Int. J. Nonlinear Anal. Appl. In Press, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.30726.4474

Semi P-function and some new related inequalities

Mahir Kadakal[a,*], İmdat İşcan[b], Huriye Kadakal[c]

[a] Bayburt University, Faculty of Applied Sciences, Department of Customs Management, Baberti Campus, 69000, Bayburt, TÜRKİYE

[b] Departments of Mathematics, Sciences and Arts Faculty, Giresun University, 28100, Giresun, TÜRKİYE

[c] Bayburt University, Faculty of Education, Department of Primary Education , Baberti Campus, 69000 Bayburt, TÜRKİYE

(Communicated by Abasalt Bodaghi)

Abstract

In this manuscript, we introduce and study the concept of semi P-functions and their some algebraic properties. Also, we compare the results obtained with both Hölder, Hölder-İşcan inequalities and power-mean, improved-power-mean integral inequalities and show that the result obtained with Hölder-İşcan and improved power-mean inequalities give a better approach than the others. Some applications to special means of real numbers are also given.

Keywords: Convex function, semi P-function, Hermite-Hadamard inequality, Hölder-İşcan inequality, improved power-mean inequality
2020 MSC: 26A51, 26D10, 26D15

1 Preliminaries and fundamentals

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [4, 7, 12, 15, 16] and the references therein. Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double integral inequality is well known as the Hermite-Hadamard integral inequality [5]. Some refinements of the Hermite-Hadamard integral inequality for convex functions have been obtained [3, 22].

In [4], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

*Corresponding author
Email addresses: mahirkadakal@gmail.com (Mahir Kadakal), imdati@yahoo.com (İmdat İşcan), huriyekadakal@hotmail.com (Huriye Kadakal)

Received: May 2023 *Accepted:* October 2023

Definition 1.1. *A nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Theorem 1.2. *Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)\,dx \leq 2\left[f(a)+f(b)\right]. \tag{1.2}$$

In [6], İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 1.3 (Hölder-İşcan integral inequality [6]). Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then

$$\begin{aligned} \int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}\Bigg\{ & \left(\int_a^b (b-x)|f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b (b-x)|g(x)|^q\,dx\right)^{\frac{1}{q}} \\ & + \left(\int_a^b (x-a)|f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b (x-a)|g(x)|^q\,dx\right)^{\frac{1}{q}} \Bigg\}. \end{aligned} \tag{1.3}$$

For more information on the Hermite-Hadamard and Hölder-İşcan integral inequalities, readers can refer to references [17, 18, 19, 20] and the references within these references. An refinement of power-mean integral inequality as a different version of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.4 (Improved power-mean integral inequality [14]). Let $q \geq 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$ then

$$\begin{aligned} \int_a^b |f(x)g(x)|\,dx \leq \frac{1}{b-a}\Bigg\{ & \left(\int_a^b (b-x)|f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b (b-x)|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}} \\ & + \left(\int_a^b (x-a)|f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b (x-a)|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}} \Bigg\} \end{aligned}$$

2 The definition of semi P-function

The main purpose of this manuscript is to introduce the concept of semi P-function and establish some results connected with the right-hand side of new inequalities similar to the Hermite-Hadamard integral inequality for semi P-function.

In recent years, many function classes such as P-function, convex, quasi-convex, log-convex, AH-convex, s-convex functions in the first and second sense and trigonometrically convex, etc. have been studied by many authors, and integral inequalities belonging to these function classes have been studied in the literature (see [2, 4, 8, 9, 10, 11, 13, 15, 21]). There are many studies on the subject. In this section, a new function class, semi-P-function definition will be given, and the relations of this function class with the above-mentioned function classes will also be given. Moreover, we prove two Hermite-Hadamard type inequalities for the semi P-functions. Moreover, we obtain some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is semi P-functions. Also, some applications to special means of positive real numbers are also given.

Definition 2.1. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called semi P-function if for all $x, y \in I$, and $t \in [0,1]$,

$$f(tx + (1-t)y) \leq (tx + (1-t)y)\left[f(x) + f(y)\right]. \tag{2.1}$$

We will denote by $SP(I)$ the class of all semi P-functions on interval I. We note that if the non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is semi P-function then $f(x) = f(tx + (1-t)x) \leq 2xf(x)$, for all $x \in I$, i.e. $(2x-1) f(x) \geq 0$, for all $x \in I$. In this case, we can say that either "$x \geq 1/2$ and $f(x) \geq 0$" or "$x \leq 1/2$ and $f(x) \leq 0$".

