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New results on f-statistical convergence of order $\tilde{\alpha}$ through triple sequences spaces

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Abstract

In this paper, we define new notions of f-statistical convergence for triple sequences of order $\tilde{\alpha}$ and strong f-Cesàro summability for triple sequences of order $\tilde{\alpha}$. Moreover we show the relationship between the spaces $w^3_{\tilde{\alpha},0}(f)$, $w^3_{\tilde{\alpha}}(f)$ and $w^3_{\tilde{\alpha},\infty}(f)$. Additionally, we show some properties of the strong f-Cesàro summability of order $\tilde{\beta}$. The main purpose of this paper is to examine the concept of f-triple statistical convergence of order α; where f-is an unbounded function and give relations between f-triple statistical convergence of order α and strong f-Cesàro summability for a triple sequence of order α.

Keywords: Triple sequences, statistical convergence, Cesàro summability method
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1 Introduction and Preliminaries

The notion of statistical convergence was originally introduced by Zygmund [41] in his monograph. Statistical convergence of number sequences was formally given by Steinhaus [30] and Fast [7] and then was reintroduced by Schoenberg [29] independently for real and complex sequences. Later on, many works have been done by using statistical convergence by Fridy [8], Connor [3], Maddox [21] and many others. Gadjiev and Orhan [9] took the initiative to establish the order of statistical convergence of a single sequence of number, and after Çolak [5] continued this idea and studied statistical convergence of order α and strong p-Cesàro summability of order α and later was extended for triple sequences by Torgut and Altin [31]. The concept of statistical convergence for triple sequences was first introduced by Şahiner et al. [27]. Besides, this topic was studied by many authors (see, for example, Granados [10], Demirci and Gürdal [6], Huban et al. [20]). There are some more studies about statistical convergence in the literature [11, 12, 13, 14, 15, 16, 26, 17, 18, 19, 32, 33]. Connor [3] has discussed the relationship between statistical convergence and strong summability with the help of modulus function. Aizpuru et al. [1] studied and showed the concept of f-statistical convergence. Bhardwaj and Dhawan [2] continued this idea and defined f-statistical convergence of order $\tilde{\alpha}$. The notion of modulus function was structured by Nakano [24]. In this paper, we define and discuss f-statistical convergence of order $\tilde{\alpha}$ for triple sequences which is an extension of the notions presented by [31]. Furthermore, we study strong p-Cesàro summability of order α and show some related inclusion relations. Moreover, we prove

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the related inclusion relations for $\tilde{\alpha}'$s in $(0,1]$. For more topics related to mentioned above, we refer the reader to [34, 35, 36, 37, 38, 39, 40].

Now, we will recall some notions which will be useful for the development of this paper.

Definition 1.1. (See [8]) The natural density of $K \subset \mathbb{N}$ is defined by

$$\delta(K) = \lim_{q\to\infty} q^{-1}|\{k \leq q : k \in K\}|,$$

where $|\{k \leq q : k \in K\}|$ denotes the number of elements of K not exceeding q.

Remark 1.2. Any finite subset of $\mathbb{N}$ has zero natural density and $\delta(K^c) = 1 - \delta(K)$.

Definition 1.3. (See [28]) A sequence (y_k) of complex numbers is said to be statistically convergent to some number L if $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\})$ has natural density zero for every $\varepsilon > 0$. L is called the statistical limit of (x_k) and written as $S\text{-}\lim x_k = L$.

Remark 1.4. Throughout this paper, S, S^3, ℓ^3_∞, c^3 and c^3_0 denote spaces of all statistically convergent sequences, the spaces of all triple sequences, the linear spaces bounded sequences, the linear spaces of convergent sequences and the linear spaces of null sequences, respectively.

Remark 1.5. $\|y\|_{(\infty,3)} = \sup\limits_{i,j,k} |x_{ijk}|$ denotes the norm, where $i, j, k \in \mathbb{N} = \{1, 2, 3, ...\}$.

Definition 1.6. (See [25]) A triple sequence $y = (y_{ijk})$ has Pringsheim limit L provided that given for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|y_{ijk} - L| < \varepsilon$ whenever $i, j, k > N$. In this case, we write $P\text{-}\lim y = L$.

