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Estimate the Bullen inequality for h-convex functions

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Abstract

This paper establishes Bullen inequalities for h-convex functions using Riemann-Liouville fractional operators. In addition, a novel fractional version with a relatively easy calculation utilizing the B-function is presented, which generalizes numerous inequalities known in the literature.

Keywords: h-convex function, B-function, Bullen inequality
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1 Introduction

The well-known Hermite-Hadamard inequality reads as follows [9], for the convex function f:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [3], Bullen improved the right side of (1.1) by the following inequality, which is known as Bullen's inequality:

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}.$$

Bullen-type inequalities are estimated for functions with convex first derivative absolute values [11, Remark 4.2].

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)\left[|f'(a)|+|f'(b)|\right]}{16}. \tag{1.2}$$

Bullen's inequalities provide an estimate of the average value of a function that is convex on both sides while simultaneously ensuring that the function is integrable. This inequality has been extensively studied in the literature, leading to numerous directions for extension and a rich mathematical literature (see [11] - [15]).

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The analysis of fractional calculations is a generalization of classical analysis, and it advanced rapidly thanks to the exciting concept of convexity. Its extensive applications in functional analysis and optimization theory have made it a very popular research area. In [16], the author introduces a novel class of functions called h-convex functions.

Definition 1.1. Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$, where $(0, 1) \subseteq J$, be a non-negative function, $h \neq 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is an h-convex function, if f is non-negative and for all $x, y \in I$, $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1 - \lambda)y) \leq h(\lambda)f(x) + h(1 - \lambda)f(y). \tag{1.3}$$

If inequality (1.3) is reversed, then f is said to be h-concave.

By setting

- $h(\lambda) = \lambda$, Definition 1.1 reduces to convex function [13].
- $h(\lambda) = 1$, Definition 1.1 reduces to P-functions [6, 14].
- $h(\lambda) = \lambda^s$, Definition 1.1 reduces to s-convex functions [2].
- $h(\lambda) = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \lambda^{\frac{1}{k}}$, Definition 1.1 reduces to polynomial n-fractional convex functions [12].

Recently, Benaissa et al. [1] presented a new class of function called B-function defined as:

Definition 1.2. Let $a < b$ and $g : (a, b) \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. The function g is a B-function, or that g belongs to the class $B(a, b)$, if for all $x \in (a, b)$, we have

$$g(x - a) + g(b - x) \leq 2\, g\left(\frac{a + b}{2}\right). \tag{1.4}$$

If the inequality (1.4) is reversed, g is called A-function, or that g belongs to the class $A(a, b)$.

If we have equality in (1.4), g is called AB-function, or that g belongs to the class $AB(a, b)$.

Corollary 1.3. Let $h : (0, 1) \to \mathbb{R}$ be a non-negative function. The function h is a B-function, if for all $\lambda \in (0, 1)$, we have

$$h(\lambda) + h(1 - \lambda) \leq 2\, h\left(\frac{1}{2}\right). \tag{1.5}$$

- The functions $h(\lambda) = \lambda$ and $h(\lambda) = 1$ are AB-function, B-function and A-function.
- The function $h(\lambda) = \lambda^s$, $s \in [0, 1]$ is B-function.
- The function $h(\lambda) = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \lambda^{\frac{1}{k}}$, $n, k \in \mathbb{N}$ is B-function.

Let $f \in L_1[a, b]$. The left- and right-sided Riemann-Liouville fractional operators with order $\beta > 0$ are defined as follows:

$$\Im^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x - t)^{\beta-1} f(t)dt, \quad x > a,$$

$$\Im^{\beta}_{b^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t - x)^{\beta-1} f(t)dt, \quad x < b.$$

Based on earlier research, we developed an additional version of Bullen inequality for h-convex functions using Riemann-Liouville integral operators.