Example 2.2. The function $f : [1, \infty) \to \mathbb{R}$, $f(x) = x$ is a semi P-function.

Example 2.3. The function $f : (-\infty, 0] \to \mathbb{R}$, $f(x) = x$ is a semi P-function.

Example 2.4. The function $f : [1, \infty) \to \mathbb{R}$, $f(x) = e^x$ is a semi P-function.

Example 2.5. For every $c \in \mathbb{R}\,(c \geq 0)$, the function $f : \left[\frac{1}{2}, \infty\right) \subset \mathbb{R} \to \mathbb{R}$, $f(x) = c$ is a semi P-function.

Remark 2.6. If $f : [1, \infty) \to [0, \infty)$ is a convex function, then f is also semi P-function. Since $t \leq ta+(1-t)b$, $1-t \leq ta + (1-t)b$, for all $a, b \in [1, \infty)$ and $t \in [0, 1]$, we get

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \leq (ta + (1-t)b)[f(a) + f(b)].$$

Remark 2.7. If $f : [1, \infty) \to [0, \infty)$ is a P-function, then f is also a semi P-function. Since, $1 \leq ta + (1-t)b$, for all $a, b \in [1, \infty)$ and $t \in [0, 1]$, we obtain

$$f(ta + (1-t)b) \leq f(a) + f(b) \leq (ta + (1-t)b)[f(a) + f(b)].$$

Remark 2.8. If $f : [1, \infty) \to [0, \infty)$ is a quasi convex function, then f is also a semi P-function. Since, $f(a) \leq (ta + (1-t)b) f(a)$, $f(b) \leq (ta + (1-t)b) f(b)$, for all $a, b \in [1, \infty)$ and $t \in [0, 1]$, we write

$$f(ta + (1-t)b) \leq \max\{f(a), f(b)\} \leq (ta + (1-t)b)[f(a) + f(b)].$$

Remark 2.9. Let $f : [1, \infty) \to [0, \infty)$ be a nonnegative and $s \in (0, 1]$. If f is a s-convex function in the first sense, then f is also a semi P-function. Since, $t^s \leq 1 \leq ta + (1-t)b$ and $1 - t^s \leq 1 \leq ta + (1-t)b$, for all $a, b \in [1, \infty)$ and $t \in [0, 1]$, we get

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t^s)f(b) = (ta + (1-t)b)[f(a) + f(b)].$$

Remark 2.10. Let $f : [1, \infty) \to [0, \infty)$ be a nonnegative and $s \in (0, 1]$. If f is a s-convex function in the second sense, then f is also a semi P-function. Since, $t^s \leq 1 \leq ta+(1-t)b$ and $(1-t)^s \leq 1 \leq ta+(1-t)b$, for all $a, b \in [1, \infty)$ and $t \in [0, 1]$, we have

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b) = (ta + (1-t)b)[f(a) + f(b)].$$

Remark 2.11. If $f : [1, \infty) \to (0, \infty]$ is a trigonometrically convex function, then f is also a semi P-function. Since, $\sin \frac{\pi t}{2} \leq 1$, $\cos \frac{\pi t}{2} \leq 1$ and $1 \leq ta + (1-t)b$, for all $a, b \in [1, \infty)$ and $t \in [0, 1]$, we can write

$$f(ta + (1-t)b) \leq \sin \frac{\pi t}{2} f(a) + \cos \frac{\pi t}{2} f(b) \leq f(a) + f(b) = (ta + (1-t)b)[f(a) + f(b)].$$

Remark 2.12. If $f : [1, \infty) \to (0, \infty]$ is a log-convex (i.e. arithmetic geometric (AG) convex) function, then f is also a semi P-function. Since, $t \leq ta + (1-t)b$, $1-t \leq ta + (1-t)b$, for $a, b \in [1, \infty)$ and $t \in [0, 1]$, we can write

$$f(ta + (1-t)b) \leq [f(a)]^t [f(b)]^{1-t} \leq (ta + (1-t)b)[f(a) + f(b)].$$

Remark 2.13. If $f : [1, \infty) \to (0, \infty]$ is an arithmetic harmonic (AH) convex function, then f is also a semi P-function. Since, $t \leq ta + (1-t)b$, $1-t \leq ta + (1-t)b$, for $a, b \in [1, \infty)$ and $t \in [0, 1]$, we can write

$$f(ta + (1-t)b) \leq \frac{f(a)f(b)}{tf(b) + (1-t)f(a)} \leq (ta + (1-t)b)[f(a) + f(b)].$$

Theorem 2.14. Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$. If f and g are semi P-functions, then $f+g$ *is a* semi P-function, $c \in \mathbb{R}$ $(c \geq 0)$, cf is a semi P-function.

