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CONVERGENCE THEOREMS OF MULTI-STEP ITERATIVE ALGORITHM WITH ERRORS FOR GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. The purpose of this paper is to study and give the necessary and sufficient condition of strong convergence of the multi-step iterative algorithm with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings to converge to common fixed points in Banach spaces. Our results extend and improve some recent results in the literature (see, e.g. [2, 3, 5, 6, 7, 8, 11, 14, 19]).

1. INTRODUCTION

It is well known that the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In 1973, Petryshyn and Williamson [11] gave necessary and sufficient conditions for Mann iterative sequence [9] to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [3] extended the results of Petryshyn and Williamson [11] and gave necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Liu [8] extended results of [3, 11] and gave necessary and sufficient conditions for Ishikawa iterative sequence with errors to converge to fixed point of asymptotically quasi-nonexpansive mappings.

In 2003, Zhou et al. [21] introduced a new class of generalized asymptotically nonexpansive mapping and gave a necessary and sufficient condition for the modified Ishikawa and Mann iterative sequences to converge to fixed points for the class of mappings. Atsushiba [1] studied the necessary and sufficient condition for the convergence of iterative sequences to a common fixed point of the finite family of asymptotically nonexpansive mappings in Banach spaces. Suzuki [15], Zeng and

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Yao [20] discussed a necessary and sufficient condition for common fixed points of two nonexpansive mappings and a finite family of nonexpansive mappings, and proved some convergence theorems for approximating a common fixed point, respectively.

In 2006, Lan [6] introduced a new class of generalized asymptotically quasi-nonexpansive mappings and gave necessary and sufficient condition for the 2-step modified Ishikawa iterative sequences to converge to fixed points for the class of mappings.

Recently, Tang and Peng [18] give the necessary and sufficient condition of strong convergence of common fixed points for a finite family of uniformly quasi-Lipschitzian mappings in banach spaces for the following iteration scheme:

Let $\{T_i : i = 1, 2, \ldots, k\}\colon K \to K$, where K is a nonempty subset of a Banach space E, be a finite family of uniformly quasi-Lipschitzian mappings. Let $x_1 \in K$, then the sequence $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= a_{kn}x_n + b_{kn}T_k^n y_{(k-1)n} + c_{kn}u_{kn}, \\ y_{(k-1)n} &= a_{(k-1)n}x_n + b_{(k-1)n}T_{k-1}^n y_{(k-2)n} + c_{(k-1)n}u_{(k-1)n}, \\ y_{(k-2)n} &= a_{(k-2)n}x_n + b_{(k-2)n}T_{k-2}^n y_{(k-3)n} + c_{(k-2)n}u_{(k-2)n}, \\ &\vdots \\ y_{2n} &= a_{2n}x_n + b_{2n}T_2^n y_{1n} + c_{2n}u_{2n} \\ y_{1n} &= a_{1n}x_n + b_{1n}T_1^n x_n + c_{1n}u_{1n}, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{a_{in}\}$, $\{b_{in}\}$, $\{c_{in}\}$ are sequences in $[0,1]$ with $a_{in} + b_{in} + c_{in} = 1$ for all $i = 1, 2, \ldots, k$ and $n \geq 1$, $\{u_{in},\ i = 1, 2, \ldots, k,\ n \geq 1\}$ are bounded sequences in K.

Remark 1.1. The iterative algorithm (1.1) is called multi-step iterative algorithm with errors. It contains well known iterations as special case. Such as, the modified Mann iteration (see, [13]), the modified Ishikawa iteration (see, [17]), the three-step iteration (see, [19]), the multi-step iteration (see, [5]).

The purpose of this paper is study the multi-step iterative algorithm with bounded errors (1.1) for a finite family of generalized asymptotically quasi-nonexpansive mappings to converge to common fixed points in Banach spaces. The results obtained in this paper extend and improve the corresponding results of [2, 3, 5, 6, 7, 8, 11, 14, 19] and many others.

2. Preliminaries

In the sequel, we need the following definitions and lemmas for our main results in this paper.

