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Fixed point theorems on generalized c-distance in ordered cone b-metric spaces

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Abstract

In this paper, we introduce a concept of a generalized c-distance in ordered cone b-metric spaces and, by using the concept, we prove some fixed point theorems in ordered cone b-metric spaces. Our results generalize the corresponding results obtained by Y. J. Cho, R. Saadati, Shenghua Wang [Y. J. Cho, R. Saadati, Shenghua Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, J. Computers and Mathematics with Application. 61 (2011), 1254-1260]. Furthermore, we give some examples and an application to support our main results.

Keywords: Fixed point; Cone b-metric spaces; Generalized c-distance.
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1. Introduction

Since the concept of a cone b-metric was introduced by N. Hussain and M. H. Shah [13], many fixed point theorems, which generalize some relative theorems on cone metric spaces (see [12]–[15]) and b-metric spaces (see [4]–[8]), have been proved for mappings on normal or non-normal cone b-metric spaces by some authors (see [11, 1] and the references contained therein). In this paper, we consider a new concept of a generalized c-distance in cone b-metric spaces, which is a generalization of c-distance of paper [7], prove theorems for some contractive type mappings in a cone b-metric space by using the generalized c-distance and give an application on the existence of solution of an integral equation.

We need the following definitions and results, consistent with [12].

Let E be a real Banach space and let P be a subset of E, int$P$ denotes the interior of P. The subset P is called a cone if and only if

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(i) P is closed, nonempty and $P \neq \{\theta\}$,

(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$,

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$. A cone P is called normal if there is a number $N > 0$ such that for all $x, y \in P$,

$$\theta \leq x \leq y \quad \text{implies} \quad \| x \| \leq N \| y \| .$$

The least positive number satisfying the above inequality is called the normal constant of P.

Definition 1.1. ([12]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(i) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.2. ([13]) Let X be a nonempty set and let $s \geq 1$ a given real number. A mapping $d : X \times X \to E$ is said to be a cone b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $\theta < d(x, y)$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a cone b-metric space.

Definition 1.3. ([13]) Let (X, d) be a cone b-metric space. Then we say that $\{x_n\}$ is:

(i) a Cauchy sequence if for every $c \in E$ with $c \gg 0$, there is $N \in \mathbb{N}$ such that for all $n, m > N,\ d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for every $c \in E$ with $c \gg 0$, there is $N \in \mathbb{N}$ such that for all $m > N,\ d(x_m, x) \ll c$ for some fixed x in X.

A cone b-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Remark 1.4. (i) If E is a real Banach space with a cone P and $\alpha \leq \lambda\alpha$, where $\alpha \in P$ and $0 < \lambda < 1$, then $\alpha = \theta$.

(ii) If $c \in \text{int}P$, $a_n \to \theta$, as $n \to \infty$. Then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Definition 1.5. ([7]) Let (X, d) be a cone metric space, then a function $q : X \times X \to E$ is called a $c-$distance on X if the following conditions are satisfied:

(q1) $\theta \le q(x, y)$ for all $x, y \in X$;

(q2) $q(x, y) \le q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) for each $x \in X$ and $n \ge 1$, if $q(x, y_n) \le u$ for some $u = u_x \in P$, then $q(x, y) \le u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Definition 1.6. ([7]) A pair (f, g) of self-mappings on a partially ordered set, $(X, \sqsubseteq)$ is said to be weakly increasing if $fx \sqsubseteq gfx$ and $gx \sqsubseteq fgx$ holds for all $x \in X$.

2. Main results

Definition 2.1. Let (X, d) be a cone metric space, then a function $q : X \times X \to E$ is called a $c-$distance on X if the following conditions are satisfied:

(q1) $\theta \le q(x, y)$ for all $x, y \in X$;

(q2) $q(x, y) \le s(q(x, y) + q(y, z))$ for all $x, y, z \in X$;

(q3) for each $x \in X$ and $n \ge 1$, if $q(x, y_n) \le u$ for some $u = u_x \in P$, then $q(x, y) \le su$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

We introduce the concept of generalized c-distance on a cone b-metric space (X, d), which is a generalization of c-distance of Yeol Je Cho,Reza Saadati and Shenghua Wang [7]. Now, we give some examples of the generalized c-distance, as follows, which is a c-distance, and generalizes the c-distance.

Example 2.2. *Let (X, d) be a cone b-metric space, let $s \ge 1$ and P be a normal cone. Put $q(x, y) = \frac{1}{s}d(u, y)$ for all $x, y \in X$, where $u \in X$ is a fixed point, then q is a generalized c-distance.*

Proof . we prove q is a generalized c-distance on X.

