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Bernstein's polynomials for convex functions and related results

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Abstract

In this paper we establish several polynomials similar to Bernstein's polynomials and several refinements of Hermite-Hadamard inequality for convex functions.

Keywords: Hermite-Hadamard inequality; Convex functions; Bernstein's polynomials.
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1. Introduction

Let us assume that the function f is continuous on $[0,1]$. Bernstein's polynomials of order $n = 0,1,2,\dots$ of the function f is defined by

$$B_n(f) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n}) \tag{1.1}$$

It is a well known fact that the sequence $\{B_n(f)\}$ converges uniformely to $f(x)$ as $n \to \infty$. A systematic study of Bernstein's polynomials of convex function was first made by Popoviciu (1961). Temple (1954) proved that a continuous function f is convex iff for every $n = 0,1,\dots$

$$B_{n+1}(f) \leq B_n(f)$$

and Arama (1960) proved that a continuous function f is convex iff, $f(x) \leq B_n(f)$ for every $x \in [0,1]$. For historical backround see [3].

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Let $f:[a,b]\to\mathbb{R}$ be a convex function, then the inequality

$$f(\frac{a+b}{2})\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.2}$$

is known as the Hermite-Hadamard inequality. In [6] the author obtained a new refinement of the Hermite-Hadamard inequality.

Theorem 1.1. Let f be a convex function on $[a,b]$. Then we have

$$f(\frac{a+b}{2})\leq x_n\leq\frac{1}{b-a}\int_a^b f(x)dx\leq y_n\leq\frac{f(a)+f(b)}{2}$$

where

$$x_n=\frac{1}{2^n}\sum_{i=1}^{2^n} f\left(a+(i-\frac{1}{2})\frac{b-a}{n}\right),$$
$$y_n=\frac{1}{2^{n+1}}\sum_{i=1}^{2^n}\left[f((1-\frac{i}{2^n})a+\frac{i}{2^n}b)+f((1-\frac{i-1}{2^n})a+\frac{i-1}{2^n}b)\right]$$

If we use the similar technic used in Theorem 1.1, we conclude that

$$\begin{aligned}
f(\frac{a+b}{2})&\leq\frac{1}{n+1}\sum_{k=0}^{n} f\left(a+\frac{2k+1}{2n+2}(b-a)\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\\
&\leq\frac{1}{2(n+1)}\sum_{k=0}^{n}[f(a+\frac{k+1}{n+1}(b-a))+f(a+\frac{k}{n+1}(b-a))]\\
&=\frac{f(a)+f(b)}{2(n+1)}+\frac{1}{n+1}\sum_{k=1}^{n} f(a+\frac{k}{n+1}(b-a))\\
&\leq\frac{f(a)+f(b)}{2}.
\end{aligned} \tag{1.3}$$

Remember that the Beta Integral is defined by

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx(a>0,b>0)$$

This integral converges for $a>0,b>0$ and we have

$$B(a,b)=\frac{(a-1)!(b-1)!}{(a+b-1)!}.$$

In this paper we establish several polynomials similar to Bernstein's polynomials for convex function. In addition we obtain several refinements of Hermite-Hadamard inequality via these polynomials and we comare some of refinements.

2. Main results

Lemma 2.1. *For all a, b and $x \in \mathbb{R}$ the following identities hold:*

$$(1) \quad \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$$

$$(2) \quad \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} = x$$

$$(3) \quad \sum_{k=0}^{n} \binom{n}{k} (\frac{x-a}{b-a})^k (\frac{b-x}{b-a})^{n-k} (a + \frac{k}{n}(b-a)) = x$$

$$(4) \quad \sum_{k=0}^{n} \binom{n}{k} (\frac{x-a}{b-a})^k (\frac{b-x}{b-a})^{n-k} (a + \frac{k+1}{n+1}(b-a)) = \frac{n}{n+1}x + \frac{b}{n+1}$$

$$(5) \quad \sum_{k=0}^{n} \binom{n}{k} (\frac{x-a}{b-a})^k (\frac{b-x}{b-a})^{n-k} (a + \frac{k}{n+1}(b-a)) = \frac{n}{n+1}x + \frac{a}{n+1}$$

$$(6) \quad \sum_{k=0}^{n} \binom{n}{k} (\frac{x-a}{b-a})^k (\frac{b-x}{b-a})^{n-k} (a + \frac{2k+1}{2n+2}(b-a)) = \frac{n}{n+1}x + \frac{a+b}{2(n+1)}$$