Definition 2.1. (see [10]) Let E be a real Banach space, K be a nonempty subset of E and $F(T)$ denotes the set of fixed points of T. A mapping $T\colon K \to K$ is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{2.1}$$

for all $x, y \in K$,

(2) quasi-nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{2.2}$$

for all $x \in K$ and $p \in F(T)$,

(3) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $r_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + r_n) \|x - y\|, \tag{2.3}$$

for all $x, y \in K$,

(4) asymptotically quasi-nonexpansive if (3) holds for all $x \in K$ and $y \in F(T)$;

(5) generalized quasi-nonexpansive with respect to $\{s_n\}$, if there exists a sequence $\{s_n\} \subset [0, 1)$ with $s_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq \|x - p\| + s_n \|x - T^n x\|, \tag{2.4}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$,

(6) generalized asymptotically quasi-nonexpansive with respect to $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq (1 + r_n) \|x - p\| + s_n \|x - T^n x\|, \tag{2.5}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

Remark 2.1. From the above definition, it is clear that:

(i) a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(ii) a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(iii) an asymptotically nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping;

(iv) a generalized quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping.

However, the converse of the above statements are not true.

Lemma 2.1. (see [16]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. (see [6]) Let K be nonempty closed subset of a Banach space E and $T\colon K \to K$ be a generalized asymptotically quasi-nonexpansive mapping with the fixed point set $F(T) \neq \emptyset$. Then $F(T)$ is closed subset in K.

3. Main Results

In this section, we prove strong convergence theorems of multi-step iterative algorithm with bounded errors for a finite family of generalized asymptotically quasi-nonexpansive mappings in a real Banach space.

Theorem 3.1. Let E be a real arbitrary Banach space, K be a nonempty closed convex subset of E. Let $\{T_i : i = 1, 2, \dots, k\}\colon K \to K$ be a finite family of generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $\sum_{n=1}^{\infty} \frac{r_{in}+2s_{in}}{1-s_{in}} < \infty$ for all $i \in \{1, 2, \dots, k\}$. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} c_{in} < \infty$, $i = 1, 2, \dots, k$. If $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \dots, k\}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.

Proof. The necessity is obvious and it is omitted. Now we prove the sufficiency. Let $p \in \mathcal{F}$, then it follows from (2.5) and for all $i \in \{2, 3, \dots, k\}$, we have

$$\begin{aligned}
\left\|y_{(i-1)n} - T_i^n y_{(i-1)n}\right\| &\leq \left\|y_{(i-1)n} - p\right\| + \left\|T_i^n y_{(i-1)n} - p\right\| \\
&\leq \left\|y_{(i-1)n} - p\right\| + (1 + r_{i_n}) \left\|y_{(i-1)n} - p\right\| \\
&\quad + s_{i_n} \left\|y_{(i-1)n} - T_i^n y_{(i-1)n}\right\| \\
&\leq (2 + r_{i_n}) \left\|y_{(i-1)n} - p\right\| + s_{i_n} \left\|y_{(i-1)n} - T_i^n y_{(i-1)n}\right\|
\end{aligned}$$

which implies that

$$\left\|y_{(i-1)n} - T_i^n y_{(i-1)n}\right\| \leq \left(\frac{2 + r_{in}}{1 - s_{in}}\right) \left\|y_{(i-1)n} - p\right\|. \tag{3.1}$$

and

$$\begin{aligned}
\|x_n - T_1^n x_n\| &\leq \|x_n - p\| + \|T_1^n x_n - p\| \\
&\leq \|x_n - p\| + (1 + r_{1n}) \|x_n - p\| + s_{1n} \|x_n - T_1^n x_n\| \\
&\leq (2 + r_{1n}) \|x_n - p\| + s_{1n} \|x_n - T_1^n x_n\|
\end{aligned}$$

which implies that

$$\|x_n - T_1^n x_n\| \leq \left(\frac{2 + r_{1n}}{1 - s_{1n}}\right) \|x_n - p\| . \tag{3.2}$$

