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L_q mean extension for the polar derivative of a polynomial

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Abstract

For a polynomial $p(z)$ of degree n, we consider an operator D_α which map a polynomial $p(z)$ into $D_\alpha p(z) := (\alpha - z)p'(z) + np(z)$ with respect to α. It was proved by Liman et al [A. Liman, R. N. Mohapatra and W. M. Shah, Inequalities for the polar derivative of a polynomial, Complex Anal. Oper. Theory, 2012] that if $p(z)$ has no zeros in $|z| < 1$ then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \geq 1,\ |\beta| \leq 1$ and $|z| = 1$,

$$|zD_\alpha p(z) + n\beta\frac{|\alpha|-1}{2}p(z)| \leq \frac{n}{2}\{[|\alpha + \beta\frac{|\alpha|-1}{2}| + |z + \beta\frac{|\alpha|-1}{2}|]\max_{|z|=1}|p(z)|.$$

In this paper, we present the integral L_q mean extension of the above inequality for the polar derivative of polynomials. Our result generalize certain well-known polynomial inequalities.

Keywords: Polynomial, Integral inequality, Polar derivative, Restricted zeros.
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1 Introduction

For a polynomial $p(z)$ of degree n, Bernstein [5], proved that

$$\max_{|z|=1}|p'(z)| \leq n\max_{|z|=1}|p(z)|. \tag{1.1}$$

The L_q mean extension of inequality(1.1) as following inequality proved by Zygmund [15] in the case $q \geq 1$ and in the case $0 < q < 1$, it is due to Arestov [1],

$$\left\{\int_0^{2\pi}|p'(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq n\left\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}, \quad 0 < q < \infty. \tag{1.2}$$

Erdös conjectured and later Lax [8] proved that if $p(z)$ having no zeros in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.3}$$

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As a generalization of inequality (1.3), with the same assumptions it is proved that

$$\Big\{\int_0^{2\pi}|p'(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}} \le nC_\gamma\Big\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}}\ ,\ \text{for } q>0, \tag{1.4}$$

where

$$C_\gamma=\Big\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\gamma}|^q d\gamma\Big\}^{\frac{-1}{q}}. \tag{1.5}$$

In the case $q \ge 1$ inequality (1.4) is proved by De-Brujin [6] and for the case $0<q<1$, it is due to Rahman and Schmeisser [12].

Also Jain [7] obtained a refinement and generalization of inequality (1.3) and proved that if $p(z)$ is a polynomial of degree n does not vanish in $|z|<1$, then for every β with $|\beta|\le 1$ and $|z|=1$,

$$|zp'(z)+\frac{n\beta}{2}p(z)| \le \frac{n}{2}\{|1+\frac{\beta}{2}|+|\frac{\beta}{2}|\}\max_{|z|=1}|p(z)|. \tag{1.6}$$

Let α be a complex number. For a polynomial $p(z)$ of degree n, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined as

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z).$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and that $D_\alpha p(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}[\frac{D_\alpha p(z)}{\alpha}]=p'(z). \tag{1.7}$$

Several researchers have explored the polar derivative of polynomials (see [10, 13, 14]). Aziz and Shah [4] extended (1.1) to the polar derivative and proved that for any α with $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \le n|\alpha|\max_{|z|=1}|p(z)|. \tag{1.8}$$

They also proved that if $p(z)\neq 0$ in $|z|<1$, then for $\alpha\in\mathbb{C}$ with $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \le \frac{n}{2}(|\alpha|+1)\max_{|z|=1}|p(z)|. \tag{1.9}$$

As an generalization of inequality (1.4) to polar derivative, Aziz and Rather [3] proved that if $p(z)$ is a polynomial of degree n does not any zeros in $|z|<1$, then for any complex number α with $|\alpha|\ge 1$,

$$\Big\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}} \le nC_\gamma(|\alpha|+1)\Big\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}}\ ,\ \text{for } q\ge 1. \tag{1.10}$$

where C_γ is in (1.5). As an improvement and generalization to the inequalities (1.10) and (1.6), for a polynomial of degree n as $p(z)$ which does not any zeros in $|z|<1$, Liman et al [9] proved that for all complex numbers $\beta,\ \alpha$ with $|\beta|\le 1,\ |\alpha|\ge 1$ and $|z|=1$,

$$|zD_\alpha p(z)+n\beta\frac{|\alpha|-1}{2}p(z)| \le \frac{n}{2}\{|\alpha+\beta\frac{|\alpha|-1}{2}|+|z+\beta\frac{|\alpha|-1}{2}|\}\max_{|z|=1}|p(z)|. \tag{1.11}$$