Proof . Let f, g be semi P-functions, then

$$\begin{aligned} (f+g)(ta+(1-t)b) &= (ta+(1-t)b)[f(a)+g(a)]+(ta+(1-t)b)[f(b)+g(b)] \\ &= (ta+(1-t)b)(f+g)(a)+(ta+(1-t)b)(f+g)(b), \end{aligned}$$

for all $a, b \in I$ and $t \in [0,1]$. Let f be semi P-function and $c \in \mathbb{R}$ $(c \geq 0)$, then

$$(cf)(ta+(1-t)b) \leq (ta+(1-t)b)[(cf)(a)+(cf)(b)],$$

for all $a, b \in I$ and $t \in [0,1]$. □

Theorem 2.15. Let $f_\alpha : I \subset [0,\infty) \to \mathbb{R}$ be an arbitrary family of semi P-functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in I : f(u) < \infty\}$ is nonempty, then J is an interval and f is a semi P-function on J.

Proof . Let $t \in [0,1]$ and $a, b \in J$ be arbitrary. Then

$$\begin{aligned} f(ta+(1-t)b) &= \sup_\alpha f_\alpha(ta+(1-t)b) \\ &\leq \sup_\alpha [(ta+(1-t)b) f_\alpha(a)+(ta+(1-t)b) f_\alpha(b)] \\ &\leq (ta+(1-t)b)\sup_\alpha f_\alpha(a)+(ta+(1-t)b)\sup_\alpha f_\alpha(b) \\ &= (ta+(1-t)b)[f(a)+f(b)] < \infty. \end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a semi P-function on J. This completes the proof of theorem. □

3 Hermite-Hadamard integral inequality for semi P-functions

The goal of this manuscript is to establish some inequalities of Hermite-Hadamard type integral inequality for semi P-functions. In this section, we will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on interval $[a,b]$.

Theorem 3.1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a semi P-function. If $a < b$ and $f \in L[a,b]$, then the following Hermite-Hadamard type inequalities hold:

$$\frac{1}{a+b} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq (a+b)\frac{f(a)+f(b)}{2}. \tag{3.1}$$

Proof . From the property of the semi P-function of f, we obtain

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}[ta+(1-t)b]+\frac{1}{2}[(1-t)a+tb]\right) = \frac{a+b}{2}[f(ta+(1-t)b)+f((1-t)a+tb)].$$

By taking integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$f\left(\frac{a+b}{2}\right) \leq \frac{a+b}{2}\left[\int_0^1 f(ta+(1-t)b)\,dt+\int_0^1 f((1-t)a+tb)\,dt\right] = \frac{a+b}{b-a}\int_a^b f(x)dx.$$

By using the property of the semi P-function f, if the variable is changed as $x = ta+(1-t)b$, then

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 (ta+(1-t)b)[f(a)+f(b)]\,dt = (a+b)\left[\frac{f(a)+f(b)}{2}\right].$$

This completes the proof of theorem. □

4 Some new inequalities for the semi P-functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is semi P-function. In this section, for shortness, we will denote by $A(a,b)$ the arithmetic mean of real numbers a and b. Dragomir and Agarwal [1] used the following lemma:

Lemma 4.1. Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I°, $a,b \in I^{\circ}$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta+(1-t)b)\,dt$$

Theorem 4.2. Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^{\circ}$ with $a < b$ and assume that $f' \in L[a,b]$. If $|f'|$ is a semi P-function on interval $[a,b]$, then the following inequality holds

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2} A(a,b)\, A(|f'(a)|, |f'(b)|). \tag{4.1}$$

Proof . Using Lemma 4.1 and the inequality

$$|f'(ta+(1-t)b)| \leq (ta+(1-t)b)\,[|f'(a)|+|f'(b)|],$$

we get

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{2}\int_0^1 |1-2t|\left((ta+(1-t)b)\,[|f'(a)|+|f'(b)|]\right)dt \\ &\leq \frac{b-a}{2}\left([|f'(a)|+|f'(b)|]\int_0^1 |1-2t|\,(ta+(1-t)b)\,dt\right) \\ &= \frac{b-a}{2} A(a,b)\, A(|f'(a)|, |f'(b)|)\end{aligned}$$

where $\int_0^1 |1-2t|\,(ta+(1-t)b)\,dt = \frac{a+b}{4}$ and A is the arithmetic mean. This completes the proof of theorem. □

