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Statistical uniform convergence in 2-normed spaces

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Abstract

The concept of statistical convergence in 2-normed spaces for double sequence was introduced in [S. Sarabadan and S. Talebi, *Statistical convergence of double sequences in* 2*-normed spaces* , Int. J. Contemp. Math. Sci. 6 (2011) 373–380]. In the first, we introduce concept strongly statistical convergence in 2-normed spaces and generalize some results. Moreover, we define the concept of statistical uniform convergence in 2-normed spaces and prove a basic theorem of uniform convergence in double sequences to the case of statistical convergence.

Keywords: statistical convergence; statistical uniform convergence; double sequences; 2-normed space
2010 MSC: Primary 40A05; Secondary 40C99, 40A99, 54A20.

1. Introduction

The concept of statistical convergence was introduced over nearly the last fifty years (Fast [3] in 1951, Schoenberg [19] in 1959), it has become an active area of research, especially in information theory, computer science, biological science, dynamical systems geographic information systems, population modeling, and motion planning in robotics. Let $(X, \|\cdot\|)$ be a normed space. Let E be subset of positive integers $\mathbb{N}$ and $j \in \mathbb{N}$. The quotient $d_j(E) = card(E \bigcap \{1, ..., j\})/j$ is called the j-th *partial density* of E . Note that d_j is a probability measure on $\mathcal{P}(\mathbb{N})$, with support $\{1, ..., j\}$.
$d(E) = \lim_{j\to\infty} d_j(E)$ is called the *natural density* of $E \subseteq \mathbb{N}$(if exists). We have:
(I1) finite subsets have natural density zero.
(I2) $d(E^c) = 1 - d(E)$ where $E^c = \mathbb{N} \setminus E$, i.e., the complement of E.
(I3) if $E_1 \subseteq E_2$ and E_1 ,E_2 have natural densities, then $d(E_1) \leq d(E_2)$, see [3, 4, 13].

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Recall that a sequence $(x_n)_n$ of elements of X is said to be *statistically convergent* to $l \in X$ if the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - l\| \geq \epsilon\}$ for each $\epsilon > 0$ has natural density zero. In other words for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} card(\{k \in \mathbb{N} : \|x_k - l\| \geq \varepsilon\}) = 0.$$

We write $st - \lim_n x_n = l$. The sequence $(x_n)_n$ is called to be *statistically Cauchy sequence* if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_n \frac{1}{n} card(\{k \leq \mathbb{N} : \|x_k - x_N\| \geq \varepsilon\}) = 0.$$

In [5] Fridy prove that a sequence $(x_n)_n$ is statistically convergence if and only if it is statistically Cauchy.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [14]. A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ is called to be *convergence in the Pringsheim's sense* if for each $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that for all $j, k \geq N$ implies $\|x_{jk} - l\| < \varepsilon$. l is called the Pringsheim limit of $(x_{jk})_{j,k}$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a set of positive integers and let $A(n, m)$ be the set of (j, k) in A such that $j \leq n$ and $k \leq m$. Then the two-dimensional concept of *natural density* can be defined as follows.

The *lower asymptotic density* of a set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{d_2}(A) = \liminf_{n,m} \frac{card(A(n, m))}{nm}.$$

If the sequence $(\frac{card(A(n,m))}{nm})_{n,m\in\mathbb{N}}$ has a limit in Pringsheim's sense then we say that A has a *double natural density* and is defined as

$$d_2(A) = \lim_{n,m} \frac{card(A(n, m))}{nm}.$$

2. Preliminary Notes

The notion of linear 2-normed spaces has been investigated by Gâhler in 1960's [6, 8] and has been developed extensively in different subjects by others[9, 2, 15, 17, 18].

Definition 2.1. Let X be a real linear space of dimension greater than 1,and $\|\cdot, \cdot\|$ be a non-negative real-valued function on $X \times X$ satisfying the following conditions:

G1)$\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.

