

# Weakly unit regular clean rings

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## Abstract

A ring  $R$  is called clean if every member of  $R$  is the sum of a self-resolved member and an invertible member. Also, we call the ring  $R$  weakly clean if every member of  $R$  can be written as the sum or difference of an invertible member and an autoregressive member. The  $a \in R$  member is called unit regular whenever  $u \in U(R)$  exists such  $a = auu$ . A ring  $R$  is called a clean unit if every member of  $R$  is the sum of an automial member and a unitary member. We call a ring  $R$  a weakly clean unit if every member of  $R$  can be written as the sum or difference of a unit regular and a self-power term. In this paper, weakly unit regular clean rings are introduced and discussed. In particular, we show that if  $\{R_i\}_{i \in I}$  is a family of commutative rings, then  $R = \prod_{i \in I} R_i$  is weakly unit regular clean if and only if every  $R_i$  is weakly clean regular unit and at most one  $R_i$ 's are not clean and regular units.

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## 1 Introduction

In this paper,  $R$  is a monojoint participating ring but not necessarily commutative. The set of all invertible members of  $R$  is represented by  $U(R)$ . The member  $u \in R$  is called idempotent, whenever  $e^2 = e$ . The set of all idempotents of  $R$  is represented by  $Id(R)$ . A ring  $R$  is called clean if every member of  $R$  is the sum of a self-resolved member and an invertible member. You can refer to [2, 3, 5, 8, 13, 16] references to more information in this field. Also, a ring  $R$  is called weakly clean if every member of  $R$  can be written as the sum or difference of an invertible member and an autoregressive member. These rings were first introduced in [1]. For more information on this topic, see [6, 10]. It is clear that every clean ring is a weakly clean ring, but the converse is not necessarily true. For example, the ring  $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$  is weakly clean but not clean. A member  $a \in R$  is called unit regular whenever  $u \in U(R)$  exists such that  $a = auu$ . It is clear that every invertible member is a regular member of the unit. The set of all unit regular members of  $R$  is denoted by  $U_r(R)$ . A ring  $R$  is called a clean unit regular ring if every member of  $R$  is the sum of a self-regular member and a unit regular member [15]. A ring  $R$  is called a weakly clean  $l$  unit if every member of  $R$  can be written as the sum or difference of a unitary regularity and a self-power term. It is clear that every clean unit regular ring is a weakly unit regular clean ring. In this paper, weakly clean regular rings are discussed. In particular, in Theorem 3.13, we show that if  $\{R_i\}_{i \in I}$  is a family of commutative rings, then  $R = \prod_{i \in I} R_i$  is weakly unit regular clean if and only if every  $R_i$  is weakly clean regular is unit and at most one of  $R_i$  is not clean unit regular.

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## 2 Every regular collective ring is a Marot ring

Here we have to express a historical fact. The beginning of the study on regular collective rings goes back to the results of Marot in [11], but this naming was done by Gilmer and Huckaba in [7]. In fact, Matsuda gave a negative answer to a question from Portelli and Spangher [14] that if a regular hybrid ring is produced by a set of regular elements, then the ring does not necessarily have the UF property.

## 3 Weakly unit regular clean rings

**Definition 3.1.** Let  $R$  be a ring. In this case, we call  $R$  weakly unit regular clean ring if every member of  $R$  can be written as the sum or difference of a unitary regularity and an arbitrary term.

**Example 3.2.** Any unit regular clean ring or any weakly clean ring is a weakly unit regular clean ring.

The following example shows that every weakly unit regular clean ring is not necessarily a clean regular unit or weakly clean ring.

**Example 3.3.** The ring  $\mathbb{Z}_{12}$  is a weakly unit regular clean ring. But not a unit regular clean ring nor a weakly unit regular clean ring, because:

$$\begin{aligned} Id(\mathbb{Z}_{12}) &= \{0.1.4\} \\ U_r(\mathbb{Z}_{12}) &= \{1.3.5.7.9.11\} \\ U(\mathbb{Z}_{12}) &= \{1.5.7.11\} \end{aligned}$$

**Lemma 3.4.** Let  $R$  be a ring. Then  $x \in R$  is weakly unit regular clean if and only if  $1 - x$  or  $1 + x$  is clean unit regular.