Since $\{u_{in},\ i = 1, 2, \dots, k,\ n \geq 1\}$ are bounded sequences in K, therefore there exists a $M > 0$, such that

$$M = \max\left\{\sup_{n\geq 1} \|u_{in} - p\| ,\ i = 1, 2, \dots, k\right\}.$$

Using (1.1), (3.1) and (3.2), we note that

$$\begin{aligned}
\|y_{1n} - p\| &= \|a_{1n}x_n + b_{1n}T_1^n x_n + c_{1n}u_{1n} - p\| \\
&= \|a_{1n}(x_n - p) + b_{1n}(T_1^n x_n - p) + c_{1n}(u_{1n} - p)\| \\
&\leq a_{1n} \|x_n - p\| + b_{1n} \|T_1^n x_n - p\| + c_{1n} \|u_{1n} - p\| \\
&\leq a_{1n} \|x_n - p\| + b_{1n} \Big[(1 + r_{1n}) \|x_n - p\| \\
&\quad + s_{1n} \|x_n - T_1^n x_n\| \Big] + c_{1n} \|u_{1n} - p\| \\
&\leq a_{1n} \|x_n - p\| + b_{1n}.\left(\frac{1 + r_{1n} + s_{1n}}{1 - s_{1n}}\right) \|x_n - p\| \\
&\quad + c_{1n} \|u_{1n} - p\| \\
&\leq (a_{1n} + b_{1n})\left(\frac{1 + r_{1n} + s_{1n}}{1 - s_{1n}}\right) \|x_n - p\| \\
&\quad + c_{1n} \|u_{1n} - p\| \\
&= (1 - c_{1n})\left(\frac{1 + r_{1n} + s_{1n}}{1 - s_{1n}}\right) \|x_n - p\| \\
&\quad + c_{1n} \|u_{1n} - p\| \\
&\leq \left(\frac{1 + r_{1n} + s_{1n}}{1 - s_{1n}}\right) \|x_n - p\| + c_{1n}M \\
&= \left[1 + \frac{r_{1n} + 2s_{1n}}{1 - s_{1n}}\right] \|x_n - p\| + c_{1n}M \\
&= (1 + t_{1n}) \|x_n - p\| + c_{1n}M \qquad (3.3)
\end{aligned}$$

where $t_{1n} = \frac{r_{1n}+2s_{1n}}{1-s_{1n}}$. Since $\sum_{n=1}^{\infty} \frac{r_{in}+2s_{in}}{1-s_{in}} < \infty$ for all $i \in \{1, 2, \dots, k\}$, it follows that $\sum_{n=1}^{\infty} t_{1n} < \infty$. Again from (1.1) and (3.3), we note that

$$\begin{aligned}
\|y_{2n}-p\| &= \|a_{2n}x_n + b_{2n}T_2^n y_{1n} + c_{2n}u_{2n} - p\| \\
&= \|a_{2n}(x_n - p) + b_{2n}(T_2^n y_{1n} - p) + c_{2n}(u_{2n} - p)\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}\|T_2^n y_{1n} - p\| + c_{2n}\|u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}\Big[(1+r_{2n})\|y_{1n} - p\| \\
&\quad + s_{2n}\|y_{1n} - T_2^n y_{1n}\|\Big] + c_{2n}\|u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}.\Big(\frac{1+r_{2n}+s_{2n}}{1-s_{2n}}\Big)\|y_{1n} - p\| \\
&\quad + c_{2n}\|u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}.\Big(1 + \frac{r_{2n}+2s_{2n}}{1-s_{2n}}\Big)\Big[(1+t_{1n})\|x_n - p\| \\
&\quad + c_{1n}M\Big] + c_{2n}\|u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}.(1+t_{2n})\Big[(1+t_{1n})\|x_n - p\| \\
&\quad + c_{1n}M\Big] + c_{2n}\|u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}.(1+t_{2n})\Big[(1+t_{1n})\|x_n - p\| + c_{1n}M\Big] \\
&\quad + c_{2n}\|u_{2n} - p\| \\
&\leq (a_{2n}+b_{2n})(1+t_{1n})(1+t_{2n})\|x_n - p\| + b_{2n}c_{1n}(1+t_{2n})M \\
&\quad + c_{2n}M \\
&= (1-c_{2n})(1+t_{1n})(1+t_{2n})\|x_n - p\| + b_{2n}c_{1n}(1+t_{2n})M \\
&\quad + c_{2n}M \\
&\leq (1+t_{1n})(1+t_{2n})\|x_n - p\| + c_{1n}(1+t_{2n})M + c_{2n}M \\
&\leq (1+t_{1n}+t_{2n})\|x_n - p\| + (c_{1n}+c_{2n})M \\
&= \Big[1 + \sum_{k=1}^{2} t_{kn}\Big]\|x_n - p\| + M\sum_{k=1}^{2} c_{kn} \qquad (3.4)
\end{aligned}$$