(q1) since $d(u, y) \ge \theta$, we have $\frac{1}{s}d(u, y) = q(x, y) \ge \theta$;

(q2) since $d(u, z) \le sd(u, y) + sd(u, z), i.e., sq(x, z) \le s^2q(x, y) + s^2q(y, z), i.e., q(x, z) \le sq(x, y) + sq(y, z)$;

(q3) is obvious;

(q4) $d(x, y) \le sd(x, u) + sd(u, y) = sd(u, x) + sd(u, y) = s^2q(z, x) + s^2q(z, y)$.

□

Remark 2.3. (1) $q(x,y) = q(y,x)$ does not necessarily hold for all $x, y \in X$.

(2) $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Example 2.4. Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0,\infty)$ and define a mapping $d : X \times X \to E$ by

$$d(x,y) = |x-y|^s, s = \{1,2\}$$

for all $x, y \in X$. Then (X,d) is a cone b-metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = y^s$ for all $x, y \in X$. Then q is a generalized c-distance. In fact (q1) and (q3) are immediate. From

$$z^s = q(x,z) \leq sq(x,y) + sq(y,z) = sy^s + sz^s,$$

it follows that (q2) holds. From $d(x,y) = |x-y|^s \leq x^s + y^s = q(z,x) + q(z,y)$, it follows that (q4) holds. Hence q is a generalized c-distance.

Example 2.5. Let

$$E = C^1_{\mathbb{R}}[0,1]$$

with

$$\|x\| = \|x\|_\infty + \|x'\|_\infty$$

and

$$P = \{x \in E : x(t) \geq 0$$

on $[0,1]\}$ (this cone is not normal). Let $X = [0,\infty)$ and define a mapping $d : X \times X \to E$ by

$$d(x,y) = |x-y|^s\varphi, s = \{1,2\}$$

for all $x, y \in X$, where $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. Then (X,d) is a cone b-metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = (x+y)^s\varphi$ for all $x, y \in X$. Then q is a generalized c-distance. In fact, (q1) and (q3) are immediate. From

$$(x+z)^s\varphi = q(x,z) \leq s(x+y)^s\varphi + s(y+z)^s\varphi = sq(x,y) + sq(y,z),$$

it follows that (q2) holds. From

$$d(x,y) = |x-y|^s\varphi \leq s(x-z)^s\varphi + s(y-z)^s\varphi \leq s(x+z)^s\varphi + s(y+z)^s\varphi = sq(z,x) + sq(z,y),$$

it follows that (q4) holds.

Theorem 2.6. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone b-metric space. Let q be a generalized c-distance on X and $f : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) there exist $a, b > 0$ with $sa + sb < 1$ such that

$$q(fx,fy) \leq aq(x,y) + bq(x,fx),$$

for all $x, y \in X$ with $x \sqsubseteq y$.

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

(iii) if (x_n) is nondecreasing with respect to $\sqsubseteq$, and converges to x, we have $x_n \sqsubseteq x$ as $n \to \infty$.

Then f has a fixed point $x' \in X$. If $v = fv$, then $q(v,v) = \theta$.

Proof . If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. Since $x_0 \sqsubseteq fx_0$ and f is nondecreasing with respect to $\sqsubseteq$, we obtain by induction,

$$x_0 \sqsubseteq fx_0 = x_1 \sqsubseteq f^2x_0 = x_2 \sqsubseteq \cdots \sqsubseteq f^nx_0 = x_n \sqsubseteq f^{n+1}x_0 = x_{n+1} \sqsubseteq \cdots .$$

Since

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \leqslant aq(x_{n-1}, x_n) + bq(x_{n-1}, fx_{n-1}) \\ &= (a+b)q(x_{n-1}, x_n), \end{aligned}$$

we have $q(x_n, x_{n+1}) \leq hq(x_{n-1}, x_n) \leq \cdots \leq h^n q(x_0, x_1)$, where $h = a + b$, for all $n \geq 1$.
Let $m > n$. Then we have

$$\begin{aligned} q(x_n, x_m) &\leq sq(x_n, x_{n+1}) + sq(x_{n+1}, x_m) \\ &\leq sq(x_n, x_{n+1}) + (s^2q(x_{n+1}, x_{n+2}) + s^2q(x_{n+2}, x_m)) \\ &\leq sq(x_n, x_{n+1}) + s^2q(x_{n+1}, x_{n+2}) + \cdots + s^{m-n}q(x_{m-1}, x_m) \\ &\leq sh^nq(x_0, x_1) + s^2h^{n+1}q(x_0, x_1) + \cdots + s^{m-n}h^{m-1}q(x_0, x_1) \\ &= \tfrac{sh^n(1-(sh)^{m-n})}{1-sh} q(x_0, x_1) \\ &\leq \tfrac{sh^n}{1-sh} q(x_0, x_1), \end{aligned}$$