2 Bullen inequalities

Lemma 2.1. If $\beta > 0$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function such that $f' \in L_1([a, b])$, then the following identity holds.

$$\begin{aligned}&\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[\Im^\beta_{\left(\frac{a+b}{2}\right)^+}f(b)+\Im^\beta_{\left(\frac{a+b}{2}\right)^-}f(a)\right]\\ =&\frac{(b-a)}{8}\int_0^1\left(1-2t^\beta\right)\left[f'\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)-f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right]dt.\end{aligned} \tag{2.1}$$

Proof . By using the integration by parts, we deduce

$$\begin{aligned}T_1=&\int_0^1\left(1-2t^\beta\right)f'\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)dt\\ =&-\left(\frac{2}{b-a}\right)\left(1-2t^\beta\right)f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\Big|_0^1-\left(\frac{4\beta}{b-a}\right)\int_0^1 t^{\beta-1}f\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)dt\\ =&\left(\frac{2}{b-a}\right)\left[f\left(\frac{a+b}{2}\right)+f(b)\right]-\left(\frac{4\beta}{b-a}\right)\left(\frac{2}{b-a}\right)^\beta\int_{\frac{a+b}{2}}^b(b-\mu)^{\beta-1}f(\mu)d\mu\\ =&\left(\frac{2}{b-a}\right)\left[f\left(\frac{a+b}{2}\right)+f(b)\right]-\left(\frac{2}{b-a}\right)^{\beta+1}2\,\Gamma(\beta+1)\,\Im^\beta_{\left(\frac{a+b}{2}\right)^+}f(b).\end{aligned}$$

Similarly,

$$\begin{aligned}T_2=&\int_0^1\left(1-2t^\beta\right)f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)dt\\ =&\left(\frac{2}{b-a}\right)\left(1-2t^\beta\right)f\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\Big|_0^1+\left(\frac{4\beta}{b-a}\right)\int_0^1 t^{\beta-1}f\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)dt\\ =&-\left(\frac{2}{b-a}\right)\left[f\left(\frac{a+b}{2}+f(a)\right)\right]+\left(\frac{4\beta}{b-a}\right)\left(\frac{2}{b-a}\right)^\beta\int_a^{\frac{a+b}{2}}(\mu-a)^{\beta-1}f(\mu)d\mu\\ =&-\left(\frac{2}{b-a}\right)\left[f\left(\frac{a+b}{2}+f(a)\right)\right]+\left(\frac{2}{b-a}\right)^{\beta+1}2\,\Gamma(\beta+1)\,\Im^\beta_{\left(\frac{a+b}{2}\right)^-}f(a).\end{aligned}$$

As a result,

$$\frac{b-a}{8}\left(T_1-T_2\right)=\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[\Im^\beta_{\left(\frac{a+b}{2}\right)^+}f(b)+\Im^\beta_{\left(\frac{a+b}{2}\right)^-}f(a)\right].$$

This gives us the desired result. □

We present now the first result on Bullen's inequality estimation.

Theorem 2.2. Let h be a B-function on $(0, 1)$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|$ is an h-convex function on $[a, b]$, then the following Bullen inequality for Riemann-Liouville fractional operators holds

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[\Im^\beta_{\left(\frac{a+b}{2}\right)^+}f(b)+\Im^\beta_{\left(\frac{a+b}{2}\right)^-}f(a)\right]\right|\le\frac{b-a}{4}h\left(\frac{1}{2}\right)K_\beta\left[|f'(a)|+|f'(b)|\right], \tag{2.2}$$

where

$$K_\beta=\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\left(\frac{2\beta}{\beta+1}\right)+\left(\frac{1-\beta}{\beta+1}\right). \tag{2.3}$$

Proof . Using the absolute value of identity (2.1) and the h-convexity of the function $|f'|$, we deduce