Theorem 4.3. Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^{\circ}$ with $a < b$, $q > 1, \frac{1}{p}+\frac{1}{q} = 1$ and assume that $f' \in L[a,b]$. If $|f'|^q$ is a semi P-function on interval $[a,b]$, then the following inequality holds

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}(a,b)\, A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right) \tag{4.2}$$

Proof . Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$|f'(ta+(1-t)b)|^q \leq (ta+(1-t)b)\left[|f'(a)|^q+|f'(b)|^q\right]$$

which is the semi P-function of $|f'|^q$, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\ &\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\ &= (b-a)\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}(a,b)\, A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right),\end{aligned}$$

where $\int_0^1 |1-2t|^p\,dt = \frac{1}{p+1}$ and A is the arithmetic mean. This completes the proof of theorem. □

Theorem 4.4. Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, $q \geq 1$ and assume that $f' \in L[a, b]$. If $|f'|^q$ is a semi P-function on the interval $[a, b]$, then the following inequality holds

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} A^{\frac{1}{q}}(a,b)\, A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right). \tag{4.3}$$

Proof . Assume first that $q > 1$. From Lemma 4.1, Hölder integral inequality and the property of the semi P-function of the function $|f'|^q$, we get

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|\, dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\int_0^1 |1-2t|\,(ta+(1-t)b)\left[|f'(a)|^q + |f'(b)|^q\right] dt \\
&= (b-a)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} A^{\frac{1}{q}}(a,b)\, A^{\frac{1}{q}}\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q\right),
\end{aligned}$$

where $\int_0^1 |1-2t|dt = \frac{1}{2}$. For $q = 1$ we use the estimates from the proof of Theorem 4.2, which also follow step by step the above estimates. This completes the proof of theorem. ☐

Corollary 4.5. Under the assumption of Theorem 4.4 with $q = 1$, we get the conclusion of Theorem 4.2.

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2} A(a,b)\, A\left(\left|f'(a)\right|, \left|f'(b)\right|\right).$$

Now, we will prove the Theorem 4.3 by using Hölder-İşcan integral inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 4.3.

Theorem 4.6. Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $f' \in L[a, b]$. If $|f'|^q$ is a semi P-function on interval $[a, b]$, then the following inequality holds

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\left(\frac{a+2b}{3}\right)^{\frac{1}{q}} + \left(\frac{2a+b}{3}\right)^{\frac{1}{q}}\right]. \tag{4.4}$$

Proof . Using Lemma 4.1, Hölder-İşcan integral inequality and the following inequality $\left|f'(ta+(1-t)b)\right|^q \leq (ta+(1-t)b)\left[|f'(a)|^q + |f'(b)|^q\right]$ which is the semi P-function of $|f'|^q$, we obtain

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq& \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{b-a}{2}\left(\int_0^1 t\,|1-2t|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
\leq& \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\,(ta+(1-t)b)\left[|f'(a)|^q + |f'(b)|^q\right] dt\right)^{\frac{1}{q}} \\
&+ \frac{b-a}{2}\left(\int_0^1 t\,|1-2t|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 t\,(ta+(1-t)b)\left[|f'(a)|^q + |f'(b)|^q\right] dt\right)^{\frac{1}{q}} \\
=& \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\left(\frac{a+2b}{3}\right)^{\frac{1}{q}} + \left(\frac{2a+b}{3}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where,

$$\int_0^1 (1-t)\,|1-2t|^p\,dt = \int_0^1 t\,|1-2t|^p\,dt = \frac{1}{2\,(p+1)},$$
$$\int_0^1 (1-t)\,(ta+(1-t)b)\,dt = \frac{a+2b}{6},$$
$$\int_0^1 t\,(ta+(1-t)b)\,dt = \frac{2a+b}{6}.$$

This completes the proof of theorem. □

Remark 4.7. The inequality (4.4) gives better results than the inequality (4.2). Let us show that

$$\frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\left(\frac{a+2b}{3}\right)^{\frac{1}{q}} + \left(\frac{2a+b}{3}\right)^{\frac{1}{q}}\right]$$
$$\leq \quad (b-a)\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}(a,b)\, A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).$$

Using concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\leq 1$ by sample calculation we get

$$\frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\left(\frac{a+2b}{3}\right)^{\frac{1}{q}} + \left(\frac{2a+b}{3}\right)^{\frac{1}{q}}\right]$$
$$\leq \quad \frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right) 2\left[\frac{1}{2}\left(\frac{a+2b}{3}\right) + \frac{1}{2}\left(\frac{2a+b}{3}\right)\right]^{\frac{1}{q}}$$
$$= \quad (b-a)\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right) A^{\frac{1}{q}}(a,b)$$

which is the required.