where $0 < sh = sa + sb < 1$, so $0 < h = a + b < 1$, we show that $\{x_n\}$ is a Cauchy sequence in X. In fact, let $c \in E$ with $\theta \ll c$ be give, since $\{\frac{sh^n}{1-sh} q(x_0, x_1)\}$ converges to θ, from Remark 1.4, then there exists a positive integer N such that $\frac{sh^n}{1-sh} q(x_0, x_1) \ll c$. for all $n \geq N$, hence choose $e = c$, then there exists a positive integer N for all $n \geq N$, such that

$$q(x_n, x_{n+1}) \ll e, q(x_n, x_m) \ll e,$$

for any $m > n > N$ and hence

$$d(x_{n+1}, x_m) \ll c.$$

Since X is complete, there exists a point $x' \in X$ such that $x_n \to x'$ as $n \to \infty$.

$$\begin{aligned} q(x_{n-1}, x_m) &\leq sq(x_{n-1}, x_n) + sq(x_n, x_m) \\ &\leq sh^{n-1}q(x_0, x_1) + \tfrac{s^2h^n}{1-sh} q(x_0, x_1), \end{aligned}$$

where $0 < h = a + b < 1$ for all $m > n > 1$, from (q3), it follows that

$$q(x_{n-1}, x') \leq s^2h^{n-1}q(x_0, x_1) + \frac{s^3h^n}{1-sh} q(x_0, x_1),$$

$$q(x_n, fx') = q(fx_{n-1}, fx') \leq aq(x_{n-1}, x') + bq(x_{n-1}, x_n).$$

For any $c \in E$ with $\theta \ll c$, there exists a positive integer n_0 such that $q(x_n, fx') \ll c$, for all $n \geq n_0$ and $q(x_n, x') \ll c$ as $n \to \infty$, by (q4) with $e = c$, it follows that $d(fx', x') \ll c$ as $n \to \infty$, this shows that $fx' = x'$.

Suppose that $v = fv$. Then we have

$$q(v, v) = q(fv, fv) \leq aq(v, v) + bq(v, fv) = (a + b)q(v, v),$$

since $a + b < 1$, we have $q(v, v) = \theta$. This completes the proof. □

Theorem 2.7. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be complete cone b-metric space. Let q be a generalized c-distance on X and $f : X \to X, g : X \to X$. be two weakly increasing mappings with respect to $\sqsubseteq$. Suppose that there exist $a, b > 0$ with $sa + sb < 1$ such that:*

(i)

$$q(fx, gy) \leq aq(x, y) + bq(x, fx)$$

and

$$q(gx, fy) \leq aq(x, y) + bq(x, gx)$$

for all comparable $x, y \in X$.

(ii) if (x_n) is nondecreasing with respect to $\sqsubseteq$, and converges to x, we have $x_n \sqsubseteq x$ as $n \to \infty$.

Then f and g have a common fixed point $x' \in X$. If $v = fv = gv$, then $q(v, v) = \theta$.

Proof . Let x_0 be an arbitrary point in X and define a sequence $\{x_n\}$ in X as follow:

$$x_{2n+1} = fx_{2n}, \quad x_{2n+2} = gx_{2n+1}$$

for all $n \geq 0$. Since f and g are weakly increasing, We have $x_1 = fx_0 \sqsubseteq gfx_0 = gx_1 = x_2$ and $x_2 = gx_1 \sqsubseteq fgx_1 = fx_2 = x_3$. Continuing this process, we have

$$x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots,$$

that is, x_n is nondecreasing. we have

$$\begin{aligned} q(x_{2n+1}, x_{2n+2}) &= q(fx_{2n}, gx_{2n+1}) \\ &\leq aq(x_{2n}, x_{2n+1}) + bq(x_{2n}, fx_{2n}) \\ &= aq(x_{2n}, x_{2n+1}) + bq(x_{2n}, x_{2n+1}) \\ &= (a+b)q(x_{2n}, x_{2n+1}), \end{aligned}$$

which implies that

$$q(x_{2n+1}, x_{2n+2}) \leq hq(x_{2n}, x_{2n+1}),$$

where $h = a + b < 1$. Similarly, it can be shown that

$$q(x_{2n+2}, x_{2n+3}) \leq hq(x_{2n+1}, x_{2n+2}).$$

Therefore, we have

$$q(x_n, x_{n+1}) \leq hq(x_{n-1}, x_n) \leq \cdots \leq h^n q(x_0, x_1).$$

Let $m > n$, as in the proof of Theorem 2.6, we have

$$q(x_n, x_m) \leq \frac{sh^n}{1 - sh} q(x_0, x_1),$$

so $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x' \in X$,such that $x_n \to x'$ as $n \to \infty$.
We have

$$\begin{aligned} q(x_{2n+1}, x_m) &\leq sq(x_{2n+1}, x_{2n+2}) + sq(x_{2n+2}, x_m) \\ &\leq sh^{2n+1} q(x_0, x_1) + \tfrac{s^2 h^{2n+2}}{1-sh} q(x_0, x_1), \end{aligned}$$

for all $m > n > 1$, where $0 < h = a + b < 1$.

