Int. J. Nonlinear Anal. Appl. 17 (2026) 6, 153–157
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.33551.5004

Weakly unit regular clean rings

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(Communicated by Seyyed Mohammad Reza Hashemi)

Abstract

A ring R is called clean if every member of R is the sum of a self-resolved member and an invertible member. Also, we call the ring R weakly clean if every member of R can be written as the sum or difference of an invertible member and an autoregressive member. The $a \in R$ member is called unit regular whenever $u \in U(R)$ exists such $a = aua$. A ring R is called a clean unit if every member of R is the sum of an autonomial member and a unitary member. We call a ring R a weakly clean unit if every member of R can be written as the sum or difference of a unit regular and a self-power term. In this paper, weakly unit regular clean rings are introduced and discussed. In particular, we show that if $\{R_i\}_{i\in I}$ is a family of commutative rings, then $R = \prod_{i\in I} R_i$ is weakly unit regular clean if and only if every R_i is weakly clean regular unit and at most one R_i's are not clean and regular units.

Keywords: clean rings, weakly clean rings, unit clean rings, weakly clean regular rings
2020 MSC: 16U99, 16S70

1 Introduction

In this paper, R is a monojoint participating ring but not necessarily commutative. The set of all invertible members of R is represented by $U(R)$. The member $u \in R$ is called idempotent, whenever $e^2 = e$. The set of all idempotents of R is represented by $Id(R)$. A ring R is called clean if every member of R is the sum of a self-resolved member and an invertible member. You can refer to [2, 3, 5, 8, 13, 16] references to more information in this field. Also, a ring R is called weakly clean if every member of R can be written as the sum or difference of an invertible member and an autoregressive member. These rings were first introduced in [1]. For more information on this topic, see [6, 10]. It is clear that every clean ring is a weakly clean ring, but the converse is not necessarily true. For example, the ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is weakly clean but not clean. A member $a \in R$ is called unit regular whenever $u \in U(R)$ exists such that $a = aua$. It is clear that every invertible member is a regular member of the unit. The set of all unit regular members of R is denoted by $U_r(R)$. A ring R is called a clean unit regular ring if every member of R is the sum of a self-regular member and a unit regular member [15]. A ring R is called a weakly clean l unit if every member of R can be written as the sum or difference of a unitary regularity and a self-power term. It is clear that every clean unit regular ring is a weakly unit regular clean ring. In this paper, weakly clean regular rings are discussed. In particular, in Theorem 3.13, we show that if $\{R_i\}_{i\in I}$ is a family of commutative rings, then $R = \prod_{i\in I} R_i$ is weakly unit regular clean if and only if every R_i is weakly clean regular is unit and at most one of R_i is not clean unit regular.

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Received: March 2024 *Accepted:* May 2024

2 Every regular collective ring is a Marot ring

Here we have to express a historical fact. The beginning of the study on regular collective rings goes back to the results of Marot in [11], but this naming was done by Gilmer and Huckaba in [7]. In fact, Matsuda gave a negative answer to a question from Portelli and Spangher [14] that if a regular hybrid ring is produced by a set of regular elements, then the ring does not necessarily have the UF property.

3 Weakly unit regular clean rings

Definition 3.1. Let R be a ring. In this case, we call R weakly unit regular clean ring if every member of R can be written as the sum or difference of a unitary regularity and an arbitrary term.

Example 3.2. Any unit regular clean ring or any weakly clean ring is a weakly unit regular clean ring.

The following example shows that every weakly unit regular clean ring is not necessarily a clean regular unit or weakly clean ring.

Example 3.3. The ring $\mathbb{Z}_{12}$ is a weakly unit regular clean ring. But not a unit regular clean ring nor a weakly unit regular clean ring, because:

$$Id(\mathbb{Z}_{12}) = \{0.1.4\}$$
$$U_r(\mathbb{Z}_{12}) = \{1.3.5.7.9.11\}$$
$$U(\mathbb{Z}_{12}) = \{1.5.7.11\}$$

Lemma 3.4. Let R be a ring. Then $x \in R$ is weakly unit regular clean if and only if $1-x$ or $1+x$ is clean unit regular.

Proof . Suppose that $x \in R$ is weakly unit regular clean. In this case, exist $a \in U_r(R)$ and $e \in Id(R)$ so that $x = a+e$ or $x = a-e$. Therefore, $1-x = -a+(1-e)$ or $1+x = a+(1-e)$ so that $-a.a \in U_r(R)$ and $1-e \in Id(R)$. As a result, $1-x$ or $1+x$ is a clean regular unit.

On the contrary, suppose that $1-x$ or $1+x$ is a clean unit regular. In this case, there exist $a \in U_r(R)$ and $e \in Id(R)$ so that $1-x = a+e$ or $1+x = a+e$. So $x = -a+(1-e)$ or $x = a-(1-e)$ so that $-a.a \in U_r$ and $1-e \in Id(R)$. as a result x is weakly unit regular clean. □

Remark 3.5. Suppose R is a ring and I is an ideal of R. Then the following example shows that if R/I is a weakly unit regular clean ring, then it is not necessary that R is a weakly unit regular clean ring.

