BILINEAR FOURIER INTEGRAL OPERATOR AND ITS BOUNDEDNESS

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ABSTRACT. We consider the bilinear Fourier integral operator

$$S_\sigma(f,g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi_1(x,\xi)} e^{i\phi_2(x,\eta)} \sigma(x,\xi,\eta) \hat{f}(\xi)\hat{g}(\eta) d\xi\, d\eta,$$

on modulation spaces. Our aim is to indicate this operator is well defined on $S(\mathbb{R}^d)$ and shall show the relationship between the bilinear operator and BFIO on modulation spaces.

1. INTRODUCTION

The notion of bilinear pseudodifferential operator

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x,\xi,\eta) \hat{f}(\xi)\hat{g}(\eta) e^{ix(\eta+\xi)} d\xi\, d\eta,$$

has considered by many authors [3], [4], [5], [6] and [7]. The purpose of this paper is doing in more general case:

$$S_\sigma(f,g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi_1(x,\xi)} e^{i\phi_2(x,\eta)} \sigma(x,\xi,\eta) \hat{f}(\xi)\hat{g}(\eta) d\xi\, d\eta,$$

as a mapping from . We prove S_σ is well defined on Schwartz spaces.and it is bounded on modulation spaces. Moreover, we obtain boundedness results for some operators with symbols which are not necessarily smooth.

2. PRELIMINARIES

2.1 we will be working on the $d-dimensional\mathbb{R}$ space $\mathbb{R}^d$.We let $S = S(\mathbb{R}^d)$ be the subspace of $C^\infty(\mathbb{R}^d)$ Schwartz rapidly decreasing functions, with its dual topology.its dual is $S' = S'(\mathbb{R}^d)$,the set of all tempered distributions on $\mathbb{R}^d$.Translation and modulation of a function f with domain $\mathbb{R}^d$ are ,respectively,
$T_x f(t) = f(t-x)$ and $M_y f(t) = e^{2\pi i y.t} f(t)$.

Fourier transform: The fourier transform of $f \in L^1(\mathbb{R}^d)$ is

$$\hat{f}(y) = \int_{\mathbb{R}^d} e^{-2\pi i y.t} f(t) dt, \quad y \in \mathbb{R}^d$$

short time fourier Transform: The short time fourier Transform (STFT)of a function f with respect to a window g is

$$V_g f(x, y) = \int_{\mathbb{R}^d} e^{-2\pi i y.t} \overline{g(t - x)} f(t) dt,$$

Weight functions: Given $s \geq 0$ a positive, continuous, and symmetric function ν is called an s-moderate weight if there exists a constant C such that

$$\nu(x + y) \leq C(1 + |x|^2)^{\frac{s}{2}} \nu(y), \quad \forall x, y \in \mathbb{R}^d.$$

Definition 2.1. (modulation space):

Given $1 \leq p, q \leq \infty$, a window $g \in S$ and a moderate weight ν defined on $\mathbb{R}^{2d}$, the modulation space $\mathcal{M}_\nu^{p,q}$ is the space of all distributions $f \in S'$ for which the following norm is finite:

$$\|f\|_{\mathcal{M}_\nu^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, y)|^p \nu(x, y) dx \right)^{\frac{q}{p}} dy \right)^{\frac{1}{q}} = \|V_g f\|_{L_\nu^{p,q}}.$$

(Refer [8] for more details.)

3. MAIN RESULTS

Definition 3.1. Here, we study the natural bilinear

$$S_\sigma(f, g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi_1(x,\xi)} e^{i\phi_2(x,\eta)} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi \, d\eta, ,$$

where it's symbol satisfies now the estimate

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \eta, \xi)| \leq C_{\alpha,\gamma,\beta,k} < \xi >^{m_1 - |\beta|\delta + |\alpha|\nu} < \eta >^{m_2 - |\rho|\gamma + |\alpha|\nu},$$

where $x, \xi, \eta \in \mathbb{R}^n$, α, β, γ are multi-indices and $C_{\alpha,\beta,\gamma,K}$ is a positive constant depends on α, β, γ. We denote by $S_{\delta,\gamma,\nu}^{m_1,m_2}(X \times \mathbb{R}^n)$ or simply $S_{\delta,\gamma,\nu}^{m_1,m_2}$ for the class of all symbols satisfying this estimate.