Recently Mir and Wani [11] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then for all $\alpha,\ \beta\in\mathbb{C}$ with $|\alpha|\ge 1,\ |\beta|\le 1$ and $0\le\theta\le 2\pi$, we have for $q>0$

$$\Big\{\int_0^{2\pi}\Big|e^{i\theta}D_\alpha p(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}p(e^{i\theta})\Big|^q d\theta\Big\}^{\frac{1}{q}} \le \quad nC_\gamma\Big(|\alpha|+1+|\beta|(|\alpha|-1)\Big)\Big\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}}, \tag{1.12}$$

where C_γ is in (1.5). Obviosly, inequalities (1.6) and (1.11) are not derived from inequality (1.12). In this paper, we will solve these problems.

More precise, in the following theorem we obtain the L_q mean extension and a refinement of the inequality (1.11).

Theorem 1.1. Let $p(z)$ be a polynomial of degree n does not vanish in $|z| < 1$, then for all $\alpha,\ \delta,\ \beta \in \mathbb{C}$ with $|\alpha| \geq 1,\ ,|\delta| \leq 1,\ |\beta| \leq 1$ and $0 \leq \theta \leq 2\pi$, we have for $q > 0$

$$\Big\{\int_0^{2\pi} \Big|e^{i\theta}D_\alpha p(e^{i\theta}) + n\beta\frac{|\alpha|-1}{2}p(e^{i\theta})\Big|^q d\theta\Big\}^{\frac{1}{q}} \leq nC_\gamma\Big(\Big|\alpha + \beta\frac{|\alpha|-1}{2}\Big| + \Big|e^{i\theta} + \beta\frac{|\alpha|-1}{2}\Big|\Big)\Big\{\int_0^{2\pi} |p(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}}. \quad (1.13)$$

where C_γ is in (1.5).

Remark 1.2. Let $q \to \infty$ then inequality (1.13) reduce to inequality (1.11).

By dividing both sides of (1.13) by $|\alpha|$ and let $|\alpha| \to \infty$, we obtain the following result that is the L_q mean extension of the inequality (1.6).

Corollary 1.3. Let $p(z)$ be a polynomial of degree n does not vanish in $|z| < 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z| = 1$, we have for $q > 0$

$$\Big\{\int_0^{2\pi} \Big|e^{i\theta}p'(e^{i\theta}) + \frac{n\beta}{2}p(e^{i\theta})\Big|^q d\theta\Big\}^{\frac{1}{q}} \leq nC_\gamma\Big(|1 + \frac{\beta}{2}| + \frac{|\beta|}{2}|\Big)\Big\{\int_0^{2\pi} |p(e^{i\theta})|^q d\theta\Big\}^{\frac{1}{q}}. \quad (1.14)$$

where C_γ is in (1.5).

Remark 1.4. Let $q \to \infty$ then inequality (1.14) reduce to inequality (1.6).

2 Lemma

We need the following Lemmas, for the proofs of the theorem. The first Lemma is due to Aziz and Rather [2].

Lemma 2.1. Let $p(z)$ be a polynomial of degree n and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for each $\gamma,\ 0 \leq \gamma < 2\pi$, and $q > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} |p'(e^{i\theta}) + e^{i\gamma}q'(e^{i\theta})|^q d\theta d\gamma \leq 2\pi n^q \int_0^{2\pi} |p(e^{i\theta})|^q d\theta.$$

The following Lemma is due to Liman et al [9].

Lemma 2.2. Let $p(z)$ be a polynomial of degree n, does not vanish in $|z| < 1$, then for $\beta,\ \alpha \in \mathbb{C}$ with $|\beta| \leq 1,\ |\alpha| \geq 1$ and $|z| = 1$, we have

$$\Big|zD_\alpha p(z) + n\beta\frac{|\alpha|-1}{2}p(z)\Big| \leq \Big|zD_\alpha q(z) + n\beta\frac{|\alpha|-1}{2}q(z)\Big|.$$

where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

3 Proof of the theorem 1.1

As $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then $p(z) = z^n\overline{q(\frac{1}{\bar{z}})}$. It can be obtained that for $0 \leq \theta \leq 2\pi$,

$$\begin{aligned} np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta}) &= e^{i(n-1)\theta}\overline{q'(e^{i\theta})}, \\ nq(e^{i\theta}) - e^{i\theta}q'(e^{i\theta}) &= e^{i(n-1)\theta}\overline{p'(e^{i\theta})}, \end{aligned} \quad (3.1)$$