Proof . (1) is obvious by binomial theorem

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = [x + (1-x)]^n = 1$$

For the proof of (2) by differentiating (1), we get

$$\sum_{k=0}^{n} \binom{n}{k} [kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1}] = \sum_{k=0}^{n} \binom{n}{k} x^{k-1}(1-x)^{n-k-1}(k - nx) = 0$$

Multiplication by $x(1-x)$ we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (k - nx) = 0$$

Hence

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (\frac{k}{n} - x) = 0$$

By using (1), we obtain

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \frac{k}{n} = x.$$

For the proof of (3) substitude x by $\frac{x-a}{b-a}$ in (2),

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k} \frac{k}{n} = \frac{x-a}{b-a}$$

Thus,

$$(b-a)\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x-a}{b-a}\right)^{k}\left(\frac{b-x}{b-a}\right)^{n-k}\frac{k}{n}+a=x$$

By using (1), we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x-a}{b-a}\right)^{k}\left(\frac{b-x}{b-a}\right)^{n-k}(b-a)\frac{k}{n}+a\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x-a}{b-a}\right)^{k}\left(\frac{b-x}{b-a}\right)^{n-k}=x$$

so

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x-a}{b-a}\right)^{k}\left(\frac{b-x}{b-a}\right)^{n-k}(a+\frac{k}{n}(b-a))=x.$$

For the proof of (4) by using (1) and (3) we have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^{k}(\frac{b-x}{b-a})^{n-k}(a+\frac{k+1}{n+1}(b-a))\\
=&a+\frac{n}{n+1}\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^{k}(\frac{b-x}{b-a})^{n-k}\frac{k+1}{n}(b-a)\\
=&a+\frac{n}{n+1}\left[\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^{k}(\frac{b-x}{b-a})^{n-k}(a+\frac{k}{n}(b-a))+(\frac{1}{n}(b-a)-a)\right]\\
=&a+\frac{n}{n+1}x+\frac{n}{n+1}(\frac{1}{n}b-\frac{1}{n}a-a)=\frac{n}{n+1}x+\frac{b}{n+1}
\end{aligned}$$

The proofs of (5) and (6) are similar to the proof of (4) and can be omitted. □

In the following theorem, when f is convex on $[a,b]$, we obtain polynomials similar to the Bernstein's polynomials that converge uniformly to $f(x)$ on $[a,b]$.

Theorem 2.2. *Let f be a convex function on $[a,b]$. Then we have*

$$\begin{aligned}
&(1)\quad f(x)\le\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^{k}(\frac{b-x}{b-a})^{n-k}f(a+\frac{k}{n}(b-a))=B_n(f)\\
&(2)\quad f(\frac{n}{n+1}x+\frac{b}{n+1})\le\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^{k}(\frac{b-x}{b-a})^{n-k}f(a+\frac{k+1}{n+1}(b-a))=C_n(f)\\
&(3)\quad f(\frac{n}{n+1}x+\frac{a}{n+1})\le\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^{k}(\frac{b-x}{b-a})^{n-k}f(a+\frac{k}{n+1}(b-a))=D_n(f)\\
&(4)\quad f(\frac{n}{n+1}x+\frac{a+b}{2(n+1)})\le\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^{k}(\frac{b-x}{b-a})^{n-k}f(a+\frac{2k+1}{2n+2}(b-a))=E_n(f)
\end{aligned}$$

and $\{B_n(f)\}$, $\{C_n(f)\}$, $\{D_n(f)\}$ and $\{E_n(f)\}$ converge uniformly on $[a,b]$ to $f(x)$ as $n\to\infty$.