Theorem 4.8. Let $f: I \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^{\circ}$ with $a<b$, $q\geq 1$ and assume that $f' \in L[a,b]$. If $|f'|^q$ is a semi P-function on the interval $[a,b]$, then the following inequality holds

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \quad \leq \quad (b-a)\left(\frac{1}{2}\right)^{3-\frac{2}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\left(\frac{a+3b}{8}\right)^{\frac{1}{q}} + \left(\frac{3a+b}{8}\right)^{\frac{1}{q}}\right]. \quad (4.5)$$

Proof . Assume first that $q>1$. Using the Lemma 4.1, improved power-mean inequality and the property of the semi P-function of $|f'|^q$, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\int_0^1 t\,|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|1-2t|\,(ta+(1-t)b)\,dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\left[|f'(a)|^q + |f'(b)|^q\right]\int_0^1 (1-t)\,|1-2t|\,(ta+(1-t)b)\,dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\left[|f'(a)|^q + |f'(b)|^q\right]\int_0^1 t\,|1-2t|\,(ta+(1-t)b)\,dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\left(\frac{a+3b}{8}\right)^{\frac{1}{q}} + \left(\frac{3a+b}{8}\right)^{\frac{1}{q}}\right],$$

where

$$\int_0^1 (1-t)\,|1-2t|\,dt = \int_0^1 t\,|1-2t|\,dt = \frac{1}{4},$$
$$\int_0^1 (1-t)\,|1-2t|\,(ta+(1-t)b)\,dt = \frac{a+3b}{16},$$
$$\int_0^1 t\,|1-2t|\,(ta+(1-t)b)\,dt = \frac{3a+b}{16}.$$

For $q=1$ we use the estimates from the proof of Theorem 4.2, which also follow step by step the above estimates. This completes the proof of theorem. □

Remark 4.9. The inequality (4.5) gives better result than the inequality (4.3). Let us show that

$$\begin{aligned}
& (b-a)\left(\frac{1}{2}\right)^{3-\frac{2}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\left(\frac{a+3b}{8}\right)^{\frac{1}{q}} + \left(\frac{3a+b}{8}\right)^{\frac{1}{q}}\right] \\
\leq\; & (b-a)\left(\frac{1}{2}\right)^{2-\frac{1}{q}} A^{\frac{1}{q}}(a,b)\, A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).
\end{aligned}$$

If we use the concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\leq 1$, we get

$$\begin{aligned}
& (b-a)\left(\frac{1}{2}\right)^{3-\frac{2}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\left(\frac{a+3b}{8}\right)^{\frac{1}{q}} + \left(\frac{3a+b}{8}\right)^{\frac{1}{q}}\right] \\
\leq\; & 2(b-a)\left(\frac{1}{2}\right)^{3-\frac{2}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\frac{1}{2}\left(\frac{a+3b}{8}\right) + \frac{1}{2}\left(\frac{3a+b}{8}\right)\right]^{\frac{1}{q}} \\
=\; & (b-a)\left(\frac{1}{2}\right)^{2-\frac{2}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)\left[\frac{1}{2}\left(\frac{a+b}{2}\right)\right]^{\frac{1}{q}} \\
=\; & \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{1}{q}} A^{\frac{1}{q}}(a,b)\, A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right),
\end{aligned}$$

which completes the proof of remark.

5 Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a, b with $b > a$:

1. The arithmetic mean
$$A := A(a,b) = \frac{a+b}{2}, \quad a,b \geq 0,$$
2. The geometric mean
$$G := G(a,b) = \sqrt{ab}, \quad a,b \geq 0$$
3. The harmonic mean
$$H := H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0,$$
4. The logarithmic mean
$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases} \;;\quad a,b > 0$$

5. The p-logarithmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R}\setminus\{-1,0\} \\ a, & a=b \end{cases} ; \quad a,b>0.$$