Remark 1.7. $y = (y_{ijk})$ is bounded if there exists a positive number M such that $|x_{ijk}| < M$ for all i, j and k, that is, $\|y\| = \sup\limits_{i,j,k\geq 0} |x_{ijk}| < \infty$.

Remark 1.8. Throughout this paper, ℓ^3_∞ denotes the collection of all bounded triple sequences.

Remark 1.9. We shall recall that convergent triple sequence need not be bounded.

Definition 1.10. Let $K \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $K(q, w, r) = \{(i, j, k) : i \leq q, j \leq w, k \leq r\}$. The triple natural density of K is defined by

$$\delta_3(K) = P - \lim_{q,w,r} \frac{1}{qwr}|K(q, w, r)|,$$

if the limit exists.

Definition 1.11. (See [27]) A triple sequence $y = (y_{ijk})$ is said to be statistically convergent to a number L if for every $\varepsilon > 0$ the set $\{(i, j, k) : i \leq q, j \leq w, k \leq r : |x_{ijk} - L| \geq \varepsilon\}$ has triple natural density zero. In this case, we write $st_3\text{-}\lim y = L$.

Remark 1.12. Throughout this paper, st_3 denotes the collection of all statistically convergent triple sequences.

Definition 1.13. (See [23]) A real-valued function f defined on $(0, \infty)$ is called a modulus if it satisfies the following properties:

1. $f(y) = 0$ if and only if $y = 0$,
2. $f(y + z) \leq f(y) + f(z)$ for every $y, z \in \mathbb{R}^+$,
3. f is increasing,
4. f is continuous from the right at 0.

Remark 1.14. We shall recall that every such function is continuous. A modulus may be unbounded or bounded, for example $f(y) = \dfrac{y}{1+y}$ and $f(y) = y^p$ with $0 < p \leq 1$.

2 Main Results

In this section, we begin introducing the notion of f-triple statistical convergence of order $\tilde{\alpha}$. We recall that throughout this paper, $a, b, c, d, e, g \in (0,1]$ in case otherwise is indicated, and for the sake of brevity we will write $\tilde{\alpha}$ instead of (a, b, c) and $\tilde{\beta}$ instead of (d, e, g). Besides, we define

1. $\tilde{\alpha} \preceq \tilde{\beta}$ if and only if $a \leq d, b \leq e$ and $c \leq g$,
2. $\tilde{\alpha} \preceq \tilde{\beta}$ if and only if $a < d, b < e$ and $c < g$,
3. $\tilde{\alpha} \cong \tilde{\beta}$ if and only if $a = d, b = e$ and $c = g$,
4. $\tilde{\alpha} \in (0,1]$ if and only if $a, b, c \in (0,1]$,
5. $\tilde{\beta} \in (0,1]$ if and only if $d, e, g \in (0,1]$,
6. $\tilde{\alpha} \cong 1$ in case $a = b = c = 1$,
7. $\tilde{\beta} \cong 1$ in case $d = e = g = 1$,
8. $\tilde{\alpha} \succ 1$ in case $a > 1, b > 1$ and $c > 1$.

Moreover, we will write $S^3_{\tilde{\alpha}}(f)$ and $S^3_{\tilde{\beta}}(f)$ to denote $S^3_{(a,b,c)}(f)$ and $S^3_{(d,e,g)}(f)$, respectively. Now, let f be an unbounded modulus function, $K \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $K(q, w, r)$ be the number of $(i, j, k) \in K$ such that $i \leq q, j \leq w$ and $k \leq r$ In case the sequence $(K(q, w, r)/qwr)$ has a limit in Pringsheim's sense, we will say that K has an $f_{\tilde{\alpha}}$-triple density and it is defined by

$$\delta^{f_3}_{\tilde{\alpha}}(K) = \lim_{q,w,r\to\infty} \frac{f(|(K(q, w, r))|)}{f(q^a w^b r^c)}.$$