Proposition 3.2. *Let*

$$S_\sigma(f, g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi_1(x,\xi)} e^{i\phi_2(x,\eta)} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi \, d\eta, \qquad (3.1)$$

be a bilinear pseudodifferential operator on $S(\mathbb{R}^d)$ and assume that there is a constant $c > 0$ satisfying the condition:$|\partial_x^{\theta_1} e^{i\phi_2(x,\eta)}| \leq C$, Then S_σ is well defined.
Proof. *One can verified that*

$$S_\sigma(f, g)(x) = \int_{\mathbb{R}^d} e^{i\phi_1(x,\xi)} \sigma_1(x, \xi) \hat{f}(\xi) d\xi$$

and refer to [2, lemma 1.1] for more detail and one can substitute $(L_1)^{k_1}e^{i\phi_1}$ instead of $e^{i\phi_1}$ and integrate by parts k_1 times. Then,

$$S_\sigma(f,g)(x) = \int_{\mathbb{R}^d} e^{i\phi_1(x,\xi)} L_1^{k_1} \sigma_1(x,\xi) \hat{f}(\xi) d\xi$$

refer to [2, Exercise 1.3]. Therefore, $L_1^{k_1}[\sigma_1(x,\xi)\hat{f}(\xi)] \in S^{m_1-k_1\rho}$. if $\rho > 0$, $\sigma_1(x,\xi) \in S_{\delta,\nu}^{m_1}$ then this integral is absolutely convergent.
Now prove that $\sigma_1(x,\xi) \in S_{\delta,\nu}^{m_1}$.

$$\sigma_1(x,\xi) = \int_{\mathbb{R}^d} e^{i\phi_2(x,\eta)} \sigma(x,\xi,\eta) \hat{g}(\eta) d\eta,$$

using again [2, lemma 1.1] and

$$\sigma_1(x,\xi) = \int_{\mathbb{R}^d} e^{i\phi_2(x,\eta)} [L_2^{k_2} \sigma(x,\xi,\eta) \hat{g}(\eta)] d\eta$$

for $\delta > 0$, this integral is absolutely convergent and $[L_2^{K_2} \sigma(x,\xi,\eta)\hat{g}(\eta)] \in S^{m_1,m_2-k_2\delta}$. Now according to the definition 3.1,

$$\partial_x^\alpha \partial_\eta^\gamma \partial_\xi^\beta \sigma(x,\eta,\xi) \in S^{m_1-|\beta|\delta+|\alpha|\nu, m_2-\rho k_2+|\alpha|\nu}$$

$$|\partial_x^\alpha \partial_\xi^\beta \sigma_1(x,\xi)| = |\partial_x^\alpha (\int e^{i\phi_2(x,\eta)} L_2^{K_2} [\partial_\xi^\alpha a(x,\eta,\xi)\hat{g}(\eta)] d\eta)|$$

$$= |\partial_x^\alpha (\int e^{i\phi_2(x,\eta)} b(x,\eta,\xi) d\eta|$$

$$= |\int \sum_{|\theta_1|+|\theta_2| \leq |\alpha|} \frac{\alpha!}{\theta_1!\theta_2!} \partial_x^{\theta_1} e^{i\phi_2(x,\eta)} \partial_x^{\theta_2} b(x,\xi,\eta) d\eta|,$$

so

$$|\partial_x^\alpha \partial_\xi^\beta \sigma_1(x,\xi)| \leq C_{\alpha,\gamma,\beta,k} \int_{\mathbb{R}^d} <\xi>^{m_1-|\beta|\delta+|\alpha|\nu} <\eta>^{m_2-\rho k_2+|\alpha|\nu} d\eta$$

and

$$\sigma_1(x,\xi) \in S_{\delta,\nu}^{m_1}.$$

Then the operator 3.1 is well defined.

Bilinear operators: We use bilinear operator associated with kernel $K \in S'(\mathbb{R}^{3d})$, which is a mapping B_K from $S(\mathbb{R}^d) \times S(\mathbb{R}^d)$ to $S'(\mathbb{R}^d)$ by

$$B_K(f,g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y,z) f(y) g(z) d(y) d(z),$$

for $f, g \in S(\mathbb{R}^d)$.
The next proposition establishes the relationship between a bilinear integral operator and a bilinear Fourier integral operator.