**Proof .** Suppose that  $x \in R$  is weakly unit regular clean. In this case, exist  $a \in U_r(R)$  and  $e \in Id(R)$  so that  $x = a + e$  or  $x = a - e$ . Therefore,  $1 - x = -a + (1 - e)$  or  $1 + x = a + (1 - e)$  so that  $-a.a \in U_r(R)$  and  $1 - e \in Id(R)$ . As a result,  $1 - x$  or  $1 + x$  is a clean regular unit.

On the contrary, suppose that  $1 - x$  or  $1 + x$  is a clean unit regular. In this case, there exist  $a \in U_r(R)$  and  $e \in Id(R)$  so that  $1 - x = a + e$  or  $1 + x = a + e$ . So  $x = -a + (1 - e)$  or  $x = a - (1 - e)$  so that  $-a.a \in U_r$  and  $1 - e \in Id(R)$ . as a result  $x$  is weakly unit regular clean.  $\square$

**Remark 3.5.** Suppose  $R$  is a ring and  $I$  is an ideal of  $R$ . Then the following example shows that if  $R/I$  is a weakly unit regular clean ring, then it is not necessary that  $R$  is a weakly unit regular clean ring.

**Example 3.6.** Let  $p$  be a prime number. Then  $\mathbb{Z}/\langle p \rangle \cong \mathbb{Z}_p$  is a weakly unit regular clean ring, but  $\mathbb{Z}$  is not a weakly unit regular clean ring.

**Definition 3.7.** Suppose that  $R$  is a ring and  $I$  is an ideal of  $R$ . In this case, we say that the eigenvalues are raised to measure  $I$ , if for each  $a \in R$  with the condition  $a - a^2 \in I$ , there is an eigenvalue member  $e \in R$  such that  $e - a \in I$  [12].

**Definition 3.8.** Suppose that  $R$  is a ring. In this case, the union of all maximal right (left) ideals of  $R$  is called the Jacobson radical of  $R$ . We denote the Jacobson radical  $R$  by the symbol  $J(R)$ .

**Lemma 3.9.** Suppose that  $R$  is a ring and  $I \subseteq J(R)$  is an ideal of  $R$ . In this case,  $R$  is weakly unit regular clean if and only if  $R/I$  is a ring weakly unit regular clean and the eigenpowers are raised to measure  $I$ .

**Proof .** Suppose that  $R$  is a weakly unit regular clean ring and  $x + I \in R/I$ , In this case,  $x \in R$  and  $x = a + e$  or  $x = a - e$  so that  $a \in U_r(R)$  and  $e \in Id(R)$ .

$$x + I = a + e + I = (a + I) + (e + I)$$

or

$$x + I = a - e + I = (a + I) - (e + I).$$

It is clear that  $e + I \in Id(R/I)$  and:

$$a + I = aua + I = (a + I)(u + I)(a + I).$$

So  $a + I \in U_r(R/I)$ . As result,  $R/I$  is a weakly unit regular clean ring and the eigenvalues are raised to the scale of  $I$ .

On the contrary, suppose that  $R/I$  is a weakly unit regular clean ring and the eigenpowers are raised to measure  $I$  and  $I x \in R$ . In this case,  $x + I = a + e + I$  or

$$x + I = a - e + I$$

so that  $a + I \in U_r(R/I)$  and  $e + I \in Id(R/I)$ . Because eigenvalues are raised to scale  $I$ ,  $e \in Id(R)$ . on the other hand:

$$x - e + I = a + I \in U_r(R/I)$$

or

$$x + e + I = a + I \in U_r(R/I).$$

So  $x - e \in U_r(R/I)$  or  $x + e \in U_r(R/I)$ . As a result,  $x = (x - e) + e$  or  $x = (x + e) - e$ , that is,  $R$  is a weakly unit regular clean ring.  $\square$

A ring  $R$  is called Abelian, if every self-sufficient member of  $R$  is central, that is, for every  $e \in Id(R)$  and every  $x \in R$ ,  $xe = ex$ .

**Proposition 3.10.** Suppose that  $R$  is an Abelian is a weakly unit regular clean ring. In this case, for each  $x \in R$ , there is an independent member  $\in Id(R)$  such that  $ex \in Id(R)$  or  $-ex \in Id(R)$ .