By adding the above equalities we have

$$n\Big(p(e^{i\theta}) + e^{i\gamma}q(e^{i\theta})\Big) - e^{i\theta}\Big(p'(e^{i\theta}) + e^{i\gamma}q'(e^{i\theta})\Big) = e^{i(n-1)\theta}\Big(\overline{q'(e^{i\theta})} + e^{i\gamma}\overline{p'(e^{i\theta})}\Big)$$

which gives

$$n\Big(p(e^{i\theta}) + e^{i\gamma}q(e^{i\theta})\Big) = e^{i\theta}\Big(p'(e^{i\theta}) + e^{i\gamma}q'(e^{i\theta})\Big) + e^{i(n-1)\theta}e^{i\gamma}\Big(\overline{p'(e^{i\theta}) + e^{i\gamma}q'(e^{i\theta})}\Big) \quad (3.2)$$

Also from (3.1) we get

$$\begin{aligned} np(e^{i\theta})-e^{i\theta}p'(e^{i\theta})+e^{i\gamma}\Big\{nq(e^{i\theta})-e^{i\theta}q'(e^{i\theta})\Big\} &= e^{i(n-1)\theta}\Big\{\overline{q'(e^{i\theta})}+e^{i\gamma}\overline{p'(e^{i\theta})}\Big\} \\ &= e^{i(n-1)\theta}e^{i\gamma}\Big\{\overline{p'(e^{i\theta})+e^{i\gamma}q'(e^{i\theta})}\Big\} \end{aligned} \tag{3.3}$$

Now we have

$$\begin{aligned} &D_\alpha p(e^{i\theta})+e^{i\gamma}D_\alpha q(e^{i\theta})=np(e^{i\theta})+(\alpha-e^{i\theta})p'(e^{i\theta})+e^{i\gamma}\Big(nq(e^{i\theta})+(\alpha-e^{i\theta})q'(e^{i\theta})\Big) \\ =&\Big\{np(e^{i\theta})-e^{i\theta}p'(e^{i\theta})+e^{i\gamma}\Big(nq(e^{i\theta})-e^{i\theta}q'(e^{i\theta})\Big)\Big\}+\alpha\Big(p'(e^{i\theta})+e^{i\gamma}q'(e^{i\theta})\Big) \\ =&e^{i(n-1)\theta}e^{i\gamma}\Big(\overline{p'(e^{i\theta})+e^{i\gamma}q'(e^{i\theta})}\Big)+\alpha\Big(p'(e^{i\theta})+e^{i\gamma}q'(e^{i\theta})\Big). \end{aligned} \tag{3.4}$$

By using (3.2) and (3.4) and taking $S(e^{i\theta})=p'(e^{i\theta})+e^{i\gamma}q'(e^{i\theta})$ we have

$$\begin{aligned} &e^{i\theta}D_\alpha p(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}p(e^{i\theta})+e^{i\gamma}\Big\{e^{i\theta}D_\alpha q(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}q(e^{i\theta})\Big\} \\ =&e^{i\theta}\Big\{D_\alpha p(e^{i\theta})+e^{i\gamma}D_\alpha q(e^{i\theta})\Big\}++\beta\frac{|\alpha|-1}{2}n\Big\{p(e^{i\theta})+e^{i\gamma}q(e^{i\theta})\Big\} \\ =&e^{i\theta}\Big\{e^{i(n-1)\theta}e^{i\gamma}\overline{S(e^{i\theta})}+\alpha S(e^{i\theta})\Big\}+\beta\frac{|\alpha|-1}{2}\Big\{e^{i\theta}S(e^{i\theta})+e^{i(n-1)\theta}e^{i\gamma}\overline{S(e^{i\theta})}\Big\} \\ =&\Big(e^{i\theta}+\beta\frac{|\alpha|-1}{2}\Big)e^{i(n-1)\theta}e^{i\gamma}\overline{S(e^{i\theta})}+\Big(\alpha+\beta\frac{|\alpha|-1}{2}\Big)e^{i\theta}S(e^{i\theta}). \end{aligned}$$

