\tilde{x}_{mne}), \tilde{x}_0)}{t}) + \frac{t_2}{t}\Phi(\frac{d_q(\Delta_{ijk}^{\beta}(\tilde{y}_{mne}), \tilde{x}_0)}{t})].$$

Thus, $\sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}(\tilde{x}_{mne} + \tilde{y}_{mne}), \tilde{x}_0)}{t}) \leq 1$ and as $t, t_1, t_2 \geq 0$, we get

$$\begin{aligned}\|(\tilde{x}_{mne} + \tilde{y}_{mne})\| &= \inf\{t : \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}(\tilde{x}_{mne} + \tilde{y}_{mne}), \tilde{x}_0)}{t}) \leq 1\} \\ &\leq \inf\{t_1 : \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{x}_{mne}, \tilde{x}_0)}{t}) \leq 1\} + \inf\{t_2 : \sum_{k,j,i,e,n,m}^{\infty} \Phi(\frac{d_q(\Delta_{ijk}^{\beta}\tilde{y}_{mne}, \tilde{x}_0)}{t}) \leq 1\} \\ &= \|(\tilde{x}_{mne})\| + \|(\tilde{y}_{mne})\|.\end{aligned}$$

□

Theorem 3.3. The vector space ${}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$ is a normed space with norm

$$\|(\tilde{x}_{mne}) = \inf\{t \in (0,\infty) : \sup_{k,j,i,e,n,m} \Phi(\frac{d_q(\Delta^{\beta}_{ijk}\tilde{x}_{mne},\tilde{0})}{t}) \leq 1\}.$$

Proof . The proof of theorem is similar to the proof of theorem 3.2, hence it is omitted. □

Theorem 3.4. For two Musielak-Orlicz functions Φ_1 and Φ_2 with Φ_1 satisfying the Δ_2-condition,

$${}^3\tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_2)} \subset^3 \tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_1*\Phi_2)}.$$

Proof . For $x \in^3 \tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_2)}$ there exists $t > 0$ such that,

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi_2(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t}) < \infty.$$

As, Φ_1 satisfies Δ_2-condition, therefore we have

$$\sum_{k,j,i,e,n,m}^{\infty} \{\Phi_1 * \Phi_2(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t})\} \leq \lambda\{\sum_{k,j,i,e,n,m}^{\infty} \Phi_1 * \Phi_2(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t})\} < \infty.$$

Thus, $x \in^3 \tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_1*\Phi_2)}$. □

Theorem 3.5. For two Musielak-Orlicz functions Φ_1 and Φ_2 we have

$$({}^3\tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_1)} \cap^3 \tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_2)}) \subset^3 \tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_1)}$$

Proof . For $\tilde{x}_{mne} \in ({}^3\tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_1)} \cap^3 \tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_2)})$, there exist $t_1, t_2 \in (0,\infty)$ such that

$$\sum_{k,j,i,e,n,m}^{\infty} \Phi_1(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t_1}) < \infty \text{ and } \sum_{k,j,i,e,n,m}^{\infty} \Phi_2(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t_2}) < \infty.$$

Now, let $t = \min\{\frac{1}{t_1}, \frac{1}{t_2}\}$, applying the properties of the modulus functions, we have

$$\sum_{k,j,i,e,n,m}^{\infty} \{(\Phi_1 + \Phi_2)(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t})\} \leq \sum_{k,j,i,e,n,m}^{\infty} \Phi_1(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t_1}) + \sum_{k,j,i,e,n,m}^{\infty} \Phi_2(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t_2}).$$

Therefore, $\sum_{k,j,i,e,n,m}^{\infty} \{(\Phi_1 + \Phi_2)(\frac{d_q(\Delta^{\beta}_{ijk}(\tilde{x}_{mne}),\tilde{x}_0)}{t})\} < \infty$. □

Next, we prove some theorems on η-dual space and show some its properties, but first we introduce the following definition.

Definition 3.6. The η-dual of order r, $r \geq 1$, of ${}^3\tilde{l}_{(\Delta^{\beta}_{ijk},\Phi)}$ is defined as follows

$${}^3\tilde{l}^{\eta}_{(\Delta^{\beta}_{ijk},\Phi)} = \{(x_{mne}) \in^3 \mathbb{S}(F) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{e=1}^{\infty} d_q(a_{mne}\tilde{x}_{mne},\tilde{x}_0)^r < \infty \text{ for all } (\tilde{x}_{mne}) \in^3 \tilde{l}_{(\Delta^{\beta}_{ijk},\Phi)}\}.$$

Remark 3.7. Similarly to Definition 3.6, we can define the η-dual of ${}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$.