From (q3), it follows that

$$q(x_{2n+1}, x') \leq s^2 h^{2n+1} q(x_0, x_1) + \frac{s^3 h^{2n+2}}{1 - sh} q(x_0, x_1).$$

Since

$$\begin{aligned} q(x_{2n+2}, fx') &= q(gx_{2n+1}, fx') \\ &\leq aq(x_{2n+1}, x') + bq(x_{2n+1}, gx_{2n+1}) \\ &= aq(x_{2n+1}, x') + bq(x_{2n+1}, x_{2n+2}), \end{aligned}$$

for any $c \in E$ with $\theta \ll c$, there exists a positive integer n_0 such that $q(x_{2n+2}, fx') \ll c$, for all $n \geq n_0$.

Since

$$q(x_{2n+1}, fx') \leq sq(x_{2n+1}, x_{2n+2}) + sq(x_{2n+2}, fx'),$$

we have
$$q(x_{2n+1}, fx') \ll c, \text{ as } n \to \infty$$
and
$$q(x_n, fx') \ll c \text{ as } n \to \infty, \quad q(x_n, x') \ll c \text{ as } n \to \infty.$$
By (q4) with $e = c$, it follows that $d(fx', x') \ll c$, as $n \to \infty$.
This shows that $fx' = x'$.

Since

$$\begin{aligned} q(x_{2n}, x_m) &\leq sq(x_{2n}, x_{2n+1}) + sq(x_{2n+1}, x_m) \\ &\leq sh^{2n} q(x_0, x_1) + \tfrac{s^2 h^{2n+1}}{1-sh} q(x_0, x_1), , \end{aligned}$$

for all $m > n > 1$, where $0 < h = a + b < 1$.

From (q3), it follows that

$$q(x_{2n}, x') \leq s^2 h^{2n} q(x_0, x_1) + \frac{s^3 h^{2n+1}}{1 - sh} q(x_0, x_1).$$

Since

$$\begin{aligned} q(x_{2n+1}, gx') &= q(fx_{2n}, gx') \leq aq(x_{2n}, x') + bq(x_{2n}, fx_{2n}) \\ &= aq(x_{2n}, x') + bq(x_{2n}, x_{2n+1}), \end{aligned}$$

we have
$$q(x_{2n+1}, gx') \ll c, \text{ as } n \to \infty.$$

Since

$$q(x_{2n}, gx') \leq sq(x_{2n}, x_{2n+1}) + sq(x_{2n+1}, gx'),$$

we have
$$q(x_{2n}, gx') \ll c, \text{ as } n \to \infty$$
and
$$q(x_n, gx') \ll c, \text{ as } n \to \infty, \quad q(x_n, x') \ll c, \text{ as } n \to \infty.$$

By (q4) with $e = c$, it follows that $d(gx', x') \ll c$, as $n \to \infty$. This shows that $gx' = x'$.

Suppose that $v = fv = gv$. Then we have

$$q(v, v) = q(fv, gv) \leq aq(v, v) + bq(v, fv) = (a + b)q(v, v).$$

Since $a + b < 1$, we have $q(v, v) = \theta$. This completes the proof. □

Remark 2.8. Compared to Theorem 3.3 in [7], Theorem 2.7 in this paper presents a method without the continuity of the mappings.

We give an example to illustrate Theorem 2.6.

Example 2.9. *Let $E=\mathbb{R}$, and $P=\{x\in E : x\geq 0\}$. Let $X=[0,1]$ and define a mapping $d: X\times X\to E, d(x,y)=|x-y|^2$ for all $x,y\in X$. Then (X,d) is a cone b-metric space. Define a mapping $q: X\times X\to E$ by $q(x,y)=y^2$ for all $x,y\in X$ and let an order relation $\sqsubseteq$ defined by $x\sqsubseteq y\Leftrightarrow x\leq y$. Then q is a generalized c-distance on X. If $f(x)=\frac{x^2}{4}$, for all $x\neq 1$ and $f(1)=\frac{1}{2}$. Let $a=\frac{1}{4}, b=\frac{1}{5}$, then f satisfies the assertion of Theorem 2.6. Moreover, 0 is a fixed point of f.*

Proof . Firstly, we prove (X,d) is a cone b-metric space.