We can see that for any set $K \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$, $\delta^{f_3}_{\tilde{\alpha}}(K)$ can be to $(1, \infty)$, but $\delta^{f_3}_{\tilde{\alpha}}(K) \leq 1$. Besides, $\delta^{f_3}_{\tilde{\alpha}}(K^c) = 1-\delta^{f_3}_{\tilde{\alpha}}(K)$ holds, but $\delta^{f_3}_{\tilde{\alpha}}(K) = 1 - \delta^{f_3}_{\tilde{\alpha}}(K)$ does not hold in general. For example, if we choose $f(y) = y^p$, $0 < p \leq 1$, $\tilde{\alpha} \in (0, 1)$, and $K = \{(i^2, j^2, k^2) : i, j, k \in \mathbb{N}\}$, then $\delta^{f_3}_{\tilde{\alpha}}(K^c) = \infty = \delta^{f_3}_{\tilde{\alpha}}(K)$.

Definition 2.1. Let $y = (y_{ijk}) \in s^3$ and $\tilde{\alpha} \in (0,1]$ be given. The sequence (y_{ijk}) is called f-triple statistically convergent of order $\tilde{\alpha}$ if there is a complex number L such that for every $\varepsilon > 0$,

$$\lim_{q,w,r\to\infty} \frac{1}{f(q^a w^b r^c)} f(|\{(i, j, k), i \leq q, j \leq w, k \leq r : |y_{ijk} - L| \geq \varepsilon\}|) = 0$$

in which case we say that y is f-triple statistically convergent of order $\tilde{\alpha}$ to L. In this case, we will write $S^3_{\tilde{\alpha}}(f)$-$\lim\limits_{i,j,k} y_{i,j,k} = L$ an we will denote the set of all f-statistically convergent triple sequences of order $\tilde{\alpha}$ by $S^3_{\tilde{\alpha}}(f)$ where f is an unbounded modulus function.

Remark 2.2. It can be easily to check that if $y = (y_{ijk})$ is f-triple statistically convergent of order $\tilde{\alpha}$ to the number L, then L is determined uniquely.

Remark 2.3. The f-triple statistical convergence of order $\tilde{\alpha}$ is well defined for $\tilde{\alpha} \in (0,1]$, but it is not well defined for $\tilde{\alpha} \succ 1$ as can be seen in the following example.

Example 2.4. Let $y = (y_{ijk})$ be defined as follows:

$$y_{ijk} = \begin{cases} 1, & \text{if } i + j + k \text{ is even,} \\ 0, & \text{if } i + j + k \text{ is odd.} \end{cases}$$

Since $\lim\limits_{c\to\infty} \dfrac{f(c)}{c} > 0$, we have

$$\lim_{q,w,r\to\infty} \frac{1}{f(q^a w^b r^c)} f(|\{(i, j, k), i \leq q, j \leq w, k \leq r : |y_{ijk} - 1| \geq \varepsilon\}|) \leq \lim_{q,w,r\to\infty} \frac{f((\frac{q}{3} + 1)(\frac{w}{3} + 1)(\frac{r}{3} + 1)}{f(q^a w^b r^c)} = 0$$

and

$$\lim_{q,w,r\to\infty} \frac{1}{f(q^a w^b r^c)} |\{(i, j, k), i \leq q, j \leq w, k \leq r : |y_{ijk} - 0| \geq \varepsilon\}| \leq \lim_{q,w,r\to\infty} \frac{f((\frac{q}{3} + 1)(\frac{w}{3} + 1)(\frac{r}{3} + 1)}{f(q^a w^b r^c)} = 0$$

for $\tilde{\alpha} \succ 1$, that is, $a > 1, b > 1$ and $c > 1$, so that $y = (y_{ijk})$ f-triple statistically converges of order $\tilde{\alpha}$ all of them to 1 and 0, i.e., $S^3_{\tilde{\alpha}}(f)$-$\lim\limits_{ijk} = 1$ and $S^3_{\tilde{\alpha}}(f)$-$\lim\limits_{ijk} = 0$. But this is impossible.

Theorem 2.5. Let f be and unbounded modulus function and $\tilde{\alpha} \in (0,1]$. Now, let $y = (y_{ijk})$ and $z = (z_{ijk})$ be any two sequences of complex numbers. Then

1. If $S^3_{\tilde{\alpha}}(f)$-$\lim y_{ijk} = y_0$ and $p \in \mathcal{C}$, then $S^3_{\tilde{\alpha}}(f)$-$\lim py_{ijk} = px_{ijk}$.
2. If $S^3_{\tilde{\alpha}}(f)$-$\lim y_{ijk} = y_0$ and If $S^3_{\tilde{\alpha}}(f)$-$\lim z_{ijk} = z_0$, then $S^3_{\tilde{\alpha}}(f)$-$\lim(y_{ijk} + z_{ijk}) = y_0 + z_0$.