6. The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b>0,$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature: $H \leq G \leq L \leq I \leq A$. It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 5.1. Let $a,b \in [1,\infty)$ with $a<b$ and $n \in (-\infty,0)\cup[1,\infty)\setminus\{-1\}$. Then, the following inequalities are obtained:

$$\frac{A^n(a,b)}{2A(a,b)} \leq L_n^n(a,b) \leq 2A(a,b)A(a^n,b^n).$$

Proof . The assertion follows from the inequalities (3.1) for the function

$$f(x) = x^n, \quad x \in [1,\infty).$$

□

Proposition 5.2. Let $a,b \in [1,\infty)$ with $a<b$. Then, the following inequalities are obtained:

$$\frac{A^{-1}(a,b)}{2A(a,b)} \leq L^{-1}(a,b) \leq 2A(a,b)H^{-1}(a,b).$$

Proof . The assertion follows from the inequalities (3.1) for the function $f(x) = x^{-1}, \quad x \in [1,\infty)$ □

References

[1] S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), 91–95.

[2] S.S. Dragomir and S. Fitzpatrick, *The Hadamard inequalities for s-convex functions in the second sense*, Demonst. Math. **32** (1999), no. 4, 687–696.

[3] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Its Applications*, RGMIA Monograph, 2002.

[4] S.S. Dragomir, J. Pečarić, and L.E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), no. 3, 335–341.

[5] J. Hadamard, *Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. **58** (1893), 171–215.

[6] İ. İşcan, *New refinements for integral and sum forms of Hölder inequality*, J. Inequal. Appl. **2019** (2019), no. 1, 1–11.

[7] İ. İşcan and M. Kunt, *Hermite-Hadamard-Fejer type inequalities for quasi-geometrically convex functions via fractional integrals*, J. Math. **2016** (2016), Article ID 6523041, 7 pages.

[8] İ. İşcan, H. Kadakal, and M. Kadakal, *Some new integral inequalities for n-times differentiable quasi-convex functions*, Sigma J. Engin. Natural Sci. **35** (2017), no. 3, 363–368.

[9] İ. İşcan, K. Kadakal, and H. Kadakal, *Some new integral inequalities for n-times differentiable log-convex functions*, New Trends Math. Sci. **5** (2017), no. 2, 10–15.

[10] İ. İşcan, K. Kadakal, and H. Kadakal, *On two times differentiable preinvex and prequasiinvex functions*, Commun. Faculty Sci. Univ. Ankara Ser. A1 Math. Statist. **68** (2019), no. 1, 950–963.

[11] H. Kadakal, *Hermite-Hadamard type inequalities for trigonometrically convex functions*, Sci. Stud. Res. Ser. Math. Inf. **28** (2018), no. 2, 19–28.

[12] H. Kadakal, *New inequalities for strongly r-convex functions*, J. Funct. Spaces **2019** (2019), Article ID 1219237, 10 pages.

[13] H. Kadakal, *On generalization of some integral inequalities for multiplicatively P-functions*, TWMS J. Appl. Engin. Math. **10** (2020), no. 4, 1023–1035.

[14] M. Kadakal, İ. İşcan, H. Kadakal, and K. Bekar, *On improvements of some integral inequalities*, Honam Math. J. **43** (2021), no. 3, 441–452.

[15] M. Kadakal, H. Kadakal and İ. İşcan, *Some new integral inequalities for n-times differentiable s-convex functions in the first sense*, Turk. J. Anal. Number Theory **5** (2017), no. 2, 63–68.

[16] S. Maden, H. Kadakal, M. Kadakal, and İ. İşcan, *Some new integral inequalities for n-times differentiable convex and concave functions*, J. Nonlinear Sci. Appl. **10** (2017), no. 12, 6141–6148.

[17] S. Özcan, *On refinements of some integral inequalities for differentiable prequasiinvex functions*, Filomat **33** (2019), no. 14, 4377–4385.

[18] S. Özcan, *Hermit-Hadamard type inequalities for multiplicatively h-convex functions*, Konuralp J. Math. **8** (2020), no. 1, 158–164.

[19] S. Özcan, *Hermite-Hadamard type inequalities for m-convex and (α, m)-convex functions*, J. Inequal. Appl. **2020** (2020), no. 1, 10 pages.

[20] S. Özcan, *Hermite-Hadamard type inequalities for multiplicatively h-preinvex functions*, Turk. J. Anal. Number Theory **9** (2021), no. 3, 65–70.

[21] B.-Y. Xi and F. Qi, *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Funct. Spaces Appl. **2012** (2012).

[22] G. Zabandan, *A new refinement of the Hermite-Hadamard inequality for convex functions*, J. Inequal. Pure Appl. Math. **10** (2009), no. 2, Article ID 45.