Proof . Since f is convex and $\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^{k}(\frac{b-x}{b-a})^{n-k}=1$, (1), (2), (3) and (4) are obvious by Lemma 2.1. $B_n(f)$ is the Bernstein's polynomials and it is a well known fact that $\{B_n(f)\}$ converges uniformly to $f(x)$ as $n \to \infty$. We have

$$\begin{aligned} f(\frac{n}{n+1}x+\frac{b}{n+1}) \leq C_n(f) &= \sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{n-k}f(\frac{n}{n+1}(a+\frac{k}{n}(b-a))+\frac{1}{n+1}b) \\ &\leq \sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{n-k}[\frac{n}{n+1}f(a+\frac{k}{n}(b-a))+\frac{1}{n+1}f(b)] \\ &= \frac{n}{n+1}B_n(f)+\frac{f(b)}{n+1} \end{aligned}$$

so

$$f(\frac{n}{n+1}x+\frac{b}{n+1}) \leq C_n(f) \leq \frac{n}{n+1}B_n(f)+\frac{f(b)}{n+1}$$

since

$$\lim_{n\to\infty} f(\frac{n}{n+1}x+\frac{b}{n+1}) = \lim_{n\to\infty}[\frac{n}{n+1}B_n(f)+\frac{f(b)}{n+1}] = f(x),$$

so $\{C_n(f)\}$ converges uniformly on $[a,b]$ to $f(x)$.

By (3) we have

$$\begin{aligned} f(\frac{n}{n+1}x+\frac{a}{n+1}) \leq D_n(f) &= \sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{n-k}f(\frac{n}{n+1}(a+\frac{k}{n}(b-a))+\frac{a}{n+1}) \\ &\leq \frac{n}{n+1}B_n(f)+\frac{f(a)}{n+1} \end{aligned}$$

so $\{D_n(f)\}$ converges uniformly to $f(x)$.

By (4) we have

$$\begin{aligned} f(\frac{n}{n+1}x+\frac{a+b}{2(n+2)}) \leq E_n(f) &= \sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{n-k}f(\frac{n}{n+1}(a+\frac{k}{n}(b-a))+\frac{a+b}{2(n+1)}) \\ &\leq \frac{n}{n+1}B_n(f)+\frac{1}{n+1}f(\frac{a+b}{2}) \end{aligned}$$

so $\{E_n(f)\}$ converges uniformly on $[a,b]$ to $f(x)$. □

In the following theorems we obtain several refinements of Hermite-Hadamard inequality by integrals inequalities and Bernstein's polynomials.

Theorem 2.3. *Let f be a convex function on $[a,b]$. Then the following inequalites hold:*

$$\begin{aligned} (1)\quad f(\frac{a+b}{2}) &\leq \frac{1}{b-a}\int_a^b f(\frac{n}{n+1}x+\frac{a+b}{2n+2})dx \\ &\leq \frac{1}{2(b-a)}[\int_a^b f(\frac{n}{n+1}x+\frac{a}{n+1})dx+\int_a^b f(\frac{n}{n+1}x+\frac{b}{n+1})dx] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \end{aligned}$$

$$\begin{aligned} (2)\quad \frac{1}{b-a}\int_a^b f(x)dx &\leq \frac{1}{b-a}\int_a^b \frac{b-x}{b-a}f(\frac{n}{n+1}x+\frac{a}{n+1})dx+\frac{1}{b-a}\int_a^b \frac{x-a}{b-a}f(\frac{n}{n+1}x+\frac{b}{n+1})dx \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

Proof . (1) By Jensen's inequality we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(\frac{n}{n+1}x+\frac{a+b}{2n+2})dx &\geq f(\frac{1}{b-a}\int_a^b (\frac{n}{n+1}x+\frac{a+b}{2n+2}))dx\\ &= f(\frac{1}{b-a}[\frac{n}{2n+2}x^2+\frac{a+b}{2n+2}x|_a^b])\\ &= f(\frac{a+b}{2}).\end{aligned}$$