Proof . Proof is straightforward. □

Remark 2.6. It is easy to verify that every convergent triple sequence is f-statistically convergent of order $\tilde{\alpha}$ to the same number, that is, $c^3 \subset S^3_{\tilde{\alpha}}(f)$ for each $\tilde{\alpha} \in (0,1]$, this is, for each pair of (a,b,c), such that $a,b,c \in (0,1]$. But the converse does not hold as can be seen in the following example.

Example 2.7. Let $y = (y_{ijk})$ be defined as follows

$$y_{ijk} = \begin{cases} 1, & if i = q^3, j = w^3, k = r^3; q,w,r = 1,2,3,..., \\ 0, & \text{otherwise.} \end{cases}$$

Taking $f(y) = y^p, (0 < p \leq 1)$. $y = (y_{ijk}) \in S^3_{\tilde{\alpha}}(f)$ for $\tilde{\alpha} \succ \frac{1}{3}$, but it is not convergent.

Theorem 2.8. Let f be an unbounded modulus function and $\tilde{\alpha}, \tilde{\beta}$ be two real numbers such that $0 \preceq \tilde{\alpha} \preceq \tilde{\beta} \preceq 1$. Then, $S^3_{\tilde{\alpha}}(f) \subset S^3_{\tilde{\beta}}(f)$ and strict inclusion may occur.

Proof . Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ be given. If $\tilde{\alpha} \leq \tilde{\beta}$, so that $(a \leq d, b \leq e$ and $c \leq g)$. Since f is increasing, then

$$\begin{aligned} & \frac{1}{f(q^d w^e r^g)} f(|\{(i,j,k), i \leq q, j \leq w, k \leq r : |y_{ijk} - L| \geq \varepsilon\}|) \\ & \leq \frac{1}{f(q^a w^b r^c)} f(|\{(i,j,k), i \leq q, j \leq w, k \leq r : |y_{ijk} - L| \geq \varepsilon\}|) \end{aligned}$$

for every $\varepsilon > 0$ and this shows $S^3_{\tilde{\alpha}}(f) \subset S^3_{\tilde{\beta}}(f)$ □

Remark 2.9. The strict inclusion may occur as can be seen in the following example.

Example 2.10. Consider the sequence $y = (y_{ijk})$ defined by

$$y_{ijk} = \begin{cases} 1, & if i = q^2, j = w^2, k = r^2; q,w,r = 1,2,3,..., \\ 0, & \text{otherwise.} \end{cases}$$

and take $f(y) = y^p$, $(0 < p \leq 1)$. Therefore, we get $y \in S^3_{\tilde{\beta}}(f)$ for $\tilde{\beta} \in (\frac{1}{2}, 1]$, but $y \notin S^3_{\tilde{\alpha}}(f)$ for $\tilde{\alpha} \in (0, \frac{1}{2}]$.

Remark 2.11. If we take $\tilde{\beta} \cong 1$ in Theorem 2.8, then we have the following results.

Corollary 2.12. If a triple sequence is f-triple statistically convergent of order $\tilde{\alpha}$ to L, for some $\tilde{\alpha}$ such that $\tilde{\alpha} \in (0,1]$, then it is f-triple statistically convergent to L, i.e. $S^3_{\tilde{\alpha}}(f) \subset S^3(f)$, and the inclusion is strict.

Proof . Proofs follows from Theorem 2.8. □

Corollary 2.13. Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ be given, then

1. $S^3_{\tilde{\alpha}}(f) = S^3_{\tilde{\beta}}(f)$ if and only if $\tilde{\alpha} \cong \tilde{\beta}$,
2. $S^3_{\tilde{\alpha}}(f) = S^3(f)$ if and only if $\tilde{\alpha} \cong 1$.

Proof . Proofs follows from Theorem 2.8. □

Remark 2.14. To show that the strict inclusion may occur, we show the following example.