Theorem 3.8. ${}^3\tilde{l}_{(\Delta^{\beta}_{ijk},\Phi)}$ and ${}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$ are convex.

Proof . We prove the theorem for the case ${}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$ and other case will follow from the similar arguments. Let $\tilde{x}_{mne}, \tilde{y}_{mne} \in^3 \tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$. Then, there exist t_1 and t_2 such that

$$\sup_{k,j,i,e,n,m} \Phi(\frac{d_q(\Delta^{\beta}_{ijk}\tilde{x}_{mne},\tilde{0})}{t_1}) < \infty \text{ and } \sup_{k,j,i,e,n,m} \Phi(\frac{d_q(\Delta^{\beta}_{ijk}\tilde{x}_{mne},\tilde{0})}{t_2}) < \infty.$$

For $0<\lambda<1$, consider $t=\max(|\lambda|t_1, |1-\lambda|t_2)$, we have

$$\begin{aligned}\sup_{k,j,i,e,n,m} \Phi(\frac{d_q(\Delta^{\beta}_{ijk}(\lambda\tilde{x}_{mne}+(1-\lambda)\tilde{y}_{mne}),\tilde{0})}{2t}) &\leq \frac{1}{2}\Phi(\frac{d_q(\Delta^{\beta}_{ijk}\lambda\tilde{x}_{mne},\tilde{0})}{t}) + \frac{1}{2}\Phi(\frac{d_q(\Delta^{\beta}_{ijk}(1-\lambda)\tilde{y}_{mne},\tilde{0})}{t})\\ &\leq \frac{1}{2}\Phi(\frac{d_q(\Delta^{\beta}_{ijk}\tilde{x}_{mne},\tilde{0})}{t_1}) + \frac{1}{2}\Phi(\frac{d_q(\Delta^{\beta}_{ijk}\tilde{y}_{mne},\tilde{0})}{t_2})\\ &< \infty.\end{aligned}$$

Therefore, $\{\lambda\tilde{x}_{mne}+(1-\lambda)\tilde{y}_{mne}\} \in^3 \tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$. □

Lemma 3.9. Let 3^w be the spaces of all triple sequences spaces, then the following statements hold:

1. E^{η} is a linear subspace of 3^w for every $E \subset 3^w$.
2. $E \subset F$ implies $F^{\eta} \subset E^{\eta}$ for every $E, f \subset 3^w$.
3. $E \subset E^{\eta\eta}$,

where, E is a non-empty subset of 3^w and E^{η} is the η-dual of E which is defined as follows

$$E^{\eta} = \{(x_{mne}) \in 3^w : \sum_m\sum_n\sum_e |x_{mne}y_{mne}|^r < \infty \text{ for all } (y_{mne}) \in E \text{ and } r \geq 1\}.$$

Proof . The proof of lemma is obvious in view of the definition of η-dual of triple sequences. □

Theorem 3.10. ${}^3\tilde{l}_{(\Delta^{\beta}_{ijk},\Phi)}$ and ${}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$ are linear spaces.

Proof . The proof is follows from Lemma 3.9 □

Theorem 3.11. Let Φ_1 and Φ_2 be Orlicz functions where Φ_1 satisfies Δ_2-condition, then ${}^3\tilde{l}^{\eta}_{(\Delta^{\beta}_{ijk},A,\Phi_1*\Phi_2)} \subset^3 \tilde{l}^{\eta}_{(\Delta^{\beta}_{ijk},A,\Phi_2)}$.

Proof . From Theorem 3.4, we have ${}^3\tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_2)} \subset^3 \tilde{l}_{(\Delta^{\beta}_{ijk},A,\Phi_1*\Phi_2)}$, using [12] and [19] the proof follows. □

Lemma 3.12. For $(\tilde{x}_{mne}) \in^3 \tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$,

$$\sup_{k,j,i,e,n,m} \Phi(\frac{d_q((m+n+e)^{-1}\Delta^{\beta}_{ijk}\tilde{x}_{mne},\tilde{0})}{t}) < \infty \;\Rightarrow \sup_{k,j,i,e,n,m} \Phi(\frac{d_q((m+n+e)^{-1}\tilde{x}_{mne},\tilde{0})}{t}) < \infty$$

