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Some new extensions of Hardy‘s inequality

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Dedicated to the Memory of Charalambos J. Papaioannou

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Abstract

In this study, by a non-negative homogeneous kernel k we prove some extensions of Hardy's inequality in two and three dimensions.

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1. Introduction

The classical Hardy inequality reads:

$$\int_0^\infty \Big(\frac{\int_0^x f(t)dt}{x}\Big)^p dx < \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f^p(x)\,dx, \qquad (p>1) \tag{1.1}$$

where, f is nonnegative function such that $f \in L^p(R_+)$ and $R_+ = (0,\infty)$. The almost dramatic period of research in at least 10 years until G.H. Hardy [1] stated and proved (1.1) was recently described in details in [2].
Another important inequality is the following:
If $p > 1$ and f is a nonnegative function such that $f \in L^p(R_+)$, then

$$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{x+y}dx\Big)^p dy < \frac{\pi}{sin(\frac{\pi}{p})}\int_0^\infty f^p(y)dy. \tag{1.2}$$

It was early known that these inequalities are in fact equivalent. Moreover, (1.2) is sometimes called Hilbert's inequality even if Hilbert himself only considered the case $p = 2$ (L^p spaces were not

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defined at that time).
We also note that (1.1) can be interpreted as the Hardy operator $H : (Hf)(x) := \frac{1}{x}\int_0^x f(t)dt$, maps L^p into L^p with the operator norm $q = \frac{p}{p-1}\Big($since, it is known that $\left(\frac{p}{p-1}\right)^p$ is the sharp constant in (1.1)$\Big)$. Similarly, (1.2) may be interpreted as also the operator $A : (Af)(y) := \int_0^\infty \frac{f(x)}{x+y}dx$ maps L^p into L^p with the operator norm $\left(\frac{\pi}{\sin\left(\pi/p\right)}\right)^p$.

In 1928 Hardy [2] proved an estimate for some integral operators as a generalization of the Schur test, from which the first "weighted" modification of Hardy's inequality (1.1) followed, namely the inequality

$$\int_0^\infty \Big(\frac{\int_0^x f(t)dt}{x}\Big)^p x^\alpha\, dx < \Big(\frac{p}{p-\alpha-1}\Big)^p \int_0^\infty f^p(x)x^\alpha\, dx, \qquad (p>1) \tag{1.3}$$

valid, with $p > 1$ and $\alpha < p-1$, for all measurable non-negative functions f(see [3] Theorem 330), where the constant $\left(\frac{p}{p-\alpha-1}\right)^p$ is the best possible. The prehistory of (1.1) up to the time when Hardy finally proved (1.1) in 1925 in [3] can be found in [6]. After that the inequality has been developed and applied in almost unbelievable ways. See for instance the books [6], [7] devoted to this subject and also the recent historical article [5] and references given therein. Recently, many extensions of hardy's inequality in higher dimensions are appeared(cf. [4], [8], [12]).

In this section, we introduce some inequalities due to B. Yang and W.T. Sulaiman. These inequalities are extended Hardy‘s inequality and Hardy-Hilbert‘s inequality. First, we give one inequality which is given by B. Yang and also is an extension of Hardy-Hilbert‘s inequality.

Theorem 1.1. *([11], Theorem 2.1). Let f and g be two nonnegative real functions and $\lambda > 2 - \min\{p,q\}$ such that*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\, dt < \infty, \qquad 0 < \int_0^\infty t^{1-\lambda} g^q(t)\, dt < \infty.$$

If $A, B > 0$, then we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\lambda}dxdy < \frac{1}{(AB)^{\frac{\lambda}{2}}}\beta\Big(\frac{p+\lambda-2}{2}, \frac{q+\lambda-2}{2}\Big)\Big(\int_0^\infty t^{1-\lambda} f^p(t)\, dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty t^{1-\lambda} g^q(t)\, dt\Big)^{\frac{1}{q}}.$$

In the following we give two inequalities which are given by W.T. Sulaiman and also are extensions of Hardy‘s inequality. We shall give some generalizations of these inequalities through the next section.

Theorem 1.2. *([9], Theorem 1). Let $f, g \geq 0$, $p > 0$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda > 1$ and $\alpha_p = p(\lambda-1)+1$. Assume that*

$$F(x) = \int_0^x f(t)\, dt, \qquad G(y) = \int_0^y g(t)\, dt.$$

Then

$$\int_0^\infty\int_0^\infty \frac{F^{\frac{\alpha_p}{p}}(x)G^{\frac{\alpha_q}{q}}(y)}{(x+y)^{2\lambda}}\, dxdy < \beta(\lambda,\lambda)\Big(\frac{\alpha_p}{\alpha_p-1}\Big)^{\frac{\alpha_p}{p}}\Big(\frac{\alpha_q}{\alpha_q-1}\Big)^{\frac{\alpha_q}{q}}\Big(\int_0^\infty f^{\alpha_p}(t)\, dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty g^{\alpha_q}(t)\, dt\Big)^{\frac{1}{q}}.$$