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\\
\leq&\frac{b-a}{8}\int_0^1\left|1-2t^{\beta}\right|\left[\left|f'\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|+\left|f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|\right]dt\\
\leq&\frac{b-a}{8}\int_0^1\left|1-2t^{\beta}\right|\times\left[h\left(\frac{t}{2}\right)|f'(a)|+h\left(1-\frac{t}{2}\right)|f'(b)|+h\left(1-\frac{t}{2}\right)|f'(a)|+h\left(\frac{t}{2}\right)|f'(b)|\right]dt\\
=&\frac{b-a}{8}\left[|f'(a)|+|f'(b)|\right]\int_0^1\left|1-2t^{\beta}\right|\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt.\end{aligned}$$

Since h is a B-function, applying inequality (1.5) for $\lambda=\frac{t}{2}$, we get the following inequality

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\\
\leq&\frac{b-a}{4}h\left(\frac{1}{2}\right)\left[|f'(a)|+|f'(b)|\right]\int_0^1\left|1-2t^{\beta}\right|dt\\
=&\frac{b-a}{4}h\left(\frac{1}{2}\right)\left[|f'(a)|+|f'(b)|\right]\left(\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\left(\frac{2\beta}{\beta+1}\right)+\left(\frac{1-\beta}{\beta+1}\right)\right).\end{aligned}$$

□

With $\beta=1$, we can derive the following Bullen inequalities for h-convex functions through Riemann integrals.

Corollary 2.3. Let h be a B-function on $(0,1)$ and assume that the assumptions of Lemma 2.1 hold. If $|f'|$ is an h-convex function on $[a,b]$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{8}h\left(\frac{1}{2}\right)\left[|f'(a)|+|f'(b)|\right].\tag{2.4}$$

Consider some specific situations involving h-convexity.

1. Applying Theorem 2.2 to $h(t)=t^s$ with $s\in[0,1]$, we get the following result.

   Corollary 2.4. Assume β and f are defined as in Theorem 2.2. If $|f'|$ is an s-convex function on $[a,b]$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{s}K_{\beta}\left[|f'(a)|+|f'(b)|\right],\tag{2.5}$$

   where K_β is determined by (2.3). For $\beta=1$, we obtain

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{8}\left(\frac{1}{2}\right)^{s}\left[|f'(a)|+|f'(b)|\right].\tag{2.6}$$

   Setting $s=1$ in inequality (2.6), we get Bullen inequality (1.2) via Riemann integral for convex function .

2. Putting $h(\lambda)=1$ in Theorem 2.2, we get the following new result for the class P-function. It is also equivalent to the situations $s=0$ in inequalities (2.5) and (2.6).

   Corollary 2.5. Assume β and f are defined as in Theorem 2.2. If $|f'|$ is a P-function on $[a,b]$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\leq\frac{b-a}{4}K_{\beta}\left[|f'(a)|+|f'(b)|\right],$$

   and

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{8}\left[|f'(a)|+|f'(b)|\right].$$

3. Setting $h(\lambda)=\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$ in Theorem 2.2.

Corollary 2.6. Assume β and f are defined as in Theorem 2.2. If $|f'|$ is an n-fractional polynomial convex function on $[a,b]$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\leq\frac{b-a}{4n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}K_{\beta}\left[|f'(a)|+|f'(b)|\right],$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|\leq\frac{b-a}{8n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\left[|f'(a)|+|f'(b)|\right]. \tag{2.7}$$

The inequality (2.7) is a novel generalization of the inequality (1.2) by just using $n=1$.

Theorem 2.7. Let h be a B-function on $(0,1)$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and assume that β,f are defined as in Lemma 2.1. If $|f'|^{p}$ is an h-convex function on $[a,b]$, we get the following Bullen type inequality

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\\&\leq\frac{b-a}{4}\left(\int_{0}^{1}\left|1-2t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|^{p}+|f'(b)|^{p}\right]^{\frac{1}{p}}\\&\leq\frac{b-a}{4}\left(\int_{0}^{1}\left|1-2t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right].\end{aligned} \tag{2.8}$$

Proof . Using the absolute value of identity (2.1), we get

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\\&\leq\frac{b-a}{8}\int_{0}^{1}\left|1-2t^{\beta}\right|\left|f'\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|dt+\frac{b-a}{8}\int_{0}^{1}\left|1-2t^{\beta}\right|\left|f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|dt,\end{aligned}$$

and applying Hölder inequality and $A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, we conclude