Proposition 3.3. *Let S_σ be a bilinear Fourier integral operator associated to a symbol. Then S_σ coincides with a bilinear integral operator B_K with kernel*

$$K(x,y,z) = \mathcal{F}_3\mathcal{F}_2\sigma(x,y,z)e^{i\phi_1(x,\xi)}e^{i\phi_2(x,\eta)}.$$

Proof. *For $f,g \in S(\mathbb{R}^d)$,*

$$
\begin{aligned}
S_\sigma(f,g)(x) &= \int_{R^d}\int_{R^d}\sigma(x,\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{i\phi_1(x,\xi)}e^{i\phi_2(x,\eta)}d\xi\,d\eta\\
&= \int\int\int\int\sigma(x,\xi,\eta)f(y)g(z)e^{-2\pi i\xi.y}e^{-2\pi i\eta.z}e^{i\phi_1(x,\xi)}e^{i\phi_2(x,\eta)}d\eta d\xi dydz\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(x,y,z)f(y)g(z)d(y)d(z)\\
&= B_K(f,g)(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
B_K(f,g)(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(x,y,z)f(y)g(z)d(y)d(z)\\
&= \int\int k(x,y,z)\mathcal{F}^{-1}\hat{f}(y)\mathcal{F}^{-1}\hat{g}(z)dydz\\
&= \int\int k(x,y,z)e^{2\pi i\xi.y}\hat{f}(\xi)e^{2\pi i\eta.z}\hat{g}(\eta)d(\eta)d(\xi).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sigma(x,\xi,\eta)e^{i\phi(x,\eta,\xi)} &= \int\int k(x,y,z)e^{2\pi i\xi.y}e^{2\pi i\eta.z}d(y)d(z)\\
&= \mathcal{F}_2^{-1}\mathcal{F}_3^{-1}k(x,\xi,\eta).
\end{aligned}
$$

In the sequel, considering

$$K(x,y,z) = \mathcal{F}_3\mathcal{F}_2\sigma(x,y,z)e^{i\phi_1(x,\xi)}e^{i\phi_2(x,\eta)},$$

proof is complete.

Remark 3.4. Considering

$$A(X,Y) = ((x_1,-y_2,-y_3),(y_1,x_2,x_3))$$

in the next Proposition we show that the symbol of the bilinear Fourier integral operator is in $\mathcal{M}^1_{\Omega_s^B}(\mathbb{R}^{3d})$ if and only if the corresponding integral kernel be as in Proposition 3.3, where $B = A^{-1}$.

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Proposition 3.5. $\sigma \in \mathcal{M}^1_{\Omega_s^B}(\mathbb{R}^{3d})$ *if and only if* $K(x,y,z) = \mathcal{F}_3\mathcal{F}_2\sigma(x,y,z)e^{i\phi_1(x,\xi)}e^{i\phi_2(x,\eta)}.$

Proof.

Let $G \in S(\mathbb{R}^{3d})$. For $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $t = (t_1, t_2, t_3) \in \mathbb{R}^{3d}$

$$V_G K(u,v) = \int_{\mathbb{R}^{3d}} e^{-2\pi i t.v} \overline{G(t-u)} K(t) dt$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}_3 \mathcal{F}_2 \sigma(t_1, t_2, t_3) e^{i\phi_1(t_1, \xi)} e^{i\phi_2(t1, \eta)} e^{-2\pi i(t_1 v_1 + t_2 v_2 + t_3 v_3)} \times$$

$$\overline{G(t_1 - u_1, t_2 - u_2, t_3 - u_3)} dt_1 dt_2 dt_3$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}_1^{-1} \hat{\sigma}(t_1, t_2, t_3) e^{i\phi_1(t_1, \xi)} e^{i\phi_2(t1, \eta)} e^{-2\pi i(t_1 v_1 + t_2 v_2 + t_3 v_3)} \times$$

$$\overline{G(t_1 - u_1, t_2 - u_2, t_3 - u_3)} dt_1 dt_2 dt_3$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{\sigma}(x, t_2, t_3) e^{i\phi_1(t_1, \xi)} e^{i\phi_2(t1, \eta)} e^{-2\pi i(t_2 v_2 + t_3 v_3)} e^{2\pi i t_1(x - v_1)} \times$$