On the other hand we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(\frac{n}{n+1}x+\frac{a+b}{2n+2})dx &= \frac{1}{b-a}\int_a^b f[\frac{1}{2}(\frac{n}{n+1}x+\frac{a}{n+1})+\frac{1}{2}(\frac{n}{n+1}x+\frac{b}{n+1})]dx\\ &\leq \frac{1}{2(b-a)}\int_a^b f(\frac{n}{n+1}x+\frac{a}{n+1})dx+\frac{1}{2(b-a)}\int_a^b f(\frac{n}{n+1}x+\frac{b}{n+1})dx\end{aligned}$$

Now we prove that

$$\frac{1}{2(b-a)}[\int_a^b f(\frac{n}{n+1}x+\frac{a}{n+1})dx+\int_a^b f(\frac{n}{n+1}x+\frac{b}{n+1})dx] \leq \frac{1}{b-a}\int_a^b f(x)dx.$$

Let

$$F(x)=\frac{1}{2}\int_a^x f(\frac{n}{n+1}t+\frac{a}{n+1})dt+\frac{1}{2}\int_a^x f(\frac{n}{n+1}t+\frac{b}{n+1})dx-\int_a^x f(t)dt$$

By change of variable we get

$$F(x)=\frac{n+1}{2n}[\int_a^{\frac{nx+a}{n+1}} f(t)dt+\int_{\frac{an+x}{n+1}}^x f(t)dt]-\int_a^x f(t)dt$$

By differentiating, we obtain

$$\begin{aligned}F'(x) &= \frac{n+1}{2n}[\frac{n}{n+1}f(\frac{nx+a}{n+1})+f(x)-\frac{1}{n+1}f(\frac{an+x}{n+1})]-f(x)\\ &= \frac{1-n}{2n}f(x)+\frac{1}{2}f(\frac{nx+a}{n+1})-\frac{1}{2n}f(\frac{an+x}{n+1})\end{aligned} \tag{2.1}$$

On the other hand, since $\frac{na+x}{n+1}\leq\frac{nx+a}{n+1}\leq x$ and f is convex, we have

$$\frac{f(\frac{nx+a}{n+1})-f(\frac{na+x}{n+1})}{\frac{nx+a}{n+1}-\frac{na+x}{n+1}} \leq \frac{f(x)-f(\frac{nx+a}{n+1})}{x-\frac{nx+a}{n+1}}$$

Hence

$$f(\frac{nx+a}{n+1})\leq\frac{1}{n}f(\frac{na+x}{n+1})+\frac{n-1}{n}f(x) \tag{2.2}$$

From (2.1) and (2.2) we deduce that $F'(x)\leq 0$. So F is decreasing on $[a,b]$ and $F(b)\leq F(a)$. Thus

$$\frac{1}{2}\int_a^b f(\frac{n}{n+1}x+\frac{a}{n+1})dx+\frac{1}{2}\int_a^b f(\frac{n}{n+1}x+\frac{b}{n+1})dx\leq\int_a^b f(x)dx$$

For the proof of (2) we have

$$f(x)=f[\frac{b-x}{b-a}(\frac{n}{n+1}x+\frac{a}{n+1})+\frac{x-a}{b-a}(\frac{n}{n+1}x+\frac{b}{n+1})]$$
$$\leq \frac{b-x}{b-a}f(\frac{n}{n+1}x+\frac{a}{n+1})+\frac{x-a}{b-a}f(\frac{n}{n+1}x+\frac{b}{n+1})$$