Example 2.15. Consider the sequence $y = (y_{ijk})$ defined by

$$y_{ijk} = \begin{cases} ijk, & if i = q^2, j = w^2, k = r^2; q, w, r = 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

Let $f(y) = \log(y + 1)$. Then, $y \in S^3_{\tilde{\alpha}}$ for $\tilde{\alpha} \in (\frac{1}{2}, 1]$ and hence $y \in S^3$. Since f is increasing,

$$\begin{aligned} & \frac{1}{f(q^a w^b r^c)} f(|\{(i,j,k), i \le q, j \le w, k \le r : |y_{ijk} - 0| \ge \varepsilon\}|) \\ \ge & \frac{1}{f(qwr)} f(|\{(i,j,k), i \le q, j \le w, k \le r : |y_{ijk} - 0| \ge \varepsilon\}|) \\ = & \frac{1}{2} \end{aligned}$$

but $y \notin S^3_{\tilde{\alpha}}(f)$.

The following diagram shows the inclusion relations among the spaces $c^3, S^3, S^3_{\tilde{\alpha}}$ and $S^3_{\tilde{\alpha}}(f)$.

Diagram I

$$\begin{array}{ccccc} c^3 & \longrightarrow & S^3_{\tilde{\alpha}}(f) & \rightarrow & S^3_{\tilde{\alpha}} \\ & & \downarrow & & \downarrow \\ & & S^3(f) & \rightarrow & S^3 \end{array}$$

Next, we define and give the relationship between the spaces $w^3_{\tilde{\alpha},0}(f)$, $w^3_{\tilde{\alpha}}(f)$ and $w^3_{\tilde{\alpha},\infty}(f)$. Besides, we prove some properties of the strong f-Cesàro summability of order $\tilde{\beta}$ which is related to strong f-Cesàro summability of order $\tilde{\alpha}$.

Definition 2.16. Let f be a modulus and $\tilde{\alpha}$ be a positive real number, then we define

$$w^3_{\tilde{\alpha},0}(f) = \left\{ y = (y_{ijk}) \in s^3 : \lim_{q,w,r \to \infty} \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|x_{ijk}|\right) = 0 \right\},$$

$$w^3_{\tilde{\alpha}}(f) = \left\{ y = (y_{ijk}) \in s^3 : \lim_{q,w,r \to \infty} \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|x_{ijk} - L|\right) = 0 \right\},$$

$$w^3_{\tilde{\alpha},\infty}(f) = \left\{ y = (y_{ijk}) \in s^3 : \sup_{q,w,r} \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|x_{ijk}|\right) < \infty \right\}.$$

Theorem 2.17. For any modulus f and $\tilde{\alpha} \succeq 0$, we have $w^3_{\tilde{\alpha},0}(f) \subset w^3_{\tilde{\alpha},\infty}(f)$.

Proof . This proof follows from Definition 2.16. □

Theorem 2.18. For any modulus f and $\tilde{\alpha} \succeq 1$, we have $w^3_{\tilde{\alpha}}(f) \subset w^3_{\tilde{\alpha},\infty}(f)$

Proof . Let $y \in w^3_{\tilde{\alpha}}(f)$, by definition of modulus function, we have

$$\frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|x_{ijk}|\right) \le \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|x_{ijk} - L|\right) + f\left(|L|\right) \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} 1,$$

and since $\tilde{\alpha} \succeq 1$ and $y \in w^3_{\tilde{\alpha}}(f)$, this implies $y \in w^3_{\tilde{\alpha},\infty}(f)$. □

Theorem 2.19. For any modulus f and $\tilde{\alpha} \succeq 1$, we have

1. $w^3_{\tilde{\alpha}} \subset w^3_{\tilde{\alpha}}(f)$,
2. $w^3_{\tilde{\alpha},0} \subset w^3_{\tilde{\alpha},0}(f)$,
3. $w^3_{\tilde{\alpha},\infty} \subset w^3_{\tilde{\alpha},\infty}(f)$.