Since $|\overline{S(e^{i\theta})}|=|S(e^{i\theta})|=|p'(e^{i\theta})+e^{i\gamma}q'(e^{i\theta})|$. This conclude that

$$\begin{aligned} &\Big|e^{i\theta}D_\alpha p(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}p(e^{i\theta})+e^{i\gamma}\Big\{e^{i\theta}D_\alpha q(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}q(e^{i\theta})\Big\}\Big| \\ \leq&\Big\{\Big|e^{i\theta}+\beta\frac{|\alpha|-1}{2}\Big|+\Big|\alpha+\beta\frac{|\alpha|-1}{2}\Big|\Big\}|S(e^{i\theta})|. \end{aligned}$$

With the Lemma 2.1, it implies that for each $q>0$,

$$\begin{aligned} &\int_0^{2\pi}\int_0^{2\pi}\Big|e^{i\theta}D_\alpha p(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}p(e^{i\theta})+e^{i\gamma}\Big\{e^{i\theta}D_\alpha q(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}q(e^{i\theta})\Big\}\Big|^q d\theta d\gamma \\ \leq&\Big(\Big|e^{i\theta}+\beta\frac{|\alpha|-1}{2}\Big|+\Big|\alpha+\beta\frac{|\alpha|-1}{2}\Big|\Big)^q\int_0^{2\pi}\int_0^{2\pi}|p'(e^{i\theta})+e^{i\gamma}q'(e^{i\theta})|^q d\theta d\gamma \\ \leq&2\pi n^q\Big(\Big|e^{i\theta}+\beta\frac{|\alpha|-1}{2}\Big|+\Big|\alpha+\beta\frac{|\alpha|-1}{2}\Big|\Big)^q\int_0^{2\pi}|p(e^{i\theta})|^q d\theta. \end{aligned}$$

This implies for each $q>0$, that

$$\int_0^{2\pi}\int_0^{2\pi}|f(\theta)+e^{i\gamma}g(\theta)|^q d\theta d\gamma\leq 2\pi n^q\Big(\Big|e^{i\theta}+\beta\frac{|\alpha|-1}{2}\Big|+\Big|\alpha+\beta\frac{|\alpha|-1}{2}\Big|\Big)^q\int_0^{2\pi}|p(e^{i\theta})|^q d\theta \tag{3.5}$$

where

$$f(\theta)=|e^{i\theta}D_\alpha p(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}p(e^{i\theta})| \tag{3.6}$$

and

$$g(\theta)=|e^{i\theta}D_\alpha q(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}q(e^{i\theta})|.$$

Now for every real γ and $r\geq 1$, from the fact that

$$|r+e^{i\gamma}|\geq|1+e^{i\gamma}|$$

implies that

$$\int_0^{2\pi} |r+e^{i\gamma}|^q d\gamma \geq \int_0^{2\pi} |1+e^{i\gamma}|^q d\gamma.$$

For $f(\theta)\neq 0$, we can take $r=\frac{|g(\theta)|}{|f(\theta)|}$, by Lemma 2.2 we have $r\geq 1$. It yields

$$\begin{aligned}\int_0^{2\pi} |f(\theta)+e^{i\gamma}g(\theta)|^q d\gamma =&|f(\theta)|^q \int_0^{2\pi} |1+e^{i\gamma}\frac{g(\theta)}{f(\theta)}|^q d\gamma\\ =&|f(\theta)|^q \int_0^{2\pi} |\frac{g(\theta)}{f(\theta)}+e^{i\gamma}|^q d\gamma\\ =&|f(\theta)|^q \int_0^{2\pi} ||\frac{g(\theta)}{f(\theta)}|+e^{i\gamma}|^q d\gamma\\ \geq&|f(\theta)|^q \int_0^{2\pi} |1+e^{i\gamma}|^q d\gamma. \end{aligned} \tag{3.7}$$

In the case $f(\theta)=0$, the inequality (3.7) is apparent. Now by substituting $f(\theta)$ from (3.6) and combining inequalities (3.5) and (3.7) we obtain

$$\begin{aligned}&\int_0^{2\pi} |1+e^{i\gamma}|^q d\gamma \int_0^{2\pi} \{|e^{i\theta}D_\alpha p(e^{i\theta})+n\beta\frac{|\alpha|-1}{2}p(e^{i\theta})|\}^q d\theta\\ \leq&2\pi n^q\left(\left|e^{i\theta}+\beta\frac{|\alpha|-1}{2}\right|+\left|\alpha+\beta\frac{|\alpha|-1}{2}\right|\right)^q \int_0^{2\pi} |p(e^{i\theta})|^q d\theta.\end{aligned}$$

The proof is completed for Theorem 1.1. □

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