G2) $\|x, y\| = \|y, x\|$ for all x, y in X.

G3)$\|\alpha x, y\| = |\alpha|\|x, y\|$ where α is real.

G4)$\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all x, y, z in X.

$\|\cdot, \cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot, \cdot\|)$ is called a linear 2-normed space.

Every linear 2-normed space $(X, \|\cdot, \cdot\|)$ of dimension different from one is a locally convex topological vector space. In fact, for a fixed $b \in X$, $p_b(x) = \|x, b\|, x \in X$ is a seminorm and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X. In addition, for all scalars α and all $x, y, z \in X$, we have the following properties:

1)$\|\cdot, \cdot\|$ is nonnegative.

2)$\|x, y\| = \|x, y + \alpha x\|$.

3)$\|x - y, y - z\| = \|x - y, x - z\|$.

Some of the basic properties of 2-norm introduce in [15].

Example 2.2. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\|$:= the area of the parallelogram spanned by the vectors x and y, which may be given clearly by the formula

$$\|x, y\| = |x_1y_2 - x_2y_1| \quad , \quad x = (x_1, x_2) \quad y = (y_1, y_2).$$

Definition 2.3. Let $(X, \|\cdot, \cdot\|)$ be 2-normed space. The sequence $(x_n)_n$ in X is said to be *convergent* to x in X if for every $z \in X$,

$$\lim_{n\to\infty} \|x_n - x, z\| = 0.$$

Definition 2.4. Let $(X, \|\cdot, \cdot\|)$ be 2-normed space. The sequence $(x_n)_n$ is a *Cauchy sequence* in a 2-normed space $(X, \|\cdot, \cdot\|)$ if

$$(\forall z \in X)(\forall \varepsilon > 0)(\exists n_o \in \mathbb{N})(\forall n, m \geq n_o) \qquad \|x_n - x_m, z\| < \varepsilon.$$

Theorem 2.5. *[8] Let $(X, \|\cdot, \cdot\|)$ be 2-normed space .The sequence $(x_n)_n$ is statistical convergent if and only if it is statistically Cauchy.*

Theorem 2.6. *[8] If $(x_n)_{n\in\mathbb{N}}$ be a sequence in 2-normed space.the sequence $(x_n)_n$ is statistical convergent to x if and only if there exists $A \subseteq \mathbb{N}$ with $d(A) = 1$ and*

$$\lim_{n\in A} x_n = x.$$

Definition 2.7. Let $(X, \|\cdot, \cdot\|)$ be 2-normed space. A double sequence $(x_{jk})_{j,k}$ in X is said to be *convergent to $l \in X$* if

$$(\forall z \in X)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall j, k \geq N) \qquad \|x_{jk} - l, z\| < \varepsilon.$$

We write it as $\qquad x_{jk} \to \|\cdot, \cdot\|_X.$

Definition 2.8. The double sequence $(x_{jk})_{j,k}$ is a *Cauchy sequence* in a 2-normed space $(X, \|\cdot, \cdot\|)$ if for every nonzero $z \in X$ and for all $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that

$$\forall j \geq n \geq N, \forall k \geq m \geq N, \qquad \|x_{jk} - x_{nm}, z\| < \varepsilon.$$

Recall that $(X, \|\cdot, \cdot\|)$ is a 2-Banach space, if every Cauchy sequence in X is convergence to some $x \in X$.