Theorem 1.3. *([9], Theorem 2). Let* $f, g, h \geq 0$, $p, q, l > 1$ *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{l} = 1$. *Also suppose that* $s, t \in \{p, q, l\}$, $\alpha_{s,t} = \lambda(1 + \frac{s}{t}) + (\lambda - 1)(p - 1)$ *and* $\lambda > 1$. *Assume that*

$$F(x) = \int_0^x f(t)\,dt, \qquad G(y) = \int_0^y g(t)\,dt, \qquad H(z) = \int_0^z h(t)\,dt.$$

If

$$0 < \int_0^\infty f^{\alpha_{p,q}}(t)\,dt < \infty, \quad 0 < \int_0^\infty g^{\alpha_{q,l}}(t)\,dt < \infty, \quad 0 < \int_0^\infty h^{\alpha_{l,p}}(t)\,dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{F^{\frac{\alpha_{p,q}}{p}}(x) G^{\frac{\alpha_{q,l}}{q}}(y) H^{\frac{\alpha_{l,p}}{l}}(z)}{(x+y+z)^{4\lambda}}\,dxdydz < K\Big(\int_0^\infty f^{\alpha_{p,q}}(t)\,dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty g^{\alpha_{q,l}}(t)\,dt\Big)^{\frac{1}{q}} \times\Big(\int_0^\infty h^{\alpha_{l,p}}(t)\,dt\Big)^{\frac{1}{l}},$$

where

$$K = \beta(\lambda, \lambda)\beta(2\lambda, 2\lambda)\Big(\frac{\alpha_{p,q}}{\alpha_{p,q} - 1}\Big)^{\frac{\alpha_{p,q}}{p}}\Big(\frac{\alpha_{q,l}}{\alpha_{q,l} - 1}\Big)^{\frac{\alpha_{q,l}}{q}}\Big(\frac{\alpha_{l,p}}{\alpha_{l,p} - 1}\Big)^{\frac{\alpha_{l,p}}{l}}.$$

2. Extensions of Hardy‘s inequality

Suppose that

$$k(x,y) = \begin{cases} \frac{1}{\Gamma(r)} \frac{(y-x)^{r-1}}{y^r} & \text{if } x < y \\ 0 & \text{if } x \geq y. \end{cases}$$

With $r = 1$ we obtain the kernel of Cesaro

$$k(x,y) = \begin{cases} \frac{1}{y} & \text{if } x < y \\ 0 & \text{if } x \geq y. \end{cases}$$

Hardy, Littlewood and Polya proved the following known theorems:

Theorem 2.1. *([3], Theorem 319). Suppose that* $p > 1$, *and k(x,y) is nonnegative and homogeneous of degree -1, and*

$$\int_0^\infty k(x,1)x^{-\frac{1}{p}}\,dx = \int_0^\infty k(1,y)y^{-\frac{1}{q}}\,dy = c.$$

Then

$$\int_0^\infty \int_0^\infty k(x,y)f(x)g(y)\,dxdy \leq c\Big(\int_0^\infty f^p(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty g^q(y)\,dy\Big)^{\frac{1}{q}}.$$

If $k(x,y)$ *is given as above, then we have*

$$c = \int_0^\infty k(x,1)x^{-\frac{1}{p}}\,dx = \int_0^1 (1-x)^{r-1}x^{-\frac{1}{p}}\frac{1}{\Gamma(r)}\,dx = \frac{\Gamma(1-\frac{1}{p})}{\Gamma(r+1-\frac{1}{p})}.$$

With $r = 1$ *we obtain* $k = \frac{p}{p-1} = q$

Theorem 2.2. *([3], Theorem 329). If* $p > 1$*,* $r > 0$ *and*

$$f_r(x) = \frac{1}{\Gamma(r)} \int_0^x (x-t)^{r-1} f(t)\, dt,$$

then

$$\int_0^\infty \Big(\frac{f_r(x)}{x^r}\Big)^p dx < \frac{\Gamma(1-\frac{1}{p})}{\Gamma(r+1-\frac{1}{p})} \int_0^\infty f^p(x)\, dx$$

unless $f \equiv 0$*. If*

$$f^r(x) = \frac{1}{\Gamma(r)} \int_x^\infty (t-x)^{r-1} f(t)\, dt,$$

then

$$\int_0^\infty \Big(f^r(x)\Big)^p dx < \Big\{\frac{\Gamma(\frac{1}{p})}{\Gamma(r+\frac{1}{p})}\Big\}^p \int_0^\infty \Big(x^r f(x)\Big)^p dx$$

unless $f \equiv 0$*. In each case the constant is the best possible.*

The function f_r is called the Riemann-Lioville integral of f of order r and the function f^r is called the Weyl integral of f of order r.