Proof . We just prove $w^3_{\tilde{\alpha}} \subset w^3_{\tilde{\alpha}}(f)$ and other cases will follow similarly. Let $y \in w^3_{\tilde{\alpha},\infty}$, thus

$$\sup_{q,w,r} \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q}\sum_{j=1}^{w}\sum_{k=1}^{r} |y_{ijk}| < \infty.$$

Now, let $\varepsilon > 0$ and take δ with $0 < \delta < 1$ such that $f(b) < \varepsilon$ for $0 \le b < \delta$. Now, write

$$\frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q}\sum_{j=1}^{w}\sum_{k=1}^{r} f\left(|y_{ijk}|\right) = \sum_1 + \sum_2 + \sum_3$$

where the first summation is over $|y_{ijk}| \le \delta$ and the second is over $|y_{ijk}| > \delta$. Then, $\sum_1 \le \varepsilon \frac{1}{(qwr)^{\tilde{\alpha}-1}}$, and for $|y_{ijk}| > \delta$ w use the fact

$$|y_{ijk}| < \frac{|y_{ijk}|}{\delta} < 1 + \left[\left|\frac{|y_{ijk}|}{\delta}\right|\right],$$

where $[|b|]$ denotes the integer part of b. Now, given $\varepsilon > 0$, by the definition of f, we have for $|y_{ijk}| > \delta$

$$f(|y_{ijk}|) \le \left(1 + \left[\left|\frac{|y_{ijk}|}{\delta}\right|\right]\right) f(1) \le 2f(1)\frac{|y_{ijk}|}{\delta}$$

and then $\sum_2 + \sum_3 \le 2f(1)\delta^{-1} \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q}\sum_{j=1}^{w}\sum_{k=1}^{r} |y_{ijk}|$, which together with $\sum_1 \le \varepsilon \frac{1}{(qwr)^{\tilde{\alpha}-1}}$ yields

$$\frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q}\sum_{j=1}^{w}\sum_{k=1}^{r} f\left(|y_{ijk}|\right) \le \varepsilon \frac{1}{(qwr)^{\tilde{\alpha}-1}} + 2f(1)\delta^{-1} \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q}\sum_{j=1}^{w}\sum_{k=1}^{r} |y_{ijk}|.$$

Since $\tilde{\alpha} \ge 1$ and $y \in w_{\tilde{\alpha},\infty}$, this implies that $y \in w_{\tilde{\alpha},\infty}(f)$. □

Theorem 2.20. For any modulus function f and $\tilde{\alpha} \succ 0$, if $\lim_{b\to\infty} \frac{f(b)}{b} > 0$, then $w^3_{\tilde{\alpha}}(f) \subset w^3_{\tilde{\alpha}}$.

Proof . Following the proof of [22, Proposition 1], we have $\lambda = \lim_{b\to\infty} \frac{f(b)}{b} = \inf\left\{\frac{f(b)}{b} : b > 0\right\}$. By definition of λ, we have $f(b) \ge \lambda b$ for all $b \ge 0$. Since $\lambda > 0$, we have $b \le \lambda^{-1} f(b)$ for all $b \ge 0$ and thus

$$\frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q}\sum_{j=1}^{w}\sum_{k=1}^{r} |y_{ijk} - L| \le \lambda^{-1} \frac{1}{(qwr)^{\tilde{\alpha}}} \sum_{i=1}^{q}\sum_{j=1}^{w}\sum_{k=1}^{r} f\left(|y_{ijk} - L|\right).$$

Therefore, this implies that $y \in w^3_{\tilde{\alpha}}(f)$ whenever $y \in w^3_{\tilde{\alpha}}$. □

Proposition 2.21. For any modulus function f such that $\lim_{b\to\infty} \frac{f(b)}{b} > 0$ and $\tilde{\alpha} \succeq 1$, then $w^3_{\tilde{\alpha}}(f) = w^3_{\tilde{\alpha}}$.

Proof . Proofs follows form Theorem 2.20. □

Theorem 2.22. For any modulus function f such that $\lim_{b\to\infty} \frac{f(b)}{b} > 0$ and $\tilde{\beta} \succeq \tilde{\alpha} \succeq 0$, then $w^3_{\tilde{\alpha}}(f) \subset w^3_{\tilde{\beta}}(f)$ and the inclusion is strict.