Definition 2.9. The double sequence $(x_{jk})_{j,k}$ in 2-normed space $(X, \|\cdot, \cdot\|)$ is said to be *statistical convergent* to $l \in X$, if for each nonzero $z \in X$, for each $\varepsilon > 0$, the set $\{(j, k) : \|x_{jk} - l, z\| \geq \varepsilon\}$ has double natural density zero; in other words:

$$\lim_{n,m} \frac{1}{nm} card(\{(j, k) : j \leq n, k \leq m, \|x_{jk} - l, z\| > \varepsilon\}) = 0.$$

In this case we write it as

$$st_2 - \lim_{j,k} x_{jk} = l.$$

Definition 2.10. A double sequence $(x_{jk})_{j,k}$ in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be *statistically Cauchy* if for every $\varepsilon > 0$, for each nonzero $z \in X$ there exists p and q such that

$$\lim_{n,m} \frac{1}{nm} card(\{(j,k), j \leq n, k \leq m : \|x_{jk} - x_{pq}, z\| \geq \varepsilon\}) = 0.$$

The next example we show that there exists double sequence $(x_{jk})_{j,k}$ is statistical convergence but is not bounded.

Example 2.11. Let $X = R^2$ with the 2-norm define by example (2.2).Define double sequence $(x_{jk})_{j,k}$ in 2-normed space $(X, \|\cdot,\cdot\|)$ by

$$x_{jk} = \begin{cases} (1,j), & j = n^2 \\ (1,1), & \text{other wise.} \end{cases} \tag{2.1}$$

The double sequence $(x_{jk})_{j,k}$ is not convergence, so is not bounded but it is *statistical convergent* to $(1,1)$.

Theorem 2.12. [17] Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space .The double sequence $(x_{jk})_{j,k}$ is *statistical convergent* to $l \in X$ if and only if there exists a subset $A = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N}$, such that $d_2(A) = 1$ and $x_{jk} \xrightarrow{\|\cdot,\cdot\|_X} l\ (on (j,k) \in A)$.

Theorem 2.13. [17] If $(x_{jk})_{j,k}$ be a double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$ The sequence $(x_{jk})_{j,k}$ is *statistical convergent* if and only if $(x_{jk})_{j,k}$ is *statistically Cauchy.*

3. Strongly statistically convergence in 2-normed spaces

Definition 3.1. Let $(x_{jk})_{j,k}$ be a double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$. The double sequence $(x_{jk})_{j,k}$ is *strongly statistically convergent* to l if there exists $A \subset \mathbb{N}$ with $d(\mathbb{N} \setminus A) = 0$ such that for every $\varepsilon > 0$, for each $0 \neq z \in X$, there exists $m, n \in A$:

$$d_2(\{(j,k) \in A \times A : \|x_{jk} - l, z\| \geq \varepsilon\}) = 0 \qquad if \qquad j \geq m, k \geq n.$$

We write $\quad Sst_2 - \lim_{j,k} x_{jk} = l.$

Definition 3.2. Let $(x_{jk})_{j,k}$ be a double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$. It is said $(x_{jk})_{j,k}$ *strongly statistically Cauchy* if there exists $A \subset \mathbb{N}$ with $d(\mathbb{N} \setminus A) = 0$ such that for every $\varepsilon > 0$, for each $0 \neq z \in X$, there exists $m, n \in A$

$$d_2(\{(j,k) \in A \times A : \|x_{pq} - x_{jk}, z\| > \varepsilon\}) = 0 \qquad if \qquad j,p \geq m, k,q \geq n.$$

Remark 3.3. If $(x_{jk})_{j,k}$ be strongly statistically convergence (Cauchy) then it is statistically convergence (Cauchy), because if $A \subset \mathbb{N}$, $d(\mathbb{N} \setminus A) = 0$, then $d_2(A \times A) = 1$.

In the next example, we show that $(x_{jk})_{j,k}$ is statistically convergence but it is not strongly statistically convergence.

