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On λ^2- Asymptotically Double Statistical Equivalent Sequences

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Abstract

This paper presents the following new definition which is a natural combination of the definition for asymptotically double equivalent, double statistically limit and double λ^2- sequences. The double sequence $\lambda^2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \leq \lambda_{m,n} + 1, \ \lambda_{m,n+1} \leq \lambda_{m,n} + 1,$$
$$\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \left\{(k,l) : \ m - \lambda_{m,n} + 1 \leq k \leq m, \ n - \lambda_{m,n} + 1 \leq l \leq n\right\}.$$

For double λ^2-sequence; the two non-negative sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be λ^2-asymptotically double statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \left| \left\{ (k,l) \in I_{m,n} : \ \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S^L_{\lambda^2}}{\backsim} y$) and simply λ^2-asymptotically double statistical equivalent if $L = 1$.

Keywords: Pringsheim Limit Point, P-convergent, Double Statistical Convergence.
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1. Introduction and preliminaries

In 1993, Marouf [2] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [5] extends these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices. Later these definitions extended to $\lambda-$sequences by Savaş and Başarır in [8]. This paper extends the definitions presented in [8] to double λ^2- sequences. In addition to these definitions, natural inclusion theorems shall also be presented.

2. Definitions and Notations

Now we give a brief history for asymptotical equivalence for single sequences and double sequences.

Definition 2.1. *(Marouf, [2]) Two non-negative sequence $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \backsim y$).

Definition 2.2. *(Fridy, [1]) The sequence $x = (x_k)$ has statistic limit L provided that for every $\varepsilon > 0$,*

$$\lim_n \frac{1}{n} \left|\{k \le n : \ |x_k - L| \ge \varepsilon\}\right| = 0$$

The next definition is natural combination of definitions (2.1) and (2.2).

Definition 2.3. *(Patterson, [5]) Two non-negative sequence $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$,*

$$\lim_n \frac{1}{n} \left|\left\{ k \le n : \ \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| = 0$$

(denoted by $x \overset{S_L}{\backsim} y$) and simply asymptotically statistical equivalent if $L = 1$.

Definition 2.4. *(Mursaleen, [3]) Let $\Lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinitiy and $\lambda_1 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$. A sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent or $S_\lambda-$convergent to L if for every $\varepsilon > 0$*

$$\lim_n \frac{1}{\lambda_n} \left|\{ k \in I_n : \ |x_k - L| \ge \varepsilon\}\right| = 0$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \cdots$.

Definition 2.5. *(Savaş and Başarır, [8]) Let $\Lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinitiy and $\lambda_1 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$. The two non-negative sequences sequences $x = (x_k)$ and $y = (y_k)$ are S_λ-asymptotically equivalent of multiple L provided that*

$$\lim_n \frac{1}{\lambda_n} \left|\left\{ k \in I_n : \ \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right| = 0$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \cdots$.

Definition 2.6. *(Savaş and Başarır, [8]) Let $\Lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinitiy and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The two non-negative sequences sequences $x = (x_k)$ and $y = (y_k)$ are strong λ-asymptotically equivalent of multiple L provided that*

$$\lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left| \frac{x_k}{y_k} - L \right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

In 1900 Pringsheim presented the following definition for the convergence of double sequences.

Definition 2.7. *(Pringsheim, [7]) A double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as "P-convergent".*

We shall denote the space of all P-convergent sequences by c^u.By a bounded double sequence we shall mean there exists a positive number K such that $|x_{k,l}| < K$ for all (k, l) and denote such bounded by $\|x\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty$. We shall also denote the set of all bounded double sequences by l_∞^u. We also note in contrast to the case for single sequence, a P-convergent double sequence need not be bounded.

Definition 2.8. *(Patterson, [6]) The two non-negative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically double equivalent of multiple L provided that for every $\varepsilon > 0$,*

$$P - \lim_{k,l,} \frac{x_{k,l}}{y_{k,l}} = L$$

(denoted by $x \overset{P}{\backsim} y$) and simply asymptotically double equivalent if $L = 1$.

Definition 2.9. *(Mursaleen and Edely, [4]) A real double sequence $x = (x_{k,l})$ is to be statistically convergent to L, provided that for each $\varepsilon > 0$*

$$P - \lim_{m,n} \frac{1}{mn} |\{(k, l) : \ k \leq m \text{ and } l \leq n, |x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case we write $S^L - \lim x = L$ or $x_{k,l} \to L \left(S^L\right)$.

Definition 2.10. *The two non-negative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically double statistical equivalent of multiple L provided that for every $\varepsilon > 0$,*

$$P - \lim_{m,n} \frac{1}{mn} \left| \left\{ (k, l) : \ k \leq m \text{ and } l \leq n, \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| = 0.$$

(denoted by $x \overset{S^L}{\backsim} y$) and simply asymptotically double statistical equivalent if $L = 1$.

Definition 2.11. *The double sequence $\lambda^2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that*

$$\lambda_{m+1,n} \leq \lambda_{m,n} + 1, \ \lambda_{m,n+1} \leq \lambda_{m,n} + 1,$$
$$\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \{(k, l) : \ m - \lambda_{m,n} + 1 \leq k \leq m, \ n - \lambda_{m,n} + 1 \leq l \leq n\}.$$

The generalized double de Vallee-Poussin mean is defined by

$$t_{m,n} = t_{m,n}\left(x_{k,l}\right) = \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} x_{k,l}.$$

Now we give some new definitions which are natural combination of definitions (2.10) and (2.11).