(1) $d(x,y)=|x-y|^2\geq 0, d(x,y)=0\Leftrightarrow x=y$;

(2) $|x-z|^2\leq 2|x-y|^2+2|y-z|^2$, we have $d(x,z)\leq 2d(x,y)+2d(y,z)$;

(3) $d(x,y)=|x-y|^2=|y-x|^2=d(y,x)$.

Next, we prove q is a generalized c-distance on X.

(q1) $q(x,y)=y^2\geq 0$;

(q2) $z^2=q(x,z)\leq 2q(x,y)+2q(y,z)=2y^2+2z^2, .ie., q(x,z)\leq 2q(x,y)+2q(y,z)$;

(q3) is obvious;

(q4) $d(x,y)=|x-y|^2\leq x^2+y^2=q(z,x)+q(z,y)$.

Finally, we prove f satisfies the assertion of Theorem 2.6.

(i) If $x=y=1$, then we have

$$q(fx,fy)=q(\frac{1}{2},\frac{1}{2})=\frac{1}{4},\ aq(x,y)=\frac{1}{4}q(1,1)=\frac{1}{4},\ bq(x,fx)=\frac{1}{20},$$

we get $q(fx,fy)\leq aq(x,y)+bq(x,fx)$.

(ii) If $x\neq 1$ and $y=1$, then we have

$$q(fx,fy)=q(\frac{x^2}{4},\frac{1}{2})=\frac{1}{4},\ aq(x,y)=\frac{1}{4}q(x,1)=\frac{1}{4},\ bq(x,fx)=bq(x,\frac{x^2}{4})=\frac{x^4}{80},$$

we get $q(fx,fy)\leq aq(x,y)+bq(x,fx)$.

(iii) If $x\neq 1, y\neq 1$, then we have

$$q(fx,fy)=q(\frac{x^2}{4},\frac{y^2}{4})=\frac{y^4}{16},\ aq(x,y)=\frac{y^2}{4},\ bq(x,fx)=\frac{x^4}{80},$$

since $0\leq y<1$, we have $\frac{y^4}{16}\leq\frac{y^2}{4}$ and $q(fx,fy)\leq aq(x,y)+bq(x,fx)$.

□

Remark 2.10. (i) (X,d) in Example 2.9 is not only a cone metric space, but also a cone b-metric space.

(ii) The mapping f in Example 2.9 is not continuous.

Theorem 2.11. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone b-metric space. Let q be a generalized c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) there exist $a, b, c, m > 0$ with $sa + sb + c + (s^2 + s)m < 1$ such that

$$q(fx, fy) \le aq(x, y) + bq(x, fx) + cq(y, fy) + mq(x, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$.

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

Then f has a fixed point $x' \in X$. If $v = fv$, then $q(v, v) = \theta$.

Proof . If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. Since $x_0 \sqsubseteq fx_0$ and f is nondecreasing with respect to $\sqsubseteq$, we obtain by induction,

$$x_0 \sqsubseteq fx_0 = x_1 \sqsubseteq f^2x_0 = x_2 \sqsubseteq \cdots \sqsubseteq f^n x_0 = x_n \sqsubseteq f^{n+1}x_0 = x_{n+1} \sqsubseteq \cdots .$$

Now, we have

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \le aq(x_{n-1}, x_n) + bq(x_{n-1}, fx_{n-1}) + cq(x_n, fx_n) + mq(x_{n-1}, fx_n) \\ &= aq(x_{n-1}, x_n) + bq(x_{n-1}, x_n) + cq(x_n, x_{n+1}) + mq(x_{n-1}, x_{n+1}) \\ &\le aq(x_{n-1}, x_n) + bq(x_{n-1}, x_n) + cq(x_n, x_{n+1}) + smq(x_{n-1}, x_n) + smq(x_n, x_{n+1}) \\ &\le \tfrac{a+b+sm}{1-c-sm} q(x_{n-1}, x_n), \end{aligned}$$

we have $q(x_n, x_{n+1}) \le hq(x_{n-1}, x_n) \le \cdots \le h^n q(x_0, x_1)$, where $h = \frac{a+b+sm}{1-c-sm}$, for all $n \ge 1$.

Let $m > n$.Then we have

$$\begin{aligned} q(x_n, x_m) &\le sq(x_n, x_{n+1}) + sq(x_{n+1}, x_m) \\ &\le sq(x_n, x_{n+1}) + (s^2q(x_{n+1}, x_{n+2}) + s^2q(x_{n+2}, x_m)) \\ &\le sq(x_n, x_{n+1}) + s^2q(x_{n+1}, x_{n+2}) + \cdots + s^{m-n}q(x_{m-1}, x_m) \\ &\le sh^n q(x_0, x_1) + s^2h^{n+1}q(x_0, x_1) + \cdots + s^{m-n}h^{m-1}q(x_0, x_1) \\ &= \tfrac{sh^n(1-(sh)^{m-n})}{1-sh} q(x_0, x_1) \\ &\le \tfrac{sh^n}{1-sh} q(x_0, x_1), \end{aligned}$$

where $0 < a + b + c + 2sm < sa + sb + c + (s^2 + s)m < 1$, so $0 < h < 1$, and $0 < sh < 1$, we show that $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists a point $x' \in X$ such that $x_n \to x'$ as $n \to \infty$.