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\\&\leq\frac{b-a}{8}\left(\int_{0}^{1}\left|1-2t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}\left(\int_{0}^{1}\left|f'\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|^{p}dt\right)^{\frac{1}{p}}\\&\quad+\frac{b-a}{8}\left(\int_{0}^{1}\left|1-2t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}\left(\int_{0}^{1}\left|f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|^{p}dt\right)^{\frac{1}{p}}\\&\leq\frac{b-a}{8}\left(\int_{0}^{1}\left|1-2t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}2^{1-\frac{1}{p}}\times\left[\int_{0}^{1}\left|f'\left(\frac{t}{2}a+\left(1-\frac{t}{2}\right)b\right)\right|^{p}dt+\int_{0}^{1}\left|f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)\right|^{p}dt\right]^{\frac{1}{p}}.\end{aligned}$$

Assuming $|f'|^{p}$ is an h-convex function, we have

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|\\&\leq\frac{b-a}{8}\left(\int_{0}^{1}\left|1-2t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}2^{\frac{1}{q}}\left[\int_{0}^{1}\left(h\left(\frac{t}{2}\right)|f'(a)|^{p}+h\left(1-\frac{t}{2}\right)|f'(b)|^{p}\right)dt\right.\\&\quad\left.+\int_{0}^{1}\left(h\left(1-\frac{t}{2}\right)|f'(a)|^{p}+h\left(\frac{t}{2}\right)|f'(b)|^{p}\right)dt\right]^{\frac{1}{p}}\\&\leq\frac{b-a}{8}\left(\int_{0}^{1}\left|1-2t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}2^{\frac{1}{q}}\left(\int_{0}^{1}\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt\right)^{\frac{1}{p}}\left[|f'(a)|^{p}+|f'(b)|^{p}\right]^{\frac{1}{p}}.\end{aligned}$$

Applying inequality (1.5) for $\lambda = \frac{t}{2}$, we get

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|$$

$$\leq \frac{b-a}{4}\left(\int_0^1\left|1-2\,t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|^{p}+|f'(b)|^{p}\right]^{\frac{1}{p}}.$$

This ends the proof the first inequality in (2.8).

For $p > 1$ and $A, B \geq 0$, we get $A^p + B^p \leq (A + B)^p$, yielding the second inequality in (2.8). □

Putting $\beta = 1$ in Theorem 2.7, we obtain

$$\left|1-2\,t\right|^{q}=\begin{cases}(1-2\,t)^{q}, & t\in(0,\frac{1}{2})\\ (2\,t-1)^{q}, & t\in(\frac{1}{2},1),\end{cases}$$

thus

$$\int_0^1\left|1-2\,t\right|^{q}dt=\int_0^{\frac{1}{2}}(1-2\,t)^{q}dt+\int_{\frac{1}{2}}^{1}(2\,t-1)^{q}dt=\frac{1}{q+1},$$

and the following Bullen inequalities via Riemann integral for h-convex function hold.

Corollary 2.8. Let h be a B-function on $(0,1)$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and assume that f is defined as in Lemma 2.1. If $|f'|^p$ is an h-convex function on $[a,b]$, we get the following Bullen type inequality

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|^{p}+|f'(b)|^{p}\right]^{\frac{1}{p}}$$

$$\leq\frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right].$$

Now, some special cases on h-convex function are established.

(1) Let $h(\lambda) = \lambda^s$ with $s \in (0, 1]$ in Theorem 2.7.