By convexity of f and inequality 1.2 we obtain

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{b-a}\int_a^b \frac{b-x}{b-a}f(\frac{n}{n+1}x+\frac{a}{n+1})dx+\frac{1}{b-a}\int_a^b \frac{x-a}{b-a}f(\frac{n}{n+1}x+\frac{b}{n+1})dx$$
$$\leq \frac{1}{b-a}\int_a^b \frac{b-x}{b-a}[\frac{n}{n+1}f(x)+\frac{1}{n+1}f(a)]dx+\frac{1}{b-a}\int_a^b \frac{x-a}{b-a}[\frac{n}{n+1}f(x)+\frac{1}{n+1}f(b)]dx$$
$$=\frac{n}{(b-a)^2(n+1)}\int_a^b (b-x+x-a)f(x)dx+\frac{1}{(b-a)^2(n+1)}\int_a^b [(b-x)f(a)+(x-a)f(b)]\,dx$$
$$=\frac{n}{(b-a)(n+1)}\int_a^b f(x)dx+\frac{1}{(b-a)^2(n+1)}[(bf(a)-af(b))(b-a)+\frac{f(b)-f(a)}{2}(b^2-a^2)]$$
$$\leq \frac{n}{n+1}\frac{f(a)+f(b)}{2}+\frac{bf(a)-af(b)}{(n+1)(b-a)}+\frac{(f(b)-f(a))(a+b)}{2(n+1)(b-a)}$$
$$=\frac{f(a)+f(b)}{2}.$$

□

Theorem 2.4. *Let f be a convex function on $[a,b]$. Then the following inequalities hold:*

(1) $$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{n+1}\sum_{k=0}^{n}(a+\frac{k}{n}(b-a)) \leq \frac{1}{n+1}\sum_{k=0}^{n}[\frac{k}{n}f(a+\frac{k+1}{n+1}(b-a))+$$
$$+\frac{n-k}{n}f(a+\frac{k}{n+1}(b-a))] \leq \frac{f(a)+f(b)}{2}$$

(2) $$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{b-a}\int_a^b \frac{b-x}{b-a}f(\frac{n}{n+1}x+\frac{a}{n+1})dx+\frac{1}{b-a}\int_a^b \frac{x-a}{b-a}f(\frac{n}{n+1}x+\frac{b}{n+1})dx$$
$$\leq \frac{1}{(n+2)(n+1)}\left[\sum_{k=0}^{n}(n-k+1)f(a+\frac{k}{n+1}(b-a))+\sum_{k=0}^{n}(k+1)f(a+\frac{k+1}{n+1}(b-a))\right]$$
$$\leq \frac{f(a)+f(b)}{2}.$$

Proof . (1) By integrating from (1) of Theorem 2.2 we have

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{b-a}\int_a^b [\sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{n-k}f(a+\frac{k}{n}(b-a))]dx$$
$$=\sum_{k=0}^{n}\binom{n}{k}f(a+\frac{k}{n}(b-a))\frac{1}{b-a}\int_a^b (\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{n-k}dx$$

On the other hand we have

$$\frac{1}{b-a}\int_a^b (\frac{x-a}{b-a})^k (\frac{b-x}{b-a})^{n-k} dx = \int_0^1 t^k (1-t)^{n-k} dt = B(k+1, n-k+1)$$
$$= \frac{k!(n-k)!}{(n+1)!} = \frac{1}{(n+1)\binom{n}{k}}$$

so

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \sum_{k=0}^{n} \binom{n}{k} f(a+\frac{k}{n}(b-a)) \frac{1}{(n+1)\binom{n}{k}} = \frac{1}{n+1}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a))$$

For the second part of (1) we have

$$\frac{1}{n+1}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a)) \leq \frac{1}{n+1}\sum_{k=0}^{n} f[\frac{k}{n}(a+\frac{k+1}{n+1}(b-a)) + (1-\frac{k}{n})(a+\frac{k}{n+1}(b-a))]$$
$$\leq \frac{1}{n+1}\sum_{k=0}^{n}[\frac{k}{n} f(a+\frac{k+1}{n+1}(b-a)) + (1-\frac{k}{n}) f(a+\frac{k}{n+1}(b-a))]$$
$$= \frac{1}{n+1}\sum_{k=0}^{n} \frac{k}{n} f(a(1-\frac{k+1}{n+1}) + \frac{k+1}{n+1}b) + (1-\frac{k}{n}) f(a(1-\frac{k}{n+1}) + b\frac{k}{n+1})$$
$$\leq \frac{1}{n+1}\sum_{k=0}^{n} \frac{k}{n}[(1-\frac{k+1}{n+1}) f(a) + \frac{k+1}{n+1} f(b)] + (1-\frac{k}{n})[(1-\frac{k}{n+1}) f(a) + \frac{k}{n+1} f(b)]$$
$$= \frac{1}{n+1}\sum_{k=0}^{n}[\frac{n(n+1)-k(n+1)}{n(n+1)} f(a) + \frac{k(n+1)}{n(n+1)} f(b)]$$
$$= \frac{f(a)+f(b)}{2}.$$