By applying the above two theorems, we state and prove new inequalities as follows.

Theorem 2.3. *Let* $f, g \geq 0$*,* $p > 1$*,* $r > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Also assume that* $k(x,y)$ *is a non-negative homogeneous function of degree* -2λ *with* $\lambda > 1$ *and that* $\alpha_p = p(\lambda - 1) + 1$*.*

If

$$f_r(x) = \frac{1}{\Gamma(r)} \int_0^x (x-t)^{r-1} f(t)\, dt,$$

$$g_r(y) = \frac{1}{\Gamma(r)} \int_0^y (y-t)^{r-1} g(t)\, dt,$$

then

$$\int_0^\infty \int_0^\infty \Big(\frac{f_r(x)}{x^{r-1}}\Big)^{\frac{\alpha_p}{p}} \Big(\frac{g_r(y)}{y^{r-1}}\Big)^{\frac{\alpha_q}{q}} k(x,y)\, dxdy < C(p,q,\lambda)\Theta(p)\Theta(q) \|f\|_{\alpha_p}^{\frac{\alpha_p}{p}} \|g\|_{\alpha_q}^{\frac{\alpha_q}{q}},$$

where

$$C(p,q,\lambda) = \Big(\int_0^\infty t^{\lambda-1} k(1,t) dt\Big)^{\frac{1}{p}} \Big(\int_0^\infty t^{\lambda-1} k(t,1) dt\Big)^{\frac{1}{q}}$$

and

$$\Theta(s) = \Big(\frac{\Gamma(1-\frac{1}{\alpha_s})}{\Gamma(r+1-\frac{1}{\alpha_s})}\Big)^{\frac{\alpha_s}{s}}.$$

Proof .

$$\int_0^\infty \int_0^\infty \Big(\frac{f_r(x)}{x^{r-1}}\Big)^{\frac{\alpha_p}{p}} \Big(\frac{g_r(y)}{y^{r-1}}\Big)^{\frac{\alpha_q}{q}} k(x,y)\, dxdy = \int_0^\infty \int_0^\infty \frac{\big(\frac{f_r(x)}{x^{r-1}}\big)^{\frac{\alpha_p}{p}} y^{\frac{\lambda-1}{p}}}{x^{\frac{\lambda-1}{q}}} k^{\frac{1}{p}}(x,y)$$
$$\times \frac{\big(\frac{g_r(y)}{y^{r-1}}\big)^{\frac{\alpha_q}{q}} x^{\frac{\lambda-1}{q}}}{y^{\frac{\lambda-1}{p}}} k^{\frac{1}{q}}(x,y)\, dxdy$$

$$\leq \Big(\int_0^\infty \int_0^\infty \frac{\big(\frac{f_r(x)}{x^{r-1}}\big)^{\alpha_p} y^{\lambda-1}}{x^{(\lambda-1)(p-1)}} k(x,y)\, dxdy \Big)^{\frac{1}{p}}$$

$$\times \Big(\int_0^\infty \int_0^\infty \frac{\big(\frac{g_r(y)}{y^{r-1}}\big)^{\alpha_q} x^{\lambda-1}}{y^{(\lambda-1)(q-1)}} k(x,y)\, dxdy \Big)^{\frac{1}{q}}$$

$$= U^{\frac{1}{p}} V^{\frac{1}{q}}.$$

If we take $y = tx$, then applying Theorem 2.2, we obtain

$$U = \Big(\int_0^\infty \Big(\frac{f_r(x)}{x^r}\Big)^{\alpha_p} dx \Big) \Big(\int_0^\infty t^{\lambda-1} k(1,t) dt \Big)$$

$$< \Big(\int_0^\infty t^{\lambda-1} k(1,t) dt \Big) \Big(\frac{\Gamma(1-\frac{1}{\alpha_p})}{\Gamma(r+1-\frac{1}{\alpha_p})} \Big)^{\alpha_p} \Big(\int_0^\infty f^{\alpha_p}(x)\, dx \Big).$$

Similarly, we can show that

$$V < \Big(\int_0^\infty t^{\lambda-1} k(t,1) dt \Big) \Big(\frac{\Gamma(1-\frac{1}{\alpha_q})}{\Gamma(r+1-\frac{1}{\alpha_q})} \Big)^{\alpha_q} \Big(\int_0^\infty g^{\alpha_q}(y)\, dy \Big).$$