Finally, the continuity of f and $fx_{n-1} = x_n \to x'$ as $n \to \infty$ imply that $fx' = x'$. Thus we prove that x' is a fixed point of f.

Suppose that $v = fv$. Then we have $q(v, v) = q(fv, fv) \le aq(v, v) + bq(v, fv) + cq(v, fv) + mq(v, fv)$

$$= (a + b + c + m)q(v, v),$$

since $0 < a + b + c + d < sa + sb + c + (s^2 + s)d < 1$, we have $q(v, v) = \theta$. This completes the proof. □

From Theorem 2.11, we easily obtain the following result.

Corollary 2.12. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:

(i) there exist $a, b, c > 0$ with $a + b + c < 1$ such that

$$q(fx, fy) \preceq aq(x, y) + bq(x, fx) + cq(y, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$.

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

Then f has a fixed point $x' \in X$. If $v = fv$, then $q(v, v) = \theta$.

Remark 2.13. Theorem 2.11 is not only to give some generalized contractive condition of Theorem 3.1 in [7] but also to generalize the spaces.

Theorem 2.14. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone b-metric space and P is a normal cone with normal constant K. Let q be a generalized c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) there exist $a, b, c, m > 0$ with $sa + sb + c + (s^2 + s)m < 1$ such that

$$q(fx, fy) \preceq aq(x, y) + bq(x, fx) + cq(y, fy) + mq(x, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$.

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

(iii) $inf\{||q(x, y)|| + ||q(x, fx)|| : x \in X\} > 0$ for all $y \in X$ with $y \neq fy$.

Then f has a fixed point $x' \in X$. If $v = fv$, then $q(v, v) = \theta$.

Proof . If we take $x_n = f^n x_0$ in the proof of Theorem 2.6, then we have

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots .$$

Moreover, $\{x_n\}$ converges to a point $x' \in X$ and

$$q(x_n, x_m) \preceq \frac{sh^n}{1 - sh} q(x_0, x_1)$$

for all $m > n \geq 1$, where $h = \dfrac{a + b + sm}{1 - c - sm} < 1$. By (q3), we have

$$q(x_n, x') \preceq \frac{s^2 h^n}{1 - sh} q(x_0, x_1)$$

for all $n \geq 1$. Since P is a normal cone with normal constant K, we have

$$||q(x_n, x_m)|| \leq \frac{Ksh^n}{1 - sh} ||q(x_0, x_1)||$$

for all $m > n > 1$ and

$$||q(x_n, x')|| \leq \frac{Ks^2 h^n}{1 - sh} ||q(x_0, x_1)||$$

for all $n \geq 1$. If $x' \neq fx'$, then, by hypotheses, we have

$$\begin{aligned} 0 &< \inf\{||q(x,x')|| + ||q(x,fx)|| : x \in X\} \\ &\leq \inf\{||q(x_n,x')|| + ||q(x_n,x_{n+1})|| : n \geq 1\} \\ &\leq \inf\{\tfrac{Ks^2h^n}{1-sh}||q(x_0,x_1)|| + \tfrac{Ksh^n}{1-sh}||q(x_0,x_1)|| : n \geq 1\} \\ &= 0. \end{aligned}$$

This is a contradiction. Therefore, we have $x' = fx'$. Suppose that $v = fv$ holds. we can prove $q(v,v) = \theta$ by the final part of the proof of Theorem 2.11. This completes the proof. □

Corollary 2.15. Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space and P is a normal cone with normal constant K. Let q be a c-distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:

(i) there exist $a,b,c > 0$ with $a+b+c < 1$ such that

$$q(fx,fy) \leq aq(x,y) + bq(x,fx) + cq(y,fy)$$

for all $x,y \in X$ with $x \sqsubseteq y$.

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

(iii) $\inf\{||q(x,y)|| + ||q(x,fx)|| : x \in X\} > 0$ for all $y \in X$ with $y \neq fy$.

Then f has a fixed point $x' \in X$. If $v = fv$, then $q(v,v) = \theta$.

We give an example to illustrate Theorem 2.14.