Proof . To show $w^3_{\tilde{\alpha}}(f) \subset w^3_{\tilde{\beta}}(f)$, it is similar to Theorem 2.8, hence we will show that inclusion is strict. Consider the sequence $y = (y_{ijk})$ defined as follows

$$y_{ijk} = \begin{cases} 1, & if i = q^2, j = w^2, k = r^2; q, w, r = 1, 2, 3, ..., \\ 0, & \text{otherwise}. \end{cases}$$

By using definition of modulus function, we have

$$\frac{1}{q^d w^e r^g} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|y_{ijk} - 0|\right) \leq \frac{\sqrt{q}\sqrt{w}\sqrt{r}}{q^d w^e r^g} f(1) = \frac{1}{q^{d-\frac{1}{2}} w^{e-\frac{1}{2}} r^{g-\frac{1}{2}}} f(1)$$

Since $\frac{1}{q^{d-\frac{1}{2}} w^{e-\frac{1}{2}} r^{g-\frac{1}{2}}} f(1) \to 0$ as $q, w, r \to \infty$ for $d, e, g > \frac{1}{2}$, $y \in w^3_{\tilde{\beta}}(f)$ for $d, e, g > \frac{1}{2}$. Furthermore,

$$\frac{1}{q^a w^b r^c} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|y_{ijk} - 0|\right) \leq \frac{\sqrt{q}\sqrt{w}\sqrt{r} - 1}{q^a w^b r^c} f(1)$$

and $\frac{\sqrt{q}\sqrt{w}\sqrt{r} - 1}{q^a w^b r^c} f(1) \to \infty$ as $q, w, r \to \infty$ for $0 < a, b, c < \frac{1}{2}$, which implies that $x \notin w^3_{\tilde{\alpha}}(f)$ for $0 < a, b, c < \frac{1}{2}$. $\square$

Next, we show the relationship between the strong f-Cesàro summability of order $\tilde{\alpha}$ and f-triple statistical convergence of order β and we prove some inclusion theorems.

Lemma 2.23. (See [22]) Let f be unbounded modulus such that there is a positive constant k such that $f(yz) \geq kf(y)f(z)$ for all $y, z \geq 0$.

Theorem 2.24. Let $0 \prec \tilde{\alpha} \preceq \tilde{\beta}$ and f be unbounded function such that there is a positive constant k such that $f(yz) \geq kf(y)f(z)$ for all $y, z \geq 0$ and $\lim_{b \to \infty} \frac{f(b)}{b} > 0$. If a sequence $y = (y_{ijk})$ is strongly triple Cesàro summable of order $\tilde{\alpha}$ with respect to f to L, then it is f-triple statistically convergent of order $\tilde{\beta}$ to L.

Proof . For any sequence $y = (y_{jk})$ and $\varepsilon > 0$, by using definition of modulus function, we have

$$\begin{aligned} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|y_{ijk} - L|\right) &\geq f\left(\sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} |y_{ijk} - L|\right) \\ &\geq f(|\{(i,j,k), i \leq q, j \leq w, k \leq r : |y_{ijk} - L| \geq \varepsilon\}|\varepsilon) \\ &\geq kf(|\{(i,j,k), i \leq q, j \leq w, k \leq r : |y_{ijk} - L| \geq \varepsilon\}|f(\varepsilon)), \end{aligned}$$

and since $\tilde{\alpha} \preceq \tilde{\beta}$,

$$\begin{aligned} \frac{1}{q^a w^b r^c} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f\left(|y_{ijk} - L|\right) &\geq \frac{1}{q^a w^b r^c} kf(|\{(i,j,k), i \leq q, j \leq w, k \leq r : |y_{ijk} - L| \geq \varepsilon\}|f(\varepsilon)) \\ &\geq \frac{1}{q^d w^e r^g} kf(|\{(i,j,k), i \leq q, j \leq w, k \leq r : |y_{ijk} - L| \geq \varepsilon\}|f(\varepsilon)) \\ &= \frac{1}{q^d w^e r^g f(q^d w^e r^g)} kf(|\{(i,j,k), i \leq q, j \leq w, k \leq r : |y_{ijk} - L| \geq \varepsilon\}|f(\varepsilon)) f(q^d w^e r^g). \end{aligned}$$