Proof . Proof is similar to [Lemma 3.2] [10], hence it is omitted. □

Theorem 3.13. $({}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)})^{\eta} =^3 \tilde{l}_{r_{(\Delta^{\beta}_{ijk})}}$, here

$${}^3\tilde{l}_{r_{(\Delta^{\beta}_{ijk})}} = \{\tilde{y}_{mne} : \sum_m\sum_n\sum_e (|\sum_{p=0}^{\infty}(-1)^p \frac{\Gamma(1-\beta)}{p!\Gamma(1-\beta-p)}|^{-1} d_q(y_{mne},\tilde{x}_0)^r) < \infty, 0<r<1\}.$$

Proof . We have ${}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)} \subset^3 \tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk}}$ and therefore ${}^3\tilde{l}_{r_{(\Delta^{\beta}_{ijk})}} \subset ({}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)})^{\eta}$. Using [26, Theorem 1] in triple sequences, we obtain $({}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)})^{\eta} =^3 \tilde{l}_{r_{(\Delta^{\beta}_{ijk}}} \subset^3 \tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$. Now, to show $({}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)})^{\eta} \subset^3 \tilde{l}_{r_{(\Delta^{\beta}_{ijk})}}$, we consider that $\tilde{y}_{mne} \in ({}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)})^{\eta}$ and $\tilde{y}_{mne} \notin^3 \tilde{l}_{r_{(\Delta^{\beta}_{ijk})}}$. Define, $\tilde{x}_{mne}$ as follows

$$\begin{cases} d_q(\tilde{x},\tilde{0}), \quad \mathit{where}\ x = \sum_{p=0}^{\infty}(-1)^p \frac{\Gamma(1-\beta)}{p!\Gamma(1-\beta-p)}\frac{1}{v^a}, \ \text{if } e=v, (n,m) \in \mathbb{N}\times\mathbb{N}, a > \frac{1}{r}; \\ 0, \qquad \text{otehrwise.} \end{cases}$$

We have, $\sup_{m,n,e,i,j,k} \Phi(\frac{d_p(\Delta^{\beta}_{ijk}\tilde{x}_{mne},\tilde{0})}{t}) < \infty$ this implies for $ra > l$ that $(\tilde{x}_{mne}) \in^3 \tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$. As $\tilde{y}_{mne} \notin^3 \tilde{l}_{r_{(\Delta^{\beta}_{ijk})}}$, then we can find a sequence $e = (e_v)$ such that $e_v \in \mathbb{N}$ with $e_1 = 1$ such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{e=e_v}^{e_{v+1}-1}\sum_{p=0}^{\infty}|(-1)^p\frac{\Gamma(1-\beta)}{p!\Gamma(1-\beta-p)}|^{-1}d_q(\tilde{y}_{mne},\tilde{x}_0)^r > v^r, \ \text{for all } v \in \mathbb{N}.$$

Now,

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{e=1}^{\infty} d_q(\tilde{y}_{mne}\tilde{x}_{mne},\tilde{x}_0) &= \sum_{v=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{e=e_v}^{e_{v+1}-1} d_q(\tilde{y}_{mne}\frac{1}{v^r},\tilde{x}_0)^r \\ &= \sum_{v=1}^{\infty}\frac{1}{v^r}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{e=e_v}^{e_{v+1}-1} d_q(\tilde{y}_{mne},\tilde{x}_0) \\ &= \sum_{v}\frac{v^r}{v^r} \to \infty\end{aligned}$$

which is a contradiction. Therefore, $({}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)})^{\eta} \subset^3 \tilde{l}_{r_{(\Delta^{\beta}_{ijk})}}$. □

Corollary 3.14. Space ${}^3\tilde{l}^{\infty}_{(\Delta^{\beta}_{ijk},\Phi)}$ is not perfect.

Theorem 3.15. The dual ${}^3\tilde{l}_{r_{(\Delta^{\beta}_{ijk})}}$ is

$${}^3\tilde{l}^{\eta}_{r_{(\Delta^{\beta}_{ijk})}} = \{\tilde{y}_{mne} : \sup_{m,n,e}(|\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma(1-\beta)}{p!\Gamma(1-\beta-p)}|^{-1}|\tilde{y}_{m,n,e}|^r) < \infty, 0 < r < 1\}.$$

Proof . The proof is similar to Theorem 3.13, hence it is omitted. □

References

[1] H. Altinok, R. Colak, and M. Et, *λ-difference sequence spaces of fuzzy numbers*, Fuzzy Sets Syst. **160** (2009), no. 21, 3128–3139.