Example 2.16. *Let $E = \mathbb{R}$, and $P = \{x \in E : x \geq 0\}$. Let $X = [0,1]$ and define a mapping $d : X \times X \to E, d(x,y) = |x-y|^2$ for all $x,y \in X$. Then (X,d) is a cone b-metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = y^2$ for all $x,y \in X$ and let an order relation $\sqsubseteq$ defined by $x \sqsubseteq y \Leftrightarrow x \leq y$. Then q is a generalized c-distance on X. If $f(x) = \frac{x^2}{4}$, for all $x \neq 1$ and $f(1) = \frac{1}{2}$. Let $a = \frac{1}{4}, b = c = d = \frac{1}{32}$, then f satisfies the assertion of Theorem* 2.14. *Moreover, 0 is a fixed point of f.*

Proof . Firstly, we prove (X,d) is a cone b-metric space.

(1) $d(x,y) = |x-y|^2 \geq 0, d(x,y) = 0 \Leftrightarrow x = y$;

(2) $|x-z|^2 \leq 2|x-y|^2 + 2|y-z|^2$, we have $d(x,z) \leq 2d(x,y) + 2d(y,z)$;

(3) $d(x,y) = |x-y|^2 = |y-x|^2 = d(y,x)$.

Next, we prove q is a generalized c-distance on X.

(q1) $q(x,y) = y^2 \geq 0$;

(q2) $z^2 = q(x,z) \leq 2q(x,y) + 2q(y,z) = 2y^2 + 2z^2, .ie., q(x,z) \leq 2q(x,y) + 2q(y,z)$;

(q3) is obvious;

(q4) $d(x,y) = |x-y|^2 \leq x^2 + y^2 = q(z,x) + q(z,y)$.

Finally, we prove f satisfies the assertion of Theorem 2.14.

(i) If $x = y = 1$, then we have

$$q(fx, fy) = q(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4},\ aq(x, y) = \frac{1}{4}q(1, 1) = \frac{1}{4},$$

we get $q(fx, fy) \leq aq(x, y) + bq(x, fx) + cq(y, fy) + dq(x, fy)$.

(ii) If $x \neq 1$ and $y = 1$, then we have

$$q(fx, fy) = q(\frac{x^2}{4}, \frac{1}{2}) = \frac{1}{4},\ aq(x, y) = \frac{1}{4}q(x, 1) = \frac{1}{4},$$

we get $q(fx, fy) \leq aq(x, y) + bq(x, fx) + cq(y, fy) + dq(x, fy)$.

(iii) If $x \neq 1, y \neq 1$, then we have

$$q(fx, fy) = q(\frac{x^2}{4}, \frac{y^2}{4}) = \frac{y^4}{16},\ aq(x, y) = \frac{y^2}{4},$$

since $0 \leq y < 1$, we have $\frac{y^4}{16} \leq \frac{y^2}{4}$ and $q(fx, fy) \leq aq(x, y) + bq(x, fx) + cq(y, fy) + dq(x, fy)$.
Finally, for any $x, y \in E$ with $y \neq Ty$, i.e., $y > 0$, we get

$$\inf\{||q(x, y)|| + ||q(x, fx)|| : x \in X\} = y^2 > 0.$$

□

3. Applications

As an application of Theorem 2.6, we will present the existence of solution of an integral equation.

Let $X = C(I, \mathbb{R}^n), E = \mathbb{R}^n, P = \{(x_1, x_2, \cdots, x_n) : x_i \geq 0, i = 1, \cdots, n\}$, and define $d : X \times X \to E$ by $d(x, y) = \{d(x, y)_i\}_{i=1}^n, d(x, y)_i = \sup\limits_{t\in I} |x(t) - y(t)|^2, i = 1, \cdots, n$, for every $x, y \in X$. Then (X, d) is a cone b-metric space and $s = 2$. Define a mapping $q : X \times X \to E$ by $q(x, y) = \{q(x, y)_i\}_{i=1}^n, q(x, y)_i = \sup\limits_{t\in I} |y(t)|^2, i = 1, \cdots, n$, for every $x, y \in X$. and let an order relation $\sqsubseteq$ defined by $x \sqsubseteq y \Leftrightarrow \sup\limits_{t\in I} |x(t)| \leq \sup\limits_{t\in I} |y(t)|$. Then q is a generalized c-distance on X.

Theorem 3.1. *Let I be the closed unit interval $[0, 1]$ in $\mathbb{R}$. Consider the following integral equation*

$$x(t) = \int_0^t g(s, x(s))ds, \quad t \in I. \tag{3.1}$$

where $g : I \times \mathbb{R}^n \to \mathbb{R}^n$ is such that $g(s, \cdot)$ is increasing for every $s \in I$.
Suppose that the following assertion hold:

$$\{(|g(s, y)_i|)\}_{i=1}^n \leq \frac{1}{2}\{|y(s)|, \cdots, |y(s)|\},$$

for every $s \in I, x, y \in X$. Then the integral equation (3.1) has a solution in $C(I, \mathbb{R}^n)$.