Thus, using the fact $\lim_{b \to \infty} \frac{f(b)}{b} > 0$ and $y \in w^3_{\tilde{\alpha}}(f)$, this implies that $y \in S^3_{\tilde{\beta}}$. $\square$

Corollary 2.25. Let f be unbounded modulus function $f(yz) \geq kf(y)f(z)$, where k is a positive constant for all $y, z \geq 0$ and $\lim_{b \to \infty} \frac{f(b)}{b} > 0$. Let $\tilde{\alpha} \in (0, 1]$. If a sequence is strongly triple Cesàro summable of order $\tilde{\alpha}$ with respect to f to L, then it is f-triple statistically convergent of order $\tilde{\alpha}$ to L.

Proof . Proof follows form Theorem 2.24 taking $\tilde{\beta} \cong \tilde{\alpha}$. $\square$

Corollary 2.26. Let f be unbounded modulus function $f(yz) \geq kf(y)f(z)$, where k is a positive constant for all $y, z \geq 0$ and $\lim_{b\to\infty} \frac{f(b)}{b} > 0$. Let $\tilde{\alpha} \in (0,1]$. If a sequence is strongly triple Cesàro summable of order $\tilde{\alpha}$ with respect to f to L, then it is f-triple statistically convergent to L.

Proof . Proof follows form Theorem 2.24 taking $1 \cong \tilde{\alpha}$. $\square$

Remark 2.27. Converse of Theorem 2.24 is not true in general as can be seen in the following example.

Example 2.28. Consider the sequence $y = (y_{ijk})$ defined as follows

$$y_{ijk} = \begin{cases} \frac{1}{\sqrt{i}\sqrt{j}\sqrt{k}}, & if i \neq q^3, j \neq w^3, k \neq r^e; q, w, r = 1, 2, 3, ..., \\ 1, & otherwise. \end{cases}$$

And an unbounded modulus function $f(y) = y$. By Definition 4.1 and Remark of [32], we can check that the result follows.

Corollary 2.29. Let $\tilde{\alpha} \in (0,1]$. Then, $w^3_{\tilde{\alpha}}(f) \subset S^3(f)$. The inclusion is strict if $\tilde{\alpha} \in (0,1)$.

Proof . From Corollaries 2.25 and 2.12, we have $w^3_{\tilde{\alpha}}(f) \subset S^3(f)$. For proving that inclusion is strict, consider the sequence $y = (y_{ijk})$ defined as follows

$$y_{ijk} = \begin{cases} 1, & if i = q^3, j = w^3, k = r^3; q, w, r = 1, 2, 3, ..., \\ 0, & otherwise. \end{cases}$$

It is clearly that $S^3(f) - \lim y_{ijk} = 0$, i.e., $y \in S^3(f)$ but $y \notin w^3_{\tilde{\alpha}}(f)$ for $\tilde{\alpha} \in (0, \frac{1}{3}]$. In fact, we can see that

$$\frac{1}{q^a w^b r^c} \sum_{i=1}^{q} \sum_{j=1}^{w} \sum_{k=1}^{r} f(|y_{ijk}|) \geq \frac{\sqrt[3]{q} - 1}{q^a} \frac{\sqrt[3]{w} - 1}{w^b} \frac{\sqrt[3]{r} - 1}{r^c}.$$

Since $\frac{\sqrt[3]{q}-1}{q^a} \to \infty$ as $q \to \infty$, $\frac{\sqrt[3]{w}-1}{w^b} \to \infty$ as $w \to \infty$ and $\frac{\sqrt[3]{r}-1}{r^c} \to \infty$ as $r \to \infty$, $w^3_{\tilde{\alpha}}(f)$ if $\tilde{\alpha} \in (0, \frac{1}{3}]$. Therefore, $y \in w^3_{\tilde{\alpha}}(f) - S^3(f)$ for $\tilde{\alpha} \in (0, \frac{1}{3}]$. $\square$

3 Conclusion

In this paper, we extended the notion of f-statistical convergence of order $\tilde{\alpha}$ for triple sequences spaces. For future works, we suggest to study this notion in a higher dimension, also we recommend to find more properties that the f-statistical convergence of order $\tilde{\alpha}$ can be had.

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