The first part of (2) in proved in Theorem 2.3 (2). For the second part, by using Lemma 2.1 we get

$$\frac{1}{b-a}\int_a^b \frac{b-x}{b-a} f(\frac{n}{n+1}x + \frac{a}{n+1})dx + \frac{1}{b-a}\int_a^b \frac{x-a}{b-a} f(\frac{n}{n+1}x + \frac{b}{n+1})dx$$
$$\leq \frac{1}{b-a}\int_a^b \sum_{k=0}^{n} \binom{n}{k} (\frac{x-a}{b-a})^k (\frac{b-x}{b-a})^{n-k+1} f(a+\frac{k}{n+1}(b-a))dx$$
$$+ \frac{1}{b-a}\int_a^b \sum_{k=0}^{n} \binom{n}{k} (\frac{x-a}{b-a})^{k+1} (\frac{b-x}{b-a})^{n-k} f(a+\frac{k+1}{n+1}(b-a))dx$$
$$= \sum_{k=0}^{n} \binom{n}{k} f(a+\frac{k}{n+1}(b-a)) \frac{1}{b-a}\int_a^b (\frac{x-a}{b-a})^k (\frac{b-x}{b-a})^{n-k+1} dx$$
$$+ \sum_{k=0}^{n} \binom{n}{k} f(a+\frac{k+1}{n+1}(b-a)) \frac{1}{b-a}\int_a^b (\frac{x-a}{b-a})^{k+1} (\frac{b-x}{b-a})^{n-k} dx.$$

So, we have

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b \frac{b-x}{b-a} f(\frac{n}{n+1}x+\frac{a}{n+1})dx+\frac{1}{b-a}\int_a^b \frac{x-a}{b-a} f(\frac{n}{n+1}x+\frac{b}{n+1})dx\\
&\leq \sum_{k=0}^{n}\binom{n}{k} f(a+\frac{k}{n+1}(b-a))\int_0^1 t^k(1-t)^{n-k+1}dt\\
&+\sum_{k=0}^{n}\binom{n}{k} f(a+\frac{k+1}{n+1}(b-a))\int_0^1 t^{k+1}(1-t)^{n-k}dt\\
&=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!} f(a+\frac{k}{n+1}(b-a))\frac{k!(n-k+1)!}{(n+2)!}\\
&+\sum_{k=0}^{n}\frac{n!}{k!(n-k)!} f(a+\frac{k+1}{n+1}(b-a))\frac{(k+1)!(n-k)!}{(n+2)!}\\
&=\sum_{k=0}^{n}\frac{n-k+1}{(n+2)(n+1)} f(a+\frac{k}{n+1}(b-a))+\sum_{k=0}^{n}\frac{k+1}{(n+2)(n+1)} f(a+\frac{k+1}{n+1}(b-a))\\
&=\sum_{k=0}^{n}\frac{n-k+1}{(n+2)(n+1)} f(a(1-\frac{k}{n+1})+\frac{k}{n+1}b)+\sum_{k=0}^{n}\frac{k+1}{(n+2)(n+1)} f(a(1-\frac{k+1}{n+1})+\frac{k+1}{n+1}b)\\
&\leq \sum_{k=0}^{n}\frac{n-k+1}{(n+2)(n+1)}[(1-\frac{k}{n+1})f(a)+\frac{k}{n+1}f(b)]+\sum_{k=0}^{n}\frac{k+1}{(n+2)(n+1)}[(1-\frac{k+1}{n+1})f(a)+\frac{k+1}{n+1}f(b)]\\
&=\sum_{k=0}^{n}\left[\frac{(n-k+1)^2}{(n+2)(n+1)^2}+\frac{(k+1)(n-k)}{(n+2)(n+1)^2}\right] f(a)+\sum_{k=0}^{n}[\frac{k(n-k+1)}{(n+2)(n+1)^2}+\frac{(k+1)^2}{(n+2)(n+1)^2}]f(b)\\
&=\frac{f(a)+f(b)}{2}.
\end{aligned}
$$

□

In the following theorem we compare some of refinements.