This completes the proof. □

Corollary 2.4. *(i) By taking $k(x,y) = \frac{1}{(x+y)^{2\lambda}}$, in above-mentioned theorem, we have*

$$C(p,q,\lambda) = \int_0^\infty \frac{t^{\lambda-1}}{(1+t)^{2\lambda}} dt$$

$$= \beta(\lambda, \lambda).$$

Now, putting $r = 1$, Theorem 1.2 is obtained.
(ii) Assume that $A, B > 0$, by taking $k(x,y) = \frac{1}{(Ax+By)^{2\lambda}}$, in Theorem 2.3 one may obtain the following generalization of Theorem 1.2:

$$\int_0^\infty \int_0^\infty \frac{\Big(\frac{f_r(x)}{x^{r-1}}\Big)^{\frac{\alpha_p}{p}} \Big(\frac{g_r(y)}{y^{r-1}}\Big)^{\frac{\alpha_q}{q}}}{(Ax+By)^{2\lambda}} dxdy < \frac{1}{(AB)^\lambda} \beta(\lambda,\lambda) \Big(\frac{\Gamma(1-\frac{1}{\alpha_p})}{\Gamma(r+1-\frac{1}{\alpha_p})} \Big)^{\frac{\alpha_p}{p}} \Big(\frac{\Gamma(1-\frac{1}{\alpha_q})}{\Gamma(r+1-\frac{1}{\alpha_q})} \Big)^{\frac{\alpha_q}{q}}$$

$$\times \Big(\int_0^\infty f^{\alpha_p}(x)\, dx \Big)^{\frac{1}{p}} \Big(\int_0^\infty g^{\alpha_q}(y)\, dy \Big)^{\frac{1}{q}}.$$

(iii) By taking $k(x,y) = \frac{1}{x^{2\lambda}+y^{2\lambda}}$, in above-mentioned theorem, we have

$$C(p,q,\lambda) = \int_0^\infty \frac{t^{\lambda-1}}{1+t^{2\lambda}} dt$$

$$= \frac{\pi}{2\lambda}.$$

This leads us to the following inequality

$$\int_0^\infty\int_0^\infty \frac{\left(\frac{f_r(x)}{x^{r-1}}\right)^{\frac{\alpha_p}{p}}\left(\frac{g_r(y)}{y^{r-1}}\right)^{\frac{\alpha_q}{q}}}{x^{2\lambda}+y^{2\lambda}}\,dxdy < \frac{\pi}{2\lambda}\Theta(p)\Theta(q)\|f\|_{\alpha_p}^{\frac{\alpha_p}{p}}\|g\|_{\alpha_q}^{\frac{\alpha_q}{q}}.$$

Theorem 2.5. *Let* $f,g,h\geq 0$, $p,q,l>1$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{l}=1$ *and* $s,t\in\{p,q,l\}$. *Also, assume that* $k=k(x,y,z)$ *is a non-negative homogeneous function of degree* -4λ *with* $\lambda>1$ *and* $\alpha_{s,t}=\lambda(1+\frac{s}{t})+(\lambda-1)(p-1)$ *and* $r>0$. *If we take*

$$f_r(x)=\frac{1}{\Gamma(r)}\int_0^x (x-t)^{r-1}f(t)\,dt\ ,$$
$$g_r(y)=\frac{1}{\Gamma(r)}\int_0^y (y-t)^{r-1}g(t)\,dt\ ,$$
$$h_r(z)=\frac{1}{\Gamma(r)}\int_0^z (z-t)^{r-1}h(t)\,dt,$$

then

$$\int_0^\infty\int_0^\infty\int_0^\infty (\frac{f_r(x)}{x^{r-1}})^{\frac{\alpha_{p,q}}{p}}(\frac{g_r(y)}{y^{r-1}})^{\frac{\alpha_{q,l}}{q}}(\frac{h_r(z)}{z^{r-1}})^{\frac{\alpha_{l,p}}{l}}k(x,y,z)\,dxdydz < C(p,q,l,\lambda)\Theta(p,q)\Theta(q,l)\Theta(l,p)$$
$$\times\|f\|_{\alpha_{p,q}}^{\frac{\alpha_{p,q}}{p}}\|g\|_{\alpha_{q,l}}^{\frac{\alpha_{q,l}}{q}}\|h\|_{\alpha_{l,p}}^{\frac{\alpha_{l,p}}{l}},$$

where

$$C(p,q,l,\lambda)=\Big(\int_0^\infty\int_0^\infty u^{\lambda-1}v^{2\lambda-1}k(1,u,v)dudv\Big)^{\frac{1}{p}}\Big(\int_0^\infty\int_0^\infty u^{\lambda-1}v^{2\lambda-1}k(u,1,v)dudv\Big)^{\frac{1}{q}}$$
$$\times\Big(\int_0^\infty\int_0^\infty u^{\lambda-1}v^{2\lambda-1}k(u,v,1)dudv\Big)^{\frac{1}{l}}$$

and

$$\Theta(p,q)=(\frac{\Gamma(1-\frac{1}{\alpha_{p,q}})}{\Gamma(r+1-\frac{1}{\alpha_{p,q}})})^{\frac{\alpha_{p,q}}{p}}.$$

Proof .