Definition 2.12. *For double λ^2-sequence; the two non-negative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be λ^2-asymptotically double statistical equivalent of multiple L provided that for every $\varepsilon > 0$,*

$$P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \left| \left\{ (k,l) \in I_{m,n} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S^L_{\lambda^2}}{\backsim} y$) and simply asymptotically double statistical equivalent if $L = 1$. Furthermore, let $S^L_{\lambda^2}$ denote the set of all sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ such that $x \overset{S^L_{\lambda^2}}{\backsim} y$.

For double λ^2-sequence; the two double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be strong λ^2-asymptotically double equivalent of multiple L provided that

$$P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| = 0,$$

(denoted by $x \overset{N^L_{\lambda^2}}{\backsim} y$) and simply strong λ^2-asymptotically double equivalent if $L = 1$. In addition, let $N^L_{\lambda^2}$ denote the set of all sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ such that $x \overset{N^L_{\lambda^2}}{\backsim} y$.

3. Main Results

Theorem 3.1. *For double λ^2-sequence;*

(**i**). *(a) If $x \overset{N^L_{\lambda^2}}{\backsim} y$ then $x \overset{S^L_{\lambda^2}}{\backsim} y$.*
(b) $N^L_{\lambda^2}$ is a proper subset of $S^L_{\lambda^2}$.

(**ii**). *If $x = (x_{k,l}) \in l^u_\infty$ and $x \overset{S^L_{\lambda^2}}{\backsim} y$ then $x \overset{N^L_{\lambda^2}}{\backsim} y$.*

(**iii**). $S^L_{\lambda^2} \cap l^u_\infty = N^L_{\lambda^2} \cap l^u_\infty$.

Proof . (**i**).

(a) If $\varepsilon > 0$ and $x \overset{N^L_{\lambda^2}}{\backsim} y$ then

$$\sum_{(k,l)\in I_{m,n}} \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \sum_{(k,l)\in I_{m,ns} \,\&\, \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon} \left| \frac{x_{k,l}}{y_{k,l}} - L \right|$$

$$\geq \varepsilon \left| \left\{ (k,l) \in I_{m,n} : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right|.$$

Therefore $x \overset{S^L_{\lambda^2}}{\backsim} y$.

(b) To show the inclusion is strict, we define $x = (x_{k,l})$ as follows:

$$x_{k,l} = \begin{pmatrix} 1 & 2 & 3 & \dots & \left[\sqrt[3]{\lambda_{m,n}}\right] & 0 & 0 & \dots \\ 2 & 2 & 3 & \dots & \left[\sqrt[3]{\lambda_{m,n}}\right] & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 2 & \left[\sqrt[3]{\lambda_{m,n}}\right] & \left[\sqrt[3]{\lambda_{m,n}}\right] & \dots & \left[\sqrt[3]{\lambda_{m,n}}\right] & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $x \overset{S^L_{\lambda^2}}{\backsim} y$ but the following fails $x \overset{N^L_{\lambda^2}}{\backsim} y$.

(ii). Suppose that $x = (x_{k,l})$ and $y = (y_{k,l})$ are in l^n_∞ and $x \overset{S^L_{\lambda^2}}{\backsim} y$. Then we can assume that

$$\left|\frac{x_{k,l}}{y_{k,l}} - L\right| < H, \text{ for all } k \text{ and } l.$$

Given $\varepsilon > 0$

$$\frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left|\frac{x_{k,l}}{y_{k,l}} - L\right| = \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n} \,\&\, \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon} \left|\frac{x_{k,l}}{y_{k,l}} - L\right|$$

$$+\frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n} \,\&\, \left|\frac{x_{k,l}}{y_{k,l}} - L\right| < \varepsilon} \left|\frac{x_{k,l}}{y_{k,l}} - L\right|$$

$$\leq \frac{H}{\lambda_{m,n}} \left|\left\{ (k,l) \in I_{m,n} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right| + \varepsilon.$$

Therefore $x \overset{N^L_{\lambda^2}}{\backsim} y$.

(iii). It follows from (i) and (ii). □

In the next theorem we prove the following relation.

Theorem 3.2. *For double λ^2-sequence; $x \overset{S^L}{\backsim} y$ implies $x \overset{S^L_{\lambda^2}}{\backsim} y$ if*

$$\liminf_{m,n} \frac{1}{\lambda_{m,n}} > 0.$$

Proof . For given $\varepsilon > 0$

$$\left\{(k,l) :\ k \leq m \text{ and } l \leq n, \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\} \supset \left\{ (k,l) \in I_{m,n} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}.$$

Therefore

$$\frac{1}{mn}\left|\left\{(k,l) :\ k \leq m \text{ and } l \leq n, \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{mn}\left|\left\{ (k,l) \in I_{m,n} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{\lambda_{m,n}}{mn}.\frac{1}{\lambda_{m,n}}\left|\left\{ (k,l) \in I_{m,n} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right|.$$

Taking the limits as $n, m \to \infty$ in Pringsheim sense and using $\liminf_{m,n} \frac{1}{\lambda_{m,n}} > 0$, we get desired result. This completes the proof. □

References

[1] J. A. Fridy, On statistical convergence, Analysis, 5 (1985), 301-313.

[2] M. Marouf, Asymptotic equivalence and summability, Internat. J. Math. Math. Sci. 16 (4) (1993), 755-762.

[3] M. Mursaleen, $\lambda-$statistical convergence, Math. Slovaca 50 (1) (2000), 111-115.

[4] M. Mursaleen and O. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (1) (2003), 223-231.

[5] R. F. Patterson, On asymptotically statististically equivalent sequences, Demonstratio Math. 36 (1) (2003), 149-153.

[6] R. F. Patterson, Some characterization of asymptotic equivalent double sequences, (in press).

[7] A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, Mathematische Annalen 53 (1900), 289-321.

[8] R. Savaş and M. Başarır, (σ, λ)-asymptotically statististically equivalent sequences, Filomat 20 (1) (2006), 35-42.