where $t_{2n} = \frac{r_{2n}+2s_{2n}}{1-s_{2n}}$. Since $\sum_{n=1}^{\infty} \frac{r_{in}+2s_{in}}{1-s_{in}} < \infty$ for all $i \in \{1, 2, \ldots, k\}$, it follows that $\sum_{n=1}^{\infty} t_{2n} < \infty$.

Repeating the above process, we get

$$\|y_{jn} - p\| \leq \Big[1 + \sum_{k=1}^{j} t_{kn}\Big]\|x_n - p\| + M\sum_{k=1}^{j} c_{kn}, \qquad (3.5)$$

for $j = 1, 2, \ldots, k-1$. In fact, (3.5) holds for $j = 1$ via inequality (3.3). By using induction, suppose that (3.5) holds for j, then for $j+1$, we see that

$$
\begin{aligned}
\|y_{(j+1)n}-p\| =&\|a_{(j+1)n}(x_n-p)+b_{(j+1)n}(T^n_{(j+1)}y_{jn}-p)\\
&+c_{(j+1)n}(u_{(j+1)n}-p)\|\\
\le& a_{(j+1)n}\|x_n-p\|+b_{(j+1)n}\|T^n_{(j+1)}y_{jn}-p\|\\
&+c_{(j+1)n}\|u_{(j+1)n}-p\|\\
\le& a_{(j+1)n}\|x_n-p\|+b_{(j+1)n}\Big[(1+r_{(j+1)n})\|y_{jn}-p\|\\
&+s_{(j+1)n}\|y_{jn}-T^n_{(j+1)}y_{jn}\|\Big]+c_{(j+1)n}\|u_{(j+1)n}-p\|\\
\le& a_{(j+1)n}\|x_n-p\|+b_{(j+1)n}\Big(\frac{1+r_{(j+1)n}+s_{(j+1)n}}{1-s_{(j+1)n}}\Big)\|y_{jn}-p\|\\
&+c_{(j+1)n}\|u_{(j+1)n}-p\|\\
\le& a_{(j+1)n}\|x_n-p\|+b_{(j+1)n}\Big(1+\frac{r_{(j+1)n}+2s_{(j+1)n}}{1-s_{(j+1)n}}\Big)\\
&\left\{\Big(1+\sum_{k=1}^{j}t_{kn}\Big)\|x_n-p\|+M\sum_{k=1}^{j}c_{kn}\right\}+c_{(j+1)n}\|u_{(j+1)n}-p\|\\
\le& a_{(j+1)n}\|x_n-p\|+b_{(j+1)n}(1+t_{(j+1)n})\\
&\left\{\Big(1+\sum_{k=1}^{j}t_{kn}\Big)\|x_n-p\|+M\sum_{k=1}^{j}c_{kn}\right\}+c_{(j+1)n}\|u_{(j+1)n}-p\|\\
\le& a_{(j+1)n}\|x_n-p\|+b_{(j+1)n}\Big(1+t_{(j+1)n}\Big)\\
&\left\{\Big(1+\sum_{k=1}^{j}t_{kn}\Big)\|x_n-p\|+M\sum_{k=1}^{j}c_{kn}\right\}+c_{(j+1)n}\|u_{(j+1)n}-p\|\\
\le&\Big(a_{(j+1)n}+b_{(j+1)n}\Big)\Big(1+\sum_{k=1}^{j}t_{kn}\Big)\Big(1+t_{(j+1)n}\Big)\|x_n-p\|\\
&+Mb_{(j+1)n}\Big(1+t_{(j+1)n}\Big)\sum_{k=1}^{j}c_{kn}+c_{(j+1)n}M\\
\le&\Big(1+\sum_{k=1}^{j}t_{kn}\Big)\Big(1+t_{(j+1)n}\Big)\|x_n-p\|\\
&+M\Big(1+t_{(j+1)n}\Big)\sum_{k=1}^{j}c_{kn}+c_{(j+1)n}M\\
\le&\Big(1+\sum_{k=1}^{j}t_{kn}+t_{(j+1)n}\Big)\|x_n-p\|\\
&+M(\sum_{k=1}^{j}c_{kn}+c_{(j+1)n})\\
=&\Big(1+\sum_{k=1}^{j+1}t_{kn}\Big)\|x_n-p\|+M\Big(\sum_{k=1}^{j+1}c_{kn}\Big). \qquad (3.6)
\end{aligned}
$$