Example 3.6. Let p be a prime number. Then $\mathbb{Z}/\langle p\rangle \cong \mathbb{Z}_p$ is a weakly unit regular clean ring, but $\mathbb{Z}$ is not a weakly unit regular clean ring.

Definition 3.7. Suppose that R is a ring and I is an ideal of R. In this case, we say that the eigenvalues are raised to measure I, if for each $a \in R$ with the condition $a-a^2 \in I$, there is an eigenvalue member $e \in R$ such that $e-a \in I$ [12].

Definition 3.8. Suppose that R is a ring. In this case, the union of all maximal right (left) ideals of R is called the Jacobson radical of R. We denote the Jacobson radical R by the symbol $J(R)$.

Lemma 3.9. Suppose that R is a ring and $I \subseteq J(R)$ is an ideal of R. In this case, R is weakly unit regular clean if and only if R/I is a ring weakly unit regular clean and the eigenpowers are raised to measure I.

Proof . Suppose that R is a weakly unit regular clean ring and $x+I \in R/I$, In this case, $x \in R$ and $x = a+e$ or $x = a-e$ so that $a \in U_r(R)$ and $e \in Id(R)$.

$$x+I = a+e+I = (a+I)+(e+I)$$

or

$$x + I = a - e + I = (a + I) - (e + I).$$

It is clear that $e + I \in Id(R/I)$ and:

$$a + I = aua + I = (a + I)(u + I)(a + I).$$

So $a + I \in U_r(R/I)$. As result, R/I is a weakly unit regular clean ring and the eigenvalues are raised to the scale of I.

On the contrary, suppose that R/I is a weakly unit regular clean ring and the eigenpowers are raised to measure I and I $x \in R$. In this case, $x + I = a + e + I$ or

$$x + I = a - e + I$$

so that $a + I \in U_r(R/I)$ and $e + I \in Id(R/I)$. Because eigenvalues are raised to scale I, $e \in Id(R)$. on the other hand:

$$x - e + I = a + I \in U_r(R/I)$$

or

$$x + e + I = a + I \in U_r(R/I).$$

So $x - e \in U_r(R/I)$ or $x + e \in U_r(R/I)$. As a result, $x = (x - e) + e$ or $x = (x + e) - e$, that is, R is a weakly unit regular clean ring. □

A ring R is called Abelian, if every self-sufficient member of R is central, that is, for every $e \in Id(R)$ and every $x \in R$, $xe = ex$.

Proposition 3.10. Suppose that R is an Abelian is a weakly unit regular clean ring. In this case, for each $x \in R$, there is an independent member $\in Id(R)$ such that $ex \in Id(R)$ or $-ex \in Id(R)$.

Proof . Suppose that $x \in R$ in this case $a \in U_r(R)$ and $e_1 \in Id(R)$ exist such that $x = a + e_1$ or $x = a - e_1$. Because $a \in U_r(R)$, according to [4, Theorem 1], there exist $e \in Id(R)$ and $u \in U(R)$ such that $a = e + u$ and $aR \cap eR = \{0\}$. Because $ae = ea \in aR \cap eR$, $ae = ea = 0$. So

$$ex = ea + ee_1$$

or

$$ex = ea - ee_1.$$

So $ex = ee_1$ or $-ex = ee_1$. Because e and e_1 are central eigenvalues, $ee_1 \in Id(R)$. as a result $ex \in Id(R)$ or $-ex \in Id(R)$. □

Theorem 3.11. Suppose that R is a weakly unit regular clean Abelian ring and $e \in Id(R)$. In this case eRe is a weakly unit regular clean ring.

Proof . Suppose that $x \in eRe \subseteq R$. In this case, there exist $a \in U_r(R)$ and $e_1 \in Id(R)$ such that $x = a + e_1$ or $x = a - e_1$. Because $x \in eRe$:

$$x = eae + ee_1e$$

or

$$x = eae - ee_1e.$$

Since R is an Abelian ring

$$x = ea + e_1e$$

or

$$x = ae - e_1e$$

on the other hand:

$$(ae)(eue)(ae) = (ae)(eue)(ea) = (ae)u(ea) = (ea)u(ea) = e(aua)e = eae \in eRe.$$

So $ae \in U_r(eRe)$. As result, eRe is a unitary weakly clean ring. □

The ring R is directly called finite, if for each $x.y \in R$, $xy = 1$ results in $yx = 1$ [9].

Proposition 3.12. Suppose that R is a directly finite and weakly unit regular clean ring such that $Id(R) = \{0.1\}$. Then R is a weakly clean regular ring.

Proof . Suppose that $x \in R$. In this case, there exist $a \in U_r(R)$ and $e \in Id(R)$ so that $x = a + e$ or $x = a - e$. If $a = 0$, then $x = e$ or $x = -e$. So:

$$x = e = (2e - 1) + (1 - e)$$

or

$$x = -e = -(2e - 1) - (1 - e)$$

As result, R is a weakly clean ring. If $a \neq 0$, then there exists $u \in U(R)$ such that $a = aua$. So $au \in Id(R) = \{0.1\}$. If $au = 0$, then $a = aua = 0$, which is a contradiction. So $au = 1$. Since R is a directly finite ring, $ua = 1$. So $a \in U(R)$. Consequently, R is a weakly clean ring. □

Theorem 3.13. Suppose that $\{R_i\}_{i\in I}$ is a family of commutative rings. In this case $R = \prod_{i\in I} R_i$ is weakly clean regular unit if and only if every R_i is weakly unit regular clean and at most one of R_i is not clean unit regular.