Proof . Define $T : X \to X$ by

$$Tx(t) = \int_0^t g(s, x(s))ds, \quad x \in X.$$

For each $x, y \in X$, we have

$$\begin{aligned} q(Tx, Ty) &= (\sup_{t\in I} |[Ty](t)|^2, \cdots, \sup_{t\in I} |[Ty](t)|^2) \\ &\leq (\sup_{t\in I}(\int_0^t |g(s, y(s)|ds)^2, \cdots, \sup_{t\in I}(\int_0^t |g(s, y(s)|ds)^2) \\ &\leq (\sup_{t\in I}(\int_0^t \tfrac{1}{2}|y(s)|ds)^2, \cdots, \sup_{t\in I}(\int_0^t \tfrac{1}{2}|y(s)|ds)^2) \\ &\leq (\tfrac{1}{4} \sup_{t\in I}(\int_0^t |y(s)|^2 ds), \cdots, \tfrac{1}{4} \sup_{t\in I}(\int_0^t |y(s)|^2 ds)) \\ &\leq (\tfrac{1}{4} \sup_{t\in I}(\int_0^t \sup |y(s)|^2 ds), \cdots, \tfrac{1}{4} \sup_{t\in I}(\int_0^t \sup |y(s)|^2 ds)) \\ &\leq \tfrac{1}{4}(\sup |y(s)|^2, \cdots, \sup |y(s)|^2) \sup_{t\in I} \int_0^t 1ds \\ &\leq \tfrac{1}{4}q(x, y) + \tfrac{1}{5}q(x, fx). \end{aligned}$$

Then according to Theorem 2.6, the integral equation (3.1) has a solution. □

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References

[1] S. M. Abusalim and M. S. Noorani, *Fixed point and common fixed point theorems on ordered cone b-metric spaces*, Abstr. Appl. Anal., 2013, Article ID 815289, 7 pages.

[2] M. Akkouchi, *Common fixed point theorems for two self mappings of a b-metric space under an implicit relation*, Hacet. J. Math. Stat. 40 (2011) 805–810. MR2952181.

[3] H. Aydi, M. F. Bota, E. Karainar and S. Moradi, *A common fixed point for weak $\varphi-$contractions on b-metric spaces*, Fixed Point Theory 13 (2012) 337–346. MR3024322.

[4] M. Boriceanu, M. Bota and A. Petrusel, *Mutivalued fractals in b-metric spaces*, Cen. Eur. J. Math. 8 (2010) 367–377.

[5] M. Bota, A. Molnar and V. Csaba, *On Ekeland's variational principle in b-metric spaces*, Fixed Point Theory 12 (2011) 21–28.

[6] S.-H. Cho and J.-S. Bae, *Common fixed point theorems for mappings satisfying property (E. A) on cone metric space*, Math. Comput. Model. 53 (2011) 945–951.

[7] Y.J. Cho, R. Saadati and S. Wang, *Common fixed point theorems on generalized distance in ordered cone metric spaces*, J. Comput. Math. Appl. 61 (2011) 1254–1260.

[8] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis 1 (1993) 5–11.

[9] M. Djordjević, D. Djorić, Z. Kadelburg, S. Radenović and D. Spasić, *Fixed point results under c-distance in tvs-cone metric spaces*, Fixed Point Theory Appl. 2011, 2011:29.

[10] Z.M. Fadail, A.G.B. Ahmad and S. Radenović, *Common fixed point and fixed point results under c-distance in cone metric spaces*, Appl. Math. Inf. Sci. Lett. 1 (2013) 47–52.

[11] H. Huang and S. Xu, *Fixed point theorems of contractive mappings in cone b-metric spaces and applications*, Fixed Point Theory Appl. 2013, 2013: 112.

[12] L.G. Huang and X. Zhang, *Cone metric space and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 332 (2007) 1468–1476.

[13] N. Hussain and M. H. Shah, *KKM mapping in cone b-metric spaces*, J. Comput. Math. Appl.62 (2011) 1677–1684.
[14] S. Janković, Z. Kadelburg and S. Radenović, *On cone metric spaces: A survey*, Nonlinear Anal. 74 (2011) 2591–2601.
[15] M. Jovanović, Z. Kadelburg and S. Radenović, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl., Volume 2010, Article ID 978121.
[16] Z. Kadelburg, S. Radenović and Vladimir Rakočević, *A note on the equivalence of some metric and cone metric fixed point results*, Appl. Math. Lett. 24 (2011) 370–374.
[17] A. Razani, V. Rakočević and Zahra Goodarzi, *Generalized $\varphi-$contraction for a pair of mappings on cone metric spaces*, Appl. Math. Comput. 217 (2011) 8899–8906.
[18] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper"Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl. 345 (2008) 719–724.