Hence (3.5) holds. It follows from (1.1) and (3.5) that

$$\begin{aligned}
\|x_{n+1}-p\| &= \big\|a_{kn}(x_n-p)+b_{kn}(T_k^n y_{(k-1)n}-p)+c_{kn}(u_{kn}-p)\big\| \\
&\le a_{kn}\|x_n-p\|+b_{kn}\big\|T_k^n y_{(k-1)n}-p\big\|+c_{kn}\|u_{kn}-p\| \\
&\le a_{kn}\|x_n-p\|+b_{kn}.\Big(\frac{1+r_{kn}+s_{kn}}{1-s_{kn}}\Big)\big\|y_{(k-1)n}-p\big\| \\
&\quad +c_{kn}\|u_{kn}-p\| \\
&\le a_{kn}\|x_n-p\|+b_{kn}.\Big(1+\frac{r_{kn}+2s_{kn}}{1-s_{kn}}\Big)\big\|y_{(k-1)n}-p\big\| \\
&\quad +c_{kn}\|u_{kn}-p\| \\
&\le a_{kn}\|x_n-p\|+b_{kn}.\Big(1+t_{kn}\Big)\Big[\Big(1+\sum_{l=1}^{k-1}t_{ln}\Big)\|x_n-p\| \\
&\quad +M\sum_{l=1}^{k-1}c_{ln}\Big]+c_{kn}M \\
&\le \Big(a_{kn}+b_{kn}\Big)\Big(1+t_{kn}\Big)\Big(1+\sum_{l=1}^{k-1}t_{ln}\Big)\|x_n-p\| \\
&\quad +M\Big(1+t_{kn}\Big)\Big(\sum_{l=1}^{k-1}c_{ln}\Big)+c_{kn}M \\
&= \Big(1-c_{kn}\Big)\Big(1+t_{kn}\Big)\Big(1+\sum_{l=1}^{k-1}t_{ln}\Big)\|x_n-p\| \\
&\quad +M\Big(1+t_{kn}\Big)\Big(\sum_{l=1}^{k-1}c_{ln}\Big)+c_{kn}M \\
&\le \Big(1+\sum_{l=1}^{k-1}t_{ln}+t_{kn}\Big)\|x_n-p\| \\
&\quad +M\Big(\sum_{l=1}^{k-1}c_{ln}+c_{kn}\Big) \\
&= \Big[1+\sum_{l=1}^{k}t_{ln}\Big]\|x_n-p\|+M\Big(\sum_{l=1}^{k}c_{ln}\Big) \\
&= (1+\theta_n)\|x_n-p\|+M\lambda_n \qquad (3.7)
\end{aligned}$$

where $\theta_n=\sum_{l=1}^{k}t_{ln}$ and $\lambda_n=\sum_{l=1}^{k}c_{ln}$. Since $\sum_{n=1}^{\infty}t_{ln}<\infty$ and $\sum_{n=1}^{\infty}c_{ln}<\infty$ for all $l=1,2,\dots,k$, it follows that $\sum_{n=1}^{\infty}\theta_n<\infty$ and $\sum_{n=1}^{\infty}\lambda_n<\infty$. Therefore from Lemma 2.1, we know that $\lim_{n\to\infty}d(x_n,\mathcal{F})=0$.