Proof . Suppose $R = \prod_{i\in I} R_i$ is weakly unit regular clean. In this case, since every R_i is a homogenous image of R, according to Lemma 3.9, every R_i is weakly clean regular. Now suppose that $i_1 \neq i_2$, R_{i_1} and R_{i_2} are not clean unit regular. In this case, there exists $x_{i_1} \in R_{i_1}$ such $x_{i_1} = a_{i_1} + e_{i_1}$ where $a_{i_1} \in U_r(R_{i_1})$ and $e_{i_1} \in Id(R_{i_1})$, But $x_{i_1} \neq a - e$ so that $a \in U_r(R_{i_1})$ and $e \in Id(R_{i_1})$. There is also $x_{i_2} \in R_{i_2}$ such $x_{i_2} = a_{i_2} - e_{i_2}$ where a $a_{i_2} \in U_r(R_{i_2}) e_{i_2} \in Id(R_{i_2})$. But $x_{i_1} \neq a + e$ so that $a \in U_r(R_{i_2})$ and $e \in Id(R_{i_2})$. Now consider $x = (x_i)$ which is defined as follows:

$$x_i = \begin{cases} x_{i_j} & i \in \{i_1, i_2\} \\ 0 & i \notin \{i_1, i_2\} \end{cases}$$

In this case $\neq a \pm e$ where a $a \in U_r(R)$ and $\in Id(R)$ which is a contradiction. As result, the verdict is upheld.

Conversely, suppose that every R_i is weakly unit regular clean. Also suppose that R_{i_0} is weakly unit regular clean and not unit and the rest of R_i are clean unit regular, we consider $x = (x_i) \in R$. In this case $x_{i_0} = a_{i_0} + e_{i_0}$ or $x_{i_0} = a_{i_0} - e_{i_0}$, where $a_{i_0} \in U_r(R_{i_0})$ and $e_{i_0} \in Id(R_{i_0})$. If $x_{i_0} = a_{i_0} + e_{i_0}$, then for each $i \neq i_0$ we assume $x_i = a_i - e_i$ where $a_i \in U_r(R_i)$ and $e_i \in Id(R_i)$. If $x_{i_0} = a_{i_0} - e_{i_0}$, then for each $i \neq i_0$ suppose that $x_i = a_i + e_i$ where $a_i \in U_r(R_i)$ and $e_i \in Id(R_i)$. So $x = a + e$ or $x = a - e$ where $a \in U_r(R)$ and $e \in Id(R)$, as result, R is a weakly unit regular clean ring. □

Let C be a subring of D. In this case, the set:

$$R[D.C] = \{(d_1.\cdots.d_n.c.c.\cdots); d_i \in D.c \in C.n \geq 1\}$$

It is a circle with component addition and multiplication.

Theorem 3.14. The ring $R[D.C]$ is weakly unit regular clean if and only if D is a clean unit regular ring and C is a weakly unit regular clean ring.

Proof . Suppose that $S = R[D.C]$ is a weakly unit regular clean ring. Then, since $D \bigoplus D$ is a summation of S, $D \bigoplus D$ is a weakly unit regular clean ring. Therefore, according to Theorem 3.13, D is a unit regular clean ring. Since C is a homogenous image of S, according to Lemma 3.9, C is a weakly unit regular clean ring.

Conversely, suppose that $x = (d_1.\cdots.d_n.c.c.\cdots) \in S$. In this case, since C is a unitary weakly unit regular clean ring, $c = a + ec = a - e$ where $a \in U_r(C)$ and $e \in Id(C)$. If $c = a + e$, then for every $1 \leq i \leq n$ we set $d_i = a_i + e_i$ where $a_i \in U_r(D)$ and $e_i \in Id(D)$. So:

$$x = (d_1.\cdots.d_n.c.c.\cdots) = (a_1.\cdots.a_n.a.a.\cdots) + (e_1.\cdots.e_n.e.e.\cdots)$$

which in $(a_1.\cdots.a_n.a.a.\cdots) \in U_r(S)$ and $(e_1.\cdots.e_n.e.e.\cdots) \in Id(S)$. If $c = a - e$, then for every $1 \leq i \leq n$, we set $d_i = a_i - e_i$, where $a_i \in U_r(D)$ and $e_i \in Id(D)$. So

$$x = (d_1.\cdots.d_n.c.c.\cdots) = (a_1.\cdots.a_n.a.a.\cdots) - (e_1.\cdots.e_n.e.e.\cdots)$$

which in $(a_1.\cdots.a_n.a.a.\cdots) \in U_r(S)$ and $(e_1.\cdots.e_n.e.e.\cdots) \in Id(S)$. As a result, $[D.C]$ is a weakly unit regular clean ring. □

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