$$\text{The left hand side of inequality} = \int_0^\infty\int_0^\infty\int_0^\infty \frac{(\frac{f_r(x)}{x^{r-1}})^{\frac{\alpha_{p,q}}{p}}y^{\frac{\lambda-1}{p}}z^{\frac{2\lambda-1}{p}}}{x^{\frac{2\lambda-1}{q}+\frac{\lambda-1}{l}}}k^{\frac{1}{p}}(x,y,z)$$
$$\times\frac{(\frac{g_r(y)}{y^{r-1}})^{\frac{\alpha_{q,r}}{q}}z^{\frac{\lambda-1}{q}}x^{\frac{2\lambda-1}{q}}}{y^{\frac{\lambda-1}{p}+\frac{2\lambda-1}{l}}}k^{\frac{1}{q}}(x,y,z)$$
$$\times\frac{(\frac{h_r(z)}{z^{r-1}})^{\frac{\alpha_{l,p}}{l}}x^{\frac{\lambda-1}{l}}y^{\frac{2\lambda-1}{l}}}{z^{\frac{2\lambda-1}{p}+\frac{\lambda-1}{q}}}k^{\frac{1}{l}}(x,y,z)\,dxdydz.$$

$$\leq \Big(\int_0^\infty\int_0^\infty\int_0^\infty \frac{(\frac{f_r(x)}{x^{r-1}})^{\alpha_{p,q}} y^{\lambda-1} z^{2\lambda-1}}{x^{p(\frac{2\lambda-1}{q}+\frac{\lambda-1}{l})}} k(x,y,z)\,dxdydz\Big)^{\frac{1}{p}}$$

$$\times\Big(\int_0^\infty\int_0^\infty\int_0^\infty \frac{(\frac{g_r(y)}{y^{r-1}})^{\alpha_{q,l}} z^{\lambda-1} x^{2\lambda-1}}{y^{q(\frac{\lambda-1}{p}+\frac{2\lambda-1}{l})}} k(x,y,z)\,dxdydz\Big)^{\frac{1}{q}}$$

$$\times\Big(\int_0^\infty\int_0^\infty\int_0^\infty \frac{(\frac{h_r(z)}{z^{r-1}})^{\alpha_{l,p}} x^{\lambda-1} y^{2\lambda-1}}{z^{r(\frac{2\lambda-1}{p}+\frac{\lambda-1}{q})}} k(x,y,z)\,dxdydz\Big)^{\frac{1}{l}}.$$

$$= L^{\frac{1}{p}} M^{\frac{1}{q}} N^{\frac{1}{l}},$$

where

$$L = \int_0^\infty\int_0^\infty\int_0^\infty \frac{(\frac{f_r(x)}{x^{r-1}})^{\alpha_{p,q}} y^{\lambda-1} z^{2\lambda-1}}{x^{p(\frac{2\lambda-1}{q}+\frac{\lambda-1}{l})}} k(x,y,z)\,dxdydz.$$

By taking $y = ux$ and $z = vx$ we have

$$L = \Big(\int_0^\infty \Big(\frac{f_r(x)}{x^r}\Big)^{\alpha_{p,q}} dx\Big)\Big(\int_0^\infty\int_0^\infty u^{\lambda-1} v^{2\lambda-1} k(1,u,v)dudv\Big).$$

By applying Theorem 2.2 we deduce that

$$L < \Big(\int_0^\infty\int_0^\infty u^{\lambda-1} v^{2\lambda-1} k(1,u,v)dudv\Big)\Big(\frac{\Gamma(1-\frac{1}{\alpha_{p,q}})}{\Gamma(r+1-\frac{1}{\alpha_{p,q}})}\Big)^{\alpha_{p,q}}\Big(\int_0^\infty f^{\alpha_{p,q}}(x)\,dx\Big).$$

Similarly, one can show that

$$M < \Big(\int_0^\infty\int_0^\infty u^{\lambda-1} v^{2\lambda-1} k(u,1,v)dudv\Big)\Big(\frac{\Gamma(1-\frac{1}{\alpha_{q,l}})}{\Gamma(r+1-\frac{1}{\alpha_{q,l}})}\Big)^{\alpha_{q,l}}\Big(\int_0^\infty g^{\alpha_{q,l}}(y)\,dy\Big),$$

and

$$N < \Big(\int_0^\infty\int_0^\infty u^{\lambda-1} v^{2\lambda-1} k(u,u,1)dudv\Big)\Big(\frac{\Gamma(1-\frac{1}{\alpha_{l,p}})}{\Gamma(r+1-\frac{1}{\alpha_{l,p}})}\Big)^{\alpha_{l,p}}\Big(\int_0^\infty h^{\alpha_{l,p}}(z)\,dz\Big).$$

This completes the proof. □

Corollary 2.6. *By taking $k(x,y,z) = \frac{1}{(Ax+By+Cz)^{4\lambda}}$ with $A, B, C > 0$, in above-mentioned theorem, the following inequality is obtained:*