Corollary 2.9. Assume β and f are defined according to Theorem 2.7. If $|f'|^p$ is a s-convex function on $[a,b]$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+\mathfrak{I}^{\beta}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\right|$$

$$\leq\frac{b-a}{4}\left(\int_0^1\left|1-2\,t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|^{p}+|f'(b)|^{p}\right]^{\frac{1}{p}}$$

$$\leq\frac{b-a}{4}\left(\int_0^1\left|1-2\,t^{\beta}\right|^{q}dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|+|f'(b)|\right], \tag{2.9}$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|^{p}+|f'(b)|^{p}\right]^{\frac{1}{p}} \tag{2.10}$$

$$\leq\frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|+|f'(b)|\right]. \tag{2.11}$$

Remark 2.10. Setting $s = 1$ in (2.10), we have the following.

If $|f'|^p$ is a convex function on $[a,b]$ and p, $q > 1$ where $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{4\,(q+1)^{\frac{1}{q}}}\left(\frac{|f'(a)|^{p}+|f'(b)|^{p}}{2}\right)^{\frac{1}{p}}. \tag{2.12}$$

(2) Setting $h(\lambda) = 1$ in Theorem 2.7, we get the following new result about the class P-function. Consider $s \to 0^+$ in the inequalities (2.9) and (2.10).

Corollary 2.11. Assume β and f are defined as in Theorem 2.7. If $|f'|^p$ is a P-function on $[a, b]$, then

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathfrak{I}^\beta_{\left(\frac{a+b}{2}\right)^+}f(b)+\mathfrak{I}^\beta_{\left(\frac{a+b}{2}\right)^-}f(a)\right]\right|\\&\leq\frac{b-a}{4}\left(\int_0^1\left|1-2\,t^\beta\right|^q dt\right)^{\frac{1}{q}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}\\&\leq\frac{b-a}{4}\left(\int_0^1\left|1-2\,t^\beta\right|^q dt\right)^{\frac{1}{q}}\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

and

$$\begin{aligned}\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|&\leq\frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}\\&\leq\frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[|f'(a)|+|f'(b)|\right].\end{aligned}$$

(3) Letting $h(\lambda) = \frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{1}{k}}$ in Theorem 2.7, we have the following new result for the n-fractional polynomial convex function.

Corollary 2.12. Assume β and f are defined as in Theorem 2.2. If $|f'|^p$ is an n-fractional polynomial convex function on $[a, b]$, then

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[\mathfrak{I}^\beta_{\left(\frac{a+b}{2}\right)^+}f(b)+\mathfrak{I}^\beta_{\left(\frac{a+b}{2}\right)^-}f(a)\right]\right|\\&\leq\frac{b-a}{4}\left(\int_0^1\left|1-2\,t^\beta\right|^q dt\right)^{\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}\\&\leq\frac{b-a}{4}\left(\int_0^1\left|1-2\,t^\beta\right|^q dt\right)^{\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

and

$$\begin{aligned}\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|&\leq\frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}\\&\leq\frac{b-a}{4}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right]. \qquad (2.13)\end{aligned}$$

The inequality (2.13) is an extension of the inequality (2.12). Just set $n = 1$.

3 Application

We consider the means for arbitrary positive numbers $b > a > 0$ as follows;

- The arithmetic mean:

$$A(a,b) = \frac{a+b}{2}.$$

- The generalized logarithmic mean:

$$L_n(a,b) = \left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)^{\frac{1}{n}}, \quad n \in \mathbf{R} - \{-1,0\}.$$

Proposition 3.1. Let $b > a > 0$, $p > 1$ and $n > 1 + \frac{1}{P}$, then the following inequality holds:

$$\left|\frac{A(a^n,b^n)+A^n(a,b)}{2} - L_n(a,b)^n\right| \leq \frac{n\,(b-a)}{4\,(q+1)^{\frac{1}{q}}}\,A\left(a^{(n-1)p},\, b^{(n-1)p}\right)^{\frac{1}{p}}. \tag{3.1}$$

Proof . Applying Remark 2.10 and taking $f(t) = t^n$ for $t > 0$, we get $f'(t) = n\,t^{n-1}$. Since

$$(|f'(t)|^p)'' = n^p\,p(n-1)\,(p(n-1)-1)t^{p(n-1)-2} > 0,$$

the function $|f'(t)|^p$ is convex. □

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