Theorem 2.5. *Let f be a convex function on $[a,b]$. Then we have*

$$
\text{(1)}\quad \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2(n+1)}+\frac{1}{n+1}\sum_{k=1}^{n} f(a+\frac{k}{n+1}(b-a))
$$
$$
\leq \frac{1}{n+1}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a)) \leq \frac{f(a)+f(b)}{2}.
$$

$$
\text{(2)}\quad f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(\frac{n}{n+1}x+\frac{a+b}{2})dx \leq \tfrac{1}{n+1}\textstyle\sum_{k=0}^{n} f(a+\frac{2k+1}{2n+2}(b-a))
$$
$$
\leq \tfrac{1}{b-a}\textstyle\int_a^b f(x)dx.
$$

Proof . (1) We have

$$\frac{1}{n+1}\sum_{k=1}^{n} f(a+\frac{k}{n+1}(b-a)) = \frac{1}{n+1}\sum_{k=1}^{n} f[\frac{k}{n+1}(a+\frac{k-1}{n}(b-a)) + (1-\frac{k}{n+1})(a+\frac{k}{n}(b-a))]$$
$$\leq \frac{1}{n+1}\sum_{k=1}^{n}[\frac{k}{n+1}f(a+\frac{k-1}{n}(b-a)) + (1-\frac{k}{n+1})f(a+\frac{k}{n}(b-a))]$$
$$= \frac{1}{n+1}[\sum_{k=1}^{n}\frac{k}{n+1}f(a+\frac{k-1}{n}(b-a)) + \sum_{k=1}^{n}(1-\frac{k}{n+1})f(a+\frac{k}{n}(b-a))$$
$$= \frac{1}{n+1}[\frac{f(a)}{n+1} + \sum_{k=2}^{n}\frac{k}{n+1}f(a+\frac{k-1}{n}(b-a)) + \sum_{k=1}^{n-1}(1-\frac{k}{n+1})f(a+\frac{k}{n}(b-a)) + \frac{f(b)}{n+1}]$$
$$= \frac{f(a)+f(b)}{(n+1)^2} + \frac{1}{n+1}[\sum_{k=1}^{n-1}\frac{k+1}{n+1}f(a+\frac{k}{n}(b-a)) + \sum_{k=1}^{n-1}\frac{n+1-k}{n+1}f(a+\frac{k}{n}(b-a))]$$
$$= \frac{f(a)+f(b)}{(n+1)^2} + \frac{1}{n+1}\sum_{k=1}^{n-1}\frac{n+2}{n+1}f(a+\frac{k}{n}(b-a))$$
$$= \frac{f(a)+f(b)}{(n+1)^2} + \frac{n+2}{(n+1)^2}[\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a) - f(a) - f(b)]$$
$$= -\frac{f(a)+f(b)}{n+1} + \frac{n+2}{(n+1)^2}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a))$$

So

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2(n+1)} + \frac{1}{n+1}\sum_{k=1}^{n} f(a+\frac{k}{n+1}(b-a))$$
$$\leq \frac{f(a)+f(b)}{2(n+1)} - \frac{f(a)+f(b)}{n+1} + \frac{n+2}{(n+1)^2}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a))$$
$$= -\frac{f(a)+f(b)}{2(n+1)} + \frac{n+2}{(n+1)^2}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a))$$
$$\leq -\frac{1}{n+1}.\frac{1}{n+1}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a)) + \frac{n+2}{(n+1)^2}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a))$$
$$= \frac{1}{n+1}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a)) \leq \frac{f(a)+f(b)}{2}.$$