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{(\frac{f_r(x)}{x^{r-1}})^{\frac{\alpha_{p,q}}{p}}(\frac{g_r(y)}{y^{r-1}})^{\frac{\alpha_{q,l}}{q}}(\frac{h_r(z)}{z^{r-1}})^{\frac{\alpha_{l,p}}{l}}}{(Ax+By+Cz)^{4\lambda}} dxdydz < \frac{1}{(ABC^2)^{\lambda}}\beta(\lambda,\lambda)\beta(2\lambda,2\lambda)\Theta(p,q)$$

$$\times\Theta(q,l)\Theta(l,p)\|f\|_{\alpha_{p,q}}^{\frac{\alpha_{p,q}}{p}}\|g\|_{\alpha_{q,l}}^{\frac{\alpha_{q,l}}{q}}\|h\|_{\alpha_{l,p}}^{\frac{\alpha_{l,p}}{l}}.$$

In special case, by taking $r = 1$, $A = B = C = 1$ we obtain Theorem 1.3 due to W.T. Sulaiman.

By similar manner, one may prove dual form of Theorems 2.3 and 2.5 which are stated in the following:

Theorem 2.7. *Let $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Also, assume that $k(x, y)$ is a non-negative homogeneous function of degree -2λ with $\lambda > 1$, $\alpha_p = p(\lambda - 1) + 1$ and $r > 0$.*

Suppose that

$$f^r(x) = \frac{1}{\Gamma(r)} \int_x^\infty (t - x)^{r-1} f(t)\, dt,$$
$$g^r(y) = \frac{1}{\Gamma(r)} \int_y^\infty (t - y)^{r-1} g(t)\, dt.$$

Then

$$\int_0^\infty \int_0^\infty \Big(xf^r(x)\Big)^{\frac{\alpha_p}{p}} \Big(yg^r(y)\Big)^{\frac{\alpha_q}{q}} k(x, y)\, dxdy < C(p, q, \lambda)\widetilde{\Theta}(p)\widetilde{\Theta}(q)\|f\|_{\alpha_p}^{\frac{\alpha_p}{p}} \|g\|_{\alpha_q}^{\frac{\alpha_q}{q}},$$

where

$$C(p, q, \lambda) = \Big(\int_0^\infty t^{\lambda-1}k(1, t)dt\Big)^{\frac{1}{p}} \Big(\int_0^\infty t^{\lambda-1}k(t, 1)dt\Big)^{\frac{1}{q}},$$

and

$$\widetilde{\Theta}(p) = \Big(\frac{\Gamma(\frac{1}{\alpha_p})}{\Gamma(r + \frac{1}{\alpha_p})}\Big)^{\frac{\alpha_p}{p}}$$

Theorem 2.8. *Let $f, g, h \geq 0$,$p, q, l > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{l} = 1$ and $s, t \in \{p, q, r\}$. Also, assume that $k(x, y, z)$ is a non-negative homogeneous function of degree -4λ with $\lambda > 1$ and $\alpha_{s,t} = \lambda(1 + \frac{s}{t}) + (\lambda - 1)(p - 1)$ and $r > 0$. Assume that*

$$f^r(x) = \frac{1}{\Gamma(r)} \int_x^\infty (t - x)^{r-1} f(t)\, dt\ ,$$
$$g^r(y) = \frac{1}{\Gamma(r)} \int_y^\infty (t - y)^{r-1} g(t)\, dt\ ,$$
$$h^r(z) = \frac{1}{\Gamma(r)} \int_z^\infty (t - z)^{r-1} h(t)\, dt.$$

Then

$$\int_0^\infty \int_0^\infty \int_0^\infty \Big(xf^r(x)\Big)^{\frac{\alpha_{p,q}}{p}} \Big(yg^r(y)\Big)^{\frac{\alpha_{q,l}}{q}} \Big(zh^r(z)\Big)^{\frac{\alpha_{l,p}}{l}} k(x, y, z)\, dxdydz < C(p, q, l, \lambda)\widetilde{\Theta}(p, q)\widetilde{\Theta}(q, l)\widetilde{\Theta}(l, p)$$
$$\times \|f\|_{\alpha_{p,q}}^{\frac{\alpha_{p,q}}{p}} \|g\|_{\alpha_{q,l}}^{\frac{\alpha_{q,l}}{q}} \|h\|_{\alpha_{l,p}}^{\frac{\alpha_{l,p}}{l}},$$

where

$$C(p, q, l, \lambda) = \Big(\int_0^\infty \int_0^\infty u^{\lambda-1}v^{2\lambda-1}k(1, u, v)dudv\Big)^{\frac{1}{p}} \Big(\int_0^\infty \int_0^\infty u^{\lambda-1}v^{2\lambda-1}k(u, 1, v)dudv\Big)^{\frac{1}{q}}$$
$$\times \Big(\int_0^\infty \int_0^\infty u^{\lambda-1}v^{2\lambda-1}k(u, v, 1)dudv\Big)^{\frac{1}{l}},$$

and

$$\widetilde{\Theta}(p, q) = \Big(\frac{\Gamma(\frac{1}{\alpha_{p,q}})}{\Gamma(r + \frac{1}{\alpha_{p,q}})}\Big)^{\frac{\alpha_{p,q}}{p}}.$$