$$\overline{G(t_1 - u_1, t_2 - u_2, t_3 - u_3)} dt_1 dt_2 dt_3$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi_1(u_1 - t_1, \xi)} e^{i\phi_2(u_1 - t1, \eta)} \hat{\sigma}(x, t_2, t_3) e^{-2\pi i(t_2 v_2 + t_3 v_3)}$$

$$\overline{G(-t_1, t_2 - u_2, t_3 - u_3)} e^{2\pi i(u_1 - t_1)(x - v_1)} dt_1 dt_2 dt_3 dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi_1(u_1 - t_1, \xi)} e^{i\phi_2(u_1 - t1, \eta)} \hat{\sigma}(x, t_2, t_3) e^{-2\pi i(t_2 v_2 + t_3 v_3)}$$

$$\overline{\mathcal{F}_1 G(x - v_1, t_2 - u_2, t_3 - u_3)} e^{-2\pi i u_1(v_1 - x)} dt_2 dt_3 dx$$

$$= e^{i\phi_1(u_1 - t_1, \xi) + i\phi_2(u_1 - t1, \eta) - 2\pi i u_1 v_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{\sigma}(x, t_2, t_3) e^{-2\pi i(x.t_2, t_3)(-u_1, v_2, v_3)}$$

$$\overline{\mathcal{F}_1 G(x, t_2, t_3) - (v_1, u_2, u_3)} dt_2 dt_3 dx$$

$$= e^{i\phi_1(u_1 - t_1, \xi) + i\phi_2(u_1 - t1, \eta) - 2\pi i u_1 v_1} \int_{\mathbb{R}^d} \hat{\sigma}(z) e^{-2\pi i(z)(-u_1, v_2, v_3)}$$

$$\overline{\mathcal{F}_1 G(z - (v_1, u_2, u_3))} dz$$

$$= e^{i\phi_1(u_1 - t_1, \xi) + i\phi_2(u_1 - t1, \eta) - 2\pi i u_1 v_1} V_{\mathcal{F}_1 G} \hat{\sigma}((v_1, u_2, u_3), (-u_1, v_2, v_3))$$

$$= e^{i\phi_1(u_1 - t_1, \xi) + i\phi_2(u_1 - t_1, \eta) - 2\pi i u_1 v_1} V_H \hat{\sigma}((v_1, u_2, u_3), (-u_1, v_2, v_3)),$$

where $H = \mathcal{F}_1 G$. Since $|V_g f(x, y)| = |V_{\hat{g}} \hat{f}(-y, x)|$ whenever the STFT can be defined, we have

$$|V_G K(u,v)| = |V_H \hat{\sigma}((v_1, u_2, u_3), (-u_1, v_2, v_3))|$$

$$= |V_{\check{H}} \sigma((u_1, -v_2, -v_3), (v_1, u_2, u_3))|$$

$$= |V_{\check{H}} \sigma(A(u,v))|.$$

Therefore,

$$\int_{\mathbb{R}^{3d}} \int_{\mathbb{R}^{3d}} |V_G K(u,v)| \Omega_s(u,v) du dv$$

$$\int_{\mathbb{R}^{3d}} \int_{\mathbb{R}^{3d}} |V_{\check{H}} \sigma(A(u,v))| \Omega_s(u,v) du dv$$

$$\int_{\mathbb{R}^{3d}} \int_{\mathbb{R}^{3d}} |V_{\check{H}} \sigma(A(u,v))| \Omega_s^{A^{-1}}(u,v) du dv$$

$$\int_{\mathbb{R}^{3d}} \int_{\mathbb{R}^{3d}} |V_{\check{H}}\sigma(u,v)|\Omega_s^B(u,v) du dv \quad < \infty$$

and the proof is complete.

Remark 3.6. In [1] Beny and Okoudjou proved that a bilinear integral operator with kernel in the modulation space is bounded. We showed that S_σ is a pseudodifferential operator and established the relationship between a bilinear integral operator and S_σ. Now we have prepared conditions for following directly Theorems $2, 3$ and 4 in [1] to deduce the boundedness of S_σ under the assumptions in Proposition 3.5.

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