For the proof of (2) by using (6) of lemma 2.1 we have

$$f(\frac{n}{n+1}x + \frac{a+b}{2}) \leq \sum_{k=0}^{n}\binom{n}{k}(\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{n-k} f(a+\frac{2k+1}{2n+2}(b-a))$$

By integrating we obtain

$$\frac{1}{b-a}\int_a^b f(\frac{n}{n+1}x+\frac{a+b}{2})dx \leq \sum_{k=0}^{n}\binom{n}{k} f(a+\frac{2k+1}{2n+2}(b-a))\frac{1}{b-a}\int_a^b (\frac{x-a}{b-a})^k(\frac{b-x}{b-a})^{n-k}dx$$
$$= \sum_{k=0}^{n}\binom{n}{k} f(a+\frac{2k+1}{2n+2}(b-a))\int_0^1 t^k(1-t)^{n-k}dt$$
$$= \sum_{k=0}^{n}\binom{n}{k} f(a+\frac{2k+1}{2n+2}(b-a))\frac{k!(n-k)!}{(n+1)!}$$
$$= \frac{1}{n+1}\sum_{k=0}^{n} f(a+\frac{2k+1}{2n+2}(b-a))$$

The other parts of (2) is clear by 1.3 and Theorem 2.3(1). □

Remark 2.6. The inequality of Theorem 2.5 is not comparable with the right side of 1.3. Because by elementary calculus we have

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{(n+2)(n+1)}[\sum_{k=0}^{n}(n-k+1)f(a+\frac{k}{n+1}(b-a))+\sum_{k=0}^{n}(k+1)f(a+\frac{k+1}{n+1}(b-a))]$$
$$= \frac{1}{(n+2)(n+1)}\sum_{k=0}^{n+1}(n+1)f(a+\frac{k}{n+1}(b-a))$$
$$= \frac{1}{n+2}\sum_{k=0}^{n+1} f(a+\frac{k}{n+1}(b-a))$$
$$= \frac{f(a)+f(b)}{n+2}+\frac{1}{n+2}\sum_{k=1}^{n} f(a+\frac{k}{n+1}(b-a)).$$

Finally we close this paper with a simple theorem for 0-convex function.

Remember that a positive function f is called 0-convex on $[a,b]$, if for each $x,y \in [a,b]$ and $t \in [0,1]$,

$$f(tx+(1-t)y) \leq [f(x)]^t[f(y)]^{1-t}$$

It is obvious 0-convex functions are log convex functions.

Theorem 2.7. *Let f be a 0-convex function on $[a,b]$ and $f(x) \geq 1$. Then we have*

$$f(\frac{a+b}{2}) \leq \left[\prod_{k=0}^{n} f(a+\frac{2k+1}{2n+1})(b-a)\right]^{\frac{1}{n+1}}$$
$$\leq e^{\frac{1}{b-a}\int_a^b \ln f(x)dx}$$
$$\leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}$$

Proof . Since f is log-convex, by inequalities 1.3 we have

$$\ln f(\frac{a+b}{2}) \leq \frac{1}{n+1}\sum_{k=0}^{n}\ln f(a+\frac{2k+1}{2n+1}(b-a)) \leq \frac{1}{b-a}\int_a^b \ln f(x)dx$$

So

$$\ln f(\frac{a+b}{2}) \leq \ln[\prod_{k=0}^{n} f(a+\frac{2k+1}{2n+1}(b-a))]^{\frac{1}{n+1}} \leq \frac{1}{b-a}\int_a^b \ln f(x)dx$$

By increasing of e^x, we get

$$f(\frac{a+b}{2}) \leq [\prod_{k=0}^{n} f(a+\frac{2k+1}{2n+1}(b-a))]^{\frac{1}{n+1}} \leq e^{\frac{1}{b-a}\int_a^b \ln f(x)dx}$$

Since e^x is convex, by Jensen's inequality we obtain

$$e^{\frac{1}{b-a}\int_a^b \ln f(x)dx} \leq \frac{1}{b-a}\int_a^b e^{\ln f(x)}dx = \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}$$

The last assertion follows from the 0-convexity of f [7, Theorem 2.3]. □

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