**Proof .** Suppose that  $x \in R$  in this case  $a \in U_r(R)$  and  $e_1 \in Id(R)$  exist such that  $x = a + e_1$  or  $x = a - e_1$ . Because  $a \in U_r(R)$ , according to [4, Theorem 1], there exist  $e \in Id(R)$  and  $u \in U(R)$  such that  $a = e + u$  and  $aR \cap eR = \{0\}$ . Because  $ae = ea \in aR \cap eR$ ,  $ae = ea = 0$ . So

$$ex = ea + ee_1$$

or

$$ex = ea - ee_1.$$

So  $ex = ee_1$  or  $-ex = ee_1$ . Because  $e$  and  $e_1$  are central eigenvalues,  $ee_1 \in Id(R)$ . as a result  $ex \in Id(R)$  or  $-ex \in Id(R)$ .  $\square$

**Theorem 3.11.** Suppose that  $R$  is a weakly unit regular clean Abelian ring and  $e \in Id(R)$ . In this case  $eRe$  is a weakly unit regular clean ring.

**Proof .** Suppose that  $x \in eRe \subseteq R$ . In this case, there exist  $a \in U_r(R)$  and  $e_1 \in Id(R)$  such that  $x = a + e_1$  or  $x = a - e_1$ . Because  $x \in eRe$ :

$$x = eae + ee_1e$$

or

$$x = eae - ee_1e.$$

Since  $R$  is an Abelian ring

$$x = ea + e_1e$$

or

$$x = ae - e_1e$$

on the other hand:

$$(ae)(eue)(ae) = (ae)(eue)(ea) = (ae)u(ea) = (ea)u(ea) = e(aua)e = eae \in eRe.$$

So  $ae \in U_r(eRe)$ . As result,  $eRe$  is a unitary weakly clean ring.  $\square$

The ring  $R$  is directly called finite, if for each  $x, y \in R$ ,  $xy = 1$  results in  $yx = 1$  [9].

**Proposition 3.12.** Suppose that  $R$  is a directly finite and weakly unit regular clean ring such that  $Id(R) = \{0,1\}$ . Then  $R$  is a weakly clean regular ring.

**Proof .** Suppose that  $x \in R$ . In this case, there exist  $a \in U_r(R)$  and  $e \in Id(R)$  so that  $x = a + e$  or  $x = a - e$ . If  $a = 0$ , then  $x = e$  or  $x = -e$ . So:

$$x = e = (2e - 1) + (1 - e)$$

or

$$x = -e = -(2e - 1) - (1 - e)$$

As result,  $R$  is a weakly clean ring. If  $a \neq 0$ , then there exists  $u \in U(R)$  such that  $a = au$ . So  $au \in Id(R) = \{0,1\}$ . If  $au = 0$ , then  $a = au = 0$ , which is a contradiction. So  $au = 1$ . Since  $R$  is a directly finite ring,  $ua = 1$ . So  $a \in U(R)$ . Consequently,  $R$  is a weakly clean ring.  $\square$

**Theorem 3.13.** Suppose that  $\{R_i\}_{i \in I}$  is a family of commutative rings. In this case  $R = \prod_{i \in I} R_i$  is weakly clean regular unit if and only if every  $R_i$  is weakly unit regular clean and at most one of  $R_i$  is not clean unit regular.

**Proof .** Suppose  $R = \prod_{i \in I} R_i$  is weakly unit regular clean. In this case, since every  $R_i$  is a homogenous image of  $R$ , according to Lemma 3.9, every  $R_i$  is weakly clean regular. Now suppose that  $i_1 \neq i_2$ ,  $R_{i_1}$  and  $R_{i_2}$  are not clean unit regular. In this case, there exists  $x_{i_1} \in R_{i_1}$  such  $x_{i_1} = a_{i_1} + e_{i_1}$  where  $a_{i_1} \in U_r(R_{i_1})$  and  $e_{i_1} \in Id(R_{i_1})$ , But  $x_{i_1} \neq a - e$  so that  $a \in U_r(R_{i_1})$  and  $e \in Id(R_{i_1})$ . There is also  $x_{i_2} \in R_{i_2}$  such  $x_{i_2} = a_{i_2} - e_{i_2}$  where a  $a_{i_2} \in U_r(R_{i_2})$  and  $e_{i_2} \in Id(R_{i_2})$ . But  $x_{i_1} \neq a + e$  so that  $a \in U_r(R_{i_2})$  and  $e \in Id(R_{i_2})$ . Now consider  $x = (x_i)$  which is defined as follows:

$$x_i = \begin{cases} x_{i_j} & i \in \{i_1, i_2\} \\ 0 & i \notin \{i_1, i_2\} \end{cases}$$