The following statement is a Hardy-type inequality which due to W.T. Sulaiman:

Theorem 2.9. *([10], Theorem 2). Let $f, g \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.*
Suppose that

$$F(x) = \int_0^x f(t)\,dt, \qquad G(y) = \int_0^y g(t)\,dt.$$

Then

$$\int_0^\infty \int_0^\infty \frac{x^{\frac{1}{p}} y^{\frac{1}{q}} F(x)G(y)}{(x+y)^4}\,dxdy < p\ \beta^{\frac{1}{p}}(p,p)\ q\ \beta^{\frac{1}{q}}(q,q)) \Big(\int_0^\infty f^p(x)\,dx\Big)^{\frac{1}{p}} \Big(\int_0^\infty g^q(y)\,dy\Big)^{\frac{1}{q}}.$$

We generalize this theorem as follows:

Theorem 2.10. *Let $f, g \geq 0$, $p > 1$ and $r > 0$. Also assume that $k(x, y)$ is a non-negative homogeneous function of degree -2λ with $\lambda \geq 2$, $\alpha_p = p(\lambda - 1)$ and $\frac{1}{p} + \frac{1}{q} = 1$. If*

$$f_r(x) = \frac{1}{\Gamma(r)} \int_0^x (x-t)^{r-1} f(t)\,dt$$

$$g_r(y) = \frac{1}{\Gamma(r)} \int_0^y (y-t)^{r-1} g(t)\,dt,$$

then

$$\int_0^\infty \int_0^\infty x^{\frac{1}{p}} y^{\frac{1}{q}} \Big(\frac{f_r(x)}{x^{r-1}}\Big)^{\frac{\alpha_p}{p}} \Big(\frac{g_r(y)}{y^{r-1}}\Big)^{\frac{\alpha_q}{q}} k(x,y)\,dxdy < C(p,q)\Theta(p)\Theta(q)\|f\|_{\alpha_p}^{\frac{\alpha_p}{p}} \|g\|_{\alpha_q}^{\frac{\alpha_q}{q}},$$

where

$$C(p,q) = \Big(\int_0^\infty t^{p-1} k^{\frac{p}{2}}(1,t)dt\Big)^{\frac{1}{p}} \Big(\int_0^\infty t^{q-1} k^{\frac{q}{2}}(t,1)dt\Big)^{\frac{1}{q}}),$$

and

$$\Theta(s) = \Big(\frac{\Gamma(1-\frac{1}{\alpha_s})}{\Gamma(r+1-\frac{1}{\alpha_s})}\Big)^{\frac{\alpha_s}{s}}.$$

Proof .

The left-hand side of inequality $= \displaystyle\int_0^\infty \int_0^\infty y^{\frac{1}{q}} \Big(\frac{f_r(x)}{x^{r-1}}\Big)^{\frac{\alpha_p}{p}} k^{\frac{1}{2}}(x,y) \times x^{\frac{1}{p}} \Big(\frac{g_r(y)}{y^{r-1}}\Big)^{\frac{\alpha_q}{q}} k^{\frac{1}{2}}(x,y)\,dxdy$

$$\leq \Big(\int_0^\infty \int_0^\infty y^{\frac{p}{q}} \Big(\frac{f_r(x)}{x^{r-1}}\Big)^{\alpha_p} k^{\frac{p}{2}}(x,y)dxdy\Big)^{\frac{1}{p}}$$

$$\times \Big(\int_0^\infty \int_0^\infty x^{\frac{q}{p}} \Big(\frac{g_r(y)}{y^{r-1}}\Big)^{\alpha_q} k^{\frac{q}{2}}(x,y)dxdy\Big)^{\frac{1}{q}}$$

$$= M^{\frac{1}{p}} N^{\frac{1}{q}}.$$

Note that $\alpha_p = p\lambda - p$ and so by taking $y = tx$ we have

$$M = \Big(\int_0^\infty \Big(\frac{f_r(x)}{x^r}\Big)^{\alpha_p} dx\Big)\Big(\int_0^\infty t^{p-1}k^{\frac{p}{2}}(1,t)dt\Big).$$

By applying Theorem 2.2 one may obtain

$$M < \Big(\int_0^\infty t^{p-1}k^{\frac{p}{2}}(1,t)dt\Big)\Big(\frac{\Gamma(1-\frac{1}{\alpha_p})}{\Gamma(r+1-\frac{1}{\alpha_p})}\Big)^{\alpha_p}\Big(\int_0^\infty f^{\alpha_p}(x)\,dx\Big).$$