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Notice that when $x>0$, $1+x\le e^x$, from (3.7) we have

$$\begin{aligned}
\|x_{n+m}-p\| &\leq (1+\theta_{n+m-1})\|x_{n+m-1}-p\|+M\lambda_{n+m-1}\\
&\leq e^{\theta_{n+m-1}}\|x_{n+m-1}-p\|+M\lambda_{n+m-1}\\
&\leq e^{\theta_{n+m-1}}\Big[e^{\theta_{n+m-2}}\|x_{n+m-2}-p\|+M\lambda_{n+m-2}\Big]\\
&\quad +M\lambda_{n+m-1}\\
&\leq e^{\Big[\theta_{n+m-1}+\theta_{n+m-2}\Big]}\|x_{n+m-2}-p\|\\
&\quad +Me^{\theta_{n+m-1}}\Big(\lambda_{n+m-1}+\lambda_{n+m-2}\Big)\\
&\leq \ldots\\
&\leq \ldots\\
&\leq \Big(e^{\sum_{k=n}^{n+m-1}\theta_k}\Big)\|x_n-p\|\\
&\quad +M\Big(e^{\sum_{k=n+1}^{n+m-1}\theta_k}\Big)\sum_{k=n}^{n+m-1}\lambda_k\\
&\leq \Big(e^{\sum_{k=n}^{n+m-1}\theta_k}\Big)\|x_n-p\|\\
&\quad +M\Big(e^{\sum_{k=n}^{n+m-1}\theta_k}\Big)\sum_{k=n}^{n+m-1}\lambda_k\\
&\leq M'\|x_n-p\|+MM'\sum_{k=n}^{n+m-1}\lambda_k
\end{aligned}\tag{3.8}$$

where $M'=e^{\sum_{k=n}^{n+m-1}\theta_k}$ and for all $p\in\mathcal{F}$, $m,n\in\mathbb{N}$. Since $\lim_{n\to\infty}d(x_n,\mathcal{F})=0$, for given $\varepsilon>0$, there exists a natural number n_1 such that for $n\geq n_1$,

$$d(x_n,\mathcal{F})<\frac{\varepsilon}{4(M'+1)}\ \text{ and }\ \sum_{k=n_1+1}^{n+m}\lambda_k<\frac{\varepsilon}{2MM'}.\tag{3.9}$$

Hence, there exists a point $q\in\mathcal{F}$ such that

$$\|x_{n_1}-q\|<\frac{\varepsilon}{2(M'+1)}.\tag{3.10}$$

By (3.8), (3.9) and (3.10), for all $n\geq n_1$ and $m\geq 1$, we have

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \|x_{n+m}-q\|+\|x_n-q\| \\ &\leq M'\,\|x_{n_1}-q\|+MM'\sum_{k=n_1}^{n+m-1}\lambda_k+\|x_{n_1}-q\| \\ &\leq (M'+1)\,\|x_{n_1}-q\|+MM'\sum_{k=n_1}^{n+m-1}\lambda_k \\ &< (M'+1).\frac{\varepsilon}{2(M'+1)}+MM'.\frac{\varepsilon}{2MM'}=\varepsilon. \qquad (3.11)\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Thus the completeness of E implies that $\{x_n\}$ must be convergent. Let $\lim_{n\to\infty} x_n = p$, that is, $\{x_n\}$ converges to p. Then $p \in K$, because K is a closed subset of E. By Lemma 2.2 we know that the set $\mathcal{F}$ is closed. From the continuity of $d(x_n, \mathcal{F})$ with

$$d(x_n,\mathcal{F}) \to 0 \quad and \quad x_n \to p \quad as \quad n\to\infty,$$

we get

$$d(p,\mathcal{F})=0$$

and so $p \in \mathcal{F} = \cap_{i=1}^k F(T_i)$, that is, p is a common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$. This completes the proof.