In this case  $\neq a \pm e$  where a  $a \in U_r(R)$  and  $e \in Id(R)$  which is a contradiction. As result, the verdict is upheld.

Conversely, suppose that every  $R_i$  is weakly unit regular clean. Also suppose that  $R_{i_0}$  is weakly unit regular clean and not unit and the rest of  $R_i$  are clean unit regular, we consider  $x = (x_i) \in R$ . In this case  $x_{i_0} = a_{i_0} + e_{i_0}$  or  $x_{i_0} = a_{i_0} - e_{i_0}$ , where  $a_{i_0} \in U_r(R_{i_0})$  and  $e_{i_0} \in Id(R_{i_0})$ . If  $x_{i_0} = a_{i_0} + e_{i_0}$ , then for each  $i \neq i_0$  we assume  $x_i = a_i - e_i$  where  $a_i \in U_r(R_i)$  and  $e_i \in Id(R_i)$ . If  $x_{i_0} = a_{i_0} - e_{i_0}$ , then for each  $i \neq i_0$  suppose that  $x_i = a_i + e_i$  where  $a_i \in U_r(R_i)$  and  $e_i \in Id(R_i)$ . So  $x = a + e$  or  $x = a - e$  where  $a \in U_r(R)$  and  $e \in Id(R)$ , as result,  $R$  is a weakly unit regular clean ring.  $\square$

Let  $C$  be a subring of  $D$ . In this case, the set:

$$R[D.C] = \{(d_1 \cdots d_n.c.c.c \cdots); d_i \in D.c \in C.n \geq 1\}$$

It is a circle with component addition and multiplication.

**Theorem 3.14.** The ring  $R[D.C]$  is weakly unit regular clean if and only if  $D$  is a clean unit regular ring and  $C$  is a weakly unit regular clean ring.

**Proof .** Suppose that  $S = R[D.C]$  is a weakly unit regular clean ring. Then, since  $D \oplus D$  is a summation of  $S$ ,  $D \oplus D$  is a weakly unit regular clean ring. Therefore, according to Theorem 3.13,  $D$  is a unit regular clean ring. Since  $C$  is a homogenous image of  $S$ , according to Lemma 3.9,  $C$  is a weakly unit regular clean ring.

Conversely, suppose that  $x = (d_1 \cdots d_n.c.c.c \cdots) \in S$ . In this case, since  $C$  is a unitary weakly unit regular clean ring,  $c = a + ec = a - e$  where  $a \in U_r(C)$  and  $e \in Id(C)$ . If  $c = a + e$ , then for every  $1 \leq i \leq n$  we set  $d_i = a_i + e_i$  where  $a_i \in U_r(D)$  and  $e_i \in Id(D)$ . So:

$$x = (d_1 \cdots d_n.c.c.c \cdots) = (a_1 \cdots a_n.a.a.a \cdots) + (e_1 \cdots e_n.e.e.e \cdots)$$

which in  $(a_1 \cdots a_n.a.a.a \cdots) \in U_r(S)$  and  $(e_1 \cdots e_n.e.e.e \cdots) \in Id(S)$ . If  $c = a - e$ , then for every  $1 \leq i \leq n$ , we set  $d_i = a_i - e_i$ , where  $a_i \in U_r(D)$  and  $e_i \in Id(D)$ . So

$$x = (d_1 \cdots d_n.c.c.c \cdots) = (a_1 \cdots a_n.a.a.a \cdots) - (e_1 \cdots e_n.e.e.e \cdots)$$

which in  $(a_1 \cdots a_n.a.a.a \cdots) \in U_r(S)$  and  $(e_1 \cdots e_n.e.e.e \cdots) \in Id(S)$ . As a result,  $[D.C]$  is a weakly unit regular clean ring.  $\square$

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