By the same way, one can show that:

$$N < \Big(\int_0^\infty t^{q-1}k^{\frac{q}{2}}(t,1)dt\Big)\Big(\frac{\Gamma(1-\frac{1}{\alpha_q})}{\Gamma(r+1-\frac{1}{\alpha_q})}\Big)^{\alpha_q}\Big(\int_0^\infty g^{\alpha_q}(y)\,dy\Big).$$

This completes the proof of the statement. □

Corollary 2.11. *By taking $r = 1$ and $k(x,y) = \frac{1}{(x+y)^4}$ in above-mentioned theorem, one may obtain Theorem 2.9.*

Proof . Note that

$$\begin{aligned} C(p,q) &= \Big(\int_0^\infty t^{p-1}k^{\frac{p}{2}}(1,t)dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty t^{q-1}k^{\frac{q}{2}}(t,1)dt\Big)^{\frac{1}{q}}) \\ &= \Big(\int_0^\infty \frac{t^{p-1}}{(1+t)^{2p}}dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty \frac{t^{q-1}}{(1+t)^{2q}}dt\Big)^{\frac{1}{q}} \\ &= \beta^{\frac{1}{p}}(p,p)\beta^{\frac{1}{q}}(q,q). \end{aligned}$$

□

Theorem 2.12. *Suppose that $f_i (1 \le i \le n)$ are nonnegative integrable functions, $p_i (1 \le i \le n)$, are nonnegative numbers such that they are not all zero and $s = \sum_1^n p_i$. Define*

$$F_i(x) = \int_0^x f_i(t)\,dt, \qquad (1 \le i \le n).$$

Then

$$\int_0^\infty \Big(\frac{F_1^{p_1}(x)\cdots F_n^{p_n}(x)}{x^s}\Big)^{\frac{p}{s}} < \Big(\frac{p}{sp-s}\Big)^p\Big(\int_0^\infty (\sum_1^n p_i f_i(x))^p\,dx\Big).$$

Proof . By the theorem of the arithmetic and geometric means one may obtain

$$\Big(F_1^{p_1}(x)\cdots F_n^{p_n}(x)\Big)^{\frac{1}{s}} \le \frac{p_1F_1(x) + \cdots + p_nF_n(x)}{s}.$$

Therefore

$$\Big(\frac{F_1^{p_1}(x)\cdots F_n^{p_n}(x)}{x^s}\Big)^{\frac{p}{s}} \le \frac{1}{s^p}\Big(\frac{p_1F_1(x)}{x} + \cdots + \frac{p_nF_n(x)}{x}\Big)^p.$$

By integrating we have

$$\int_0^\infty \Big(\frac{F_1^{p_1}(x)\cdots F_n^{p_n}(x)}{x^s}\Big)^{\frac{p}{s}} \le \frac{1}{s^p}\int_0^\infty \Big(\frac{p_1F_1(x)}{x}+\cdots+\frac{p_nF_n(x)}{x}\Big)^p\,dx$$

$$< \frac{1}{s^p}\Big(\frac{p}{p-1}\Big)^p\Big(\int_0^\infty (\sum_1^n p_if_i(x))^p\,dx\Big).$$

The last inequality is obtained by applying the Hardy's inequality. □

Corollary 2.13. *By taking $p_1 = 1$ and $p_2 = \cdots = p_n = 0$ in Theorem 2.12, the Hardy's inequality is obtained.*

Theorem 2.14. *Let $n \ge 2$, $f_i(1 \le i \le n)$ be nonnegative integrable functions and $p_i > 1(1 \le i \le n)$. Define*

$$F_i(x) = \int_0^x f_i(t)\,dt, \qquad (1 \le i \le n).$$

Then

$$\int_0^\infty \frac{F_1(x)\cdots F_n(x)}{x^n}\,dx \le \sum_1^n \frac{1}{p_i}\Big(\frac{p_i}{p_i-1}\Big)^{p_i}\Big(\int_0^\infty f_i^{p_i}(x)\,dx\Big).$$

Proof . By Hölder's inequality we have

$$\frac{F_1(x)\cdots F_n(x)}{x^n} \le \frac{\Big(\frac{F_1(x)}{x}\Big)^{p_1}}{p_1}+\cdots+\frac{\Big(\frac{F_n(x)}{x}\Big)^{p_n}}{p_n}.$$

Now, by integrating and applying the Hardy's inequality the assertion is proved. □

Corollary 2.15. *By taking $p_i = p$ and $F_i = F$ in Theorem 2.14 one may obtain*

$$\frac{1}{n}\Big(\int_0^\infty \Big(\frac{F(x)}{x}\Big)^n\,dx\Big) \le \frac{p^{p-1}}{(p-1)^p}\int_0^\infty f^p(x)\,dx.$$

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