Theorem 3.2. Let E be a real arbitrary Banach space, K be a nonempty closed convex subset of E. Let $\{T_i : i = 1, 2, \ldots, k\}\colon K \to K$ be a finite family of generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $\sum_{n=1}^{\infty} \frac{r_{in}+2s_{in}}{1-s_{in}} < \infty$ for all $i \in \{1, 2, \ldots, k\}$. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} c_{in} < \infty$, $i = 1, 2, \ldots, k$. If $\mathcal{F} = \cap_{i=1}^k F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i : i = 1, 2, \ldots, k\}$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.

Proof. The proof of Theorem 3.2 follows from Lemma 2.1 and Theorem 3.1.

Theorem 3.3. Let E be a real arbitrary Banach space, K be a nonempty closed convex subset of E. Let $\{T_i : i = 1, 2, \ldots, k\}\colon K \to K$ be a finite family of generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $\sum_{n=1}^{\infty} \frac{r_{in}+2s_{in}}{1-s_{in}} < \infty$ for all $i \in \{1, 2, \ldots, k\}$. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} c_{in} < \infty$, $i = 1, 2, \ldots, k$. If $\mathcal{F} = \cap_{i=1}^k F(T_i) \neq \emptyset$. Suppose that the mappings $\{T_i : i = 1, 2, \ldots, k\}$ satisfy the following conditions:

(C_1) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in \{1, 2, \ldots, k\}$;

(C_2) there exists a constant $A > 0$ such that $\{\|x_n - T_i x_n\|\} \geq Ad(x_n, \mathcal{F})$ for all $i \in \{1, 2, \ldots, k\}$ and for all $n \geq 1$.

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, k\}$.

Proof. From condition (C_1) and (C_2), we have $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, it follows as in the proof of Theorem 3.1, that $\{x_n\}$ must converges strongly to a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, k\}$.

Remark 3.1. Theorem 3.1 extends the corresponding result of Khan et al. [5] and Tang and Peng [18] to the case of more general class of asymptotically quasi-nonexpansive or uniformly quasi-Lipschitzian mappings considered in this paper.

Remark 3.2. Theorem 3.1 also extend and improve the corresponding results of [7, 8, 12, 14, 19]. Especially Theorem 3.1 extend and improve Theorem 1 and 2 in [8], Theorem 1 in [7] and Theorem 3.2 in [14] in the following ways:

(1) The asymptotically quasi-nonexpansive mapping in [7], [8] and [14] is replaced by finite family of generalized asymptotically quasi-nonexpansive mappings.

(2) The usual Ishikawa iteration scheme in [7], the usual modified Ishikawa iteration scheme with errors in [8] and the usual modified Ishikawa iteration scheme with errors for two mappings are extended to the multi-step iteration scheme with errors for a finite family of mappings.

Remark 3.3. Theorem 3.2 extend and improve Theorem 3 in [8] and Theorem 3.3 extend and improve Theorem 3 in [7] in the following aspects:

(1) The asymptotically quasi-nonexpansive mapping in [7] and [8] is replaced by finite family of generalized asymptotically quasi-nonexpansive mappings.

(2) The usual Ishikawa iteration scheme in [7] and the usual modified Ishikawa iteration scheme with errors in [8] are extended to the multi-step iteration scheme with errors for a finite family of mappings.

Remark 3.4. Theorem 3.1 also extends the corresponding result of [12] to the case of more general class of uniformly quasi-Lipschitzian mapping and multi-step iteration scheme with errors for a finite family of mappings and also it extends the corresponding result of [19] to the case of more general class of asymptotically non-expansive mappings and multi-step iteration scheme with errors for a finite family of mappings considered in this paper.

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