[2] P. Baliarsingh, *A unifying approach to the difference operators and their applications*, Bol. Soc. Parana. Mat. **33** (2013), no. 1, 49–56.

[3] P. Baliarsingh, *Some new difference sequence spaces of fractional order and their dual spaces*, Appl. Math. Comput. **219** (2013), no. 18, 9737–9742.

[4] P. Baliarsingh, *On difference double sequence spaces of fractional order*, Indian J. Math. **58** (2016), no. 3, 287–310.

[5] P.Baliarsingh, *On a fractional difference operator*, Alexandria Eng. J. **55** (2016), no. 2, 1811–1816.

[6] P. Baliarsingh, *On double difference operators via four dimensional matrices*, J. Indian Math. Soc. **83** (2016), no. 3-4, 209–219.

[7] P. Baliarsingh and S. Dutta, *On the classes of fractional order difference sequence spaces and their matrix transformations*, Appl. Math. Comput. **250** (2015), 665–674.

[8] P. Baliarsingh and L. Nayak, *Anote on fractional difference operators*, Alexandria. Eng. J. **57** (2018), no. 2, 1051–1054.

[9] P. Diamond and P. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets Syst. **35** (1990)), no. 2, 241–249.

[10] H. Dutta, *Generalized difference sequence spaces defined by Orlicz functions and their Kothe-Toeplitz and null duals*. Thai J. Math. **8** (2012), no. 1, 11–19.

[11] M. Et, E. Savas, and H. Altınok, *On some difference sequence spaces of fuzzy numbers*, Soft Comput. **20** (2016), no. 11, 4395–4401.

[12] S. Gahler, *2-metric spaces and their topological structure*, Math. Nachr. **26** (1963), no. 1-4, 115–148.

[13] C. Granados, *A generalization of the strongly Cesáro ideal convergence through double sequence spaces*, Int. J. Appl. Math. **34** (2021), no. 3, 525–533.

[14] C. Granados, *New notion of triple sequences on ideal spaces in metric spaces*, Adv. Theory Nonlinear Anal. Appl. **5** (2021), no. 3, 363–368.

[15] C. Granados and A. Dhital, *Statistical convergence of double sequences in neutrosophic normed spaces*, Neutrosophic Sets Syst. **42** (2021), 333–344.

[16] A. Indumathi, N. Subramanian, and B. Hazarika, *On the Borel summability method for convergence of triple sequences of Bernstein-Stancu operators of fuzzy numbers*, Soft Comput. **25** (2021), 683–697.

[17] P. Kumar, V. Kumar, and S.S. Bhatia, *Multiple sequences of fuzzy numbers and their statistical convergence*, Math. Sci. **6** (2012), no. 2, 1–7.

[18] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Isr. J. Math. **10** (1971), no. 3, 379–390.

[19] E. Malkowsky, M. Mursaleen, and S. Suantai, *The dual spaces of sets of difference sequences of order m and matrix transformations*, Acta Math. Sin. Engl. Ser. **23** (2007), no. 3, 521–532.

[20] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets Syst. **33** (1989), no. 1, 123–126.

[21] A. Pringsheim, *Zur theorie der zweifach unendlichen zahlenfolgen*, Math. Ann. **53** (1900), no. 3, 289–321.

[22] S. Saha and S. Roy, *New classes of statistically pre-Cauchy triple sequences of fuzzy numbers defined by Orlicz function*, J. Math. Soc. **85** (2018), no. 3-4, 411–421.

[23] E. Savas and M. Gurdal, *Certain summability methods in intuitionistic fuzzy normed spaces*, J. Intell. Fuzzy Syst. **27** (2014), no. 4, 1621–1629.

[24] P. Sharma, *Fractional difference double sequences of fuzzy numbers*, Afrika Mate. **32** (2021), no. 7, 1465–1473.

[25] S.K. Singh, S. Mishra, and P. Sharma, *On difference double sequences of fractional order*, AIP Conf. Proc. 1860, AIP Publishing LLC, Melville, 2017, pp. 020028-020031.

[26] B.C. Tripathy and B. Sarma, *On some classes of difference double sequence spaces*, Fasc. Math. **41** (2009), 135–142.