Example 3.4. Let $A_1 = \{3, 6, 9, ...\}, A_2 = \{9, 18, 24, ...\}, ..., A_j = \{k.3^j : k \in \mathbb{N}\}$. We have $d(A_j) = \frac{1}{3^j}$ if $j \in \mathbb{N}$. Suppose $A = \{(i, j) : i \in A_j\}$. Hence $d_2(A) = 0$. Now let $(x_{jk})_{j,k}$ be strongly statistically convergence so there exists $K \subset \mathbb{N}$ with $d(\mathbb{N} \setminus K) = 0$, $K \times K \subset A$. Take $j \in K$ to be fixed, then for every $i \in K$, $(i, j) \in K \times K \subset A$; hence $i \in A_j, K \subset A_j$. We have $d(A_j^c) \leq d(K^c)$, thus we get a contradiction, because $1 - \frac{1}{3^j} \leq 0$.

If $X = R^2$ with the 2-norm define by Example 2.2 $(0, l)$ be a fixed vector and the double sequence $(x_{jk})_{j,k}$ in X by

$$x_{jk} = \begin{cases} (0, l) & \text{if } (j, k) \in A \\ (0, 0) & \text{if } otherwise. \end{cases}$$

Hence $(x_{jk})_{j,k}$ is statistically convergent to $(0, l)$ But it is not strongly statistically convergence.

In [18], the authors defined: The sequence of function $(f_n)_n$ is said to uniform convergent to a function f (on $(X, \|\cdot, \cdot\|)$), if
$(\forall x \in X)(\forall 0 \neq z \in X)(\forall \varepsilon > 0)(\exists n_o \in \mathbb{N})n \geq n_o \qquad \|f_n(x) - f(x), z\| < \varepsilon$. Now, if consider $X = \mathbb{N}$, $f_j(k) = x_{jk}$. We define convergence uniformly for double sequence in 2-normed space:

Definition 3.5. Double sequence $(x_{jk})_{j,k}$ is *uniformly convergent* on j to $(x_{jo})_j$ in $(X, \|\cdot, \cdot\|)$ if

$$(\forall j \in \mathbb{N})(\forall 0 \neq z \in X)(\forall \varepsilon > 0)(\exists k_o \in \mathbb{N})k \geq k_o \qquad \|x_{jk} - x_{jo}, z\| < \varepsilon.$$

Lemma 3.6. Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space and double sequence $(x_{jk})_{j,k}$ be uniformly convergent on j to $(x_{jo})_j$ then for each j sequence $(x_{jk})_k$ is Cauchy.

Proof . By define for each $\varepsilon > 0, 0 \neq z \in X$, there exists n_o such that for each $k \geq n_o$ $\|x_{jk} - x_{jo}, z\| < \frac{\varepsilon}{2}$. Similarly to the same there exists m_o such that for each $q \geq m_o$ $\|x_{jq} - x_{jo}, z\| < \frac{\varepsilon}{2}$. Now, consider $N = \max\{n_o, m_o\}$. For each $p, q \geq N$ we have

$$\|x_{jp} - x_{jq}, z\| \leq \|x_{jp} - x_{jo}, z\| + \|x_{jo} - x_{jq}, z\| < \varepsilon.$$

□

Theorem 3.7. *Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space and $(x_{jk})_{j,k}$ be a double sequence in 2-normed space $(X, \|\cdot, \cdot\|)$ such that for each j ,$\lim_k x_{jk} = x_{jo}$ and for each k ,$\lim_j x_{jk} = x_{ok}$. Then the following statements are equivalents:*
(a) $\lim_k x_{jk} = x_{jo}$, *uniformly on j.*
(b) $\lim_j x_{jk} = x_{ok}$, *uniformly on k.*
(c) $(x_{jk})_{j,k}$ *is Cauchy in Pringsheim's sense.*

Proof . Assume that (a) holds and we prove (b).
For every $\varepsilon > 0$ for each $0 \neq z \in X$ there exists k_o such that $p, q \geq k_o$ and for each j, $\|x_{jp} - x_{jq}, z\| < \frac{\varepsilon}{4}$. So we have if $p, q \geq k_o$, $\|x_{op} - x_{oq}, z\| < \frac{\varepsilon}{4}$. Fix $p > k_o$, since $\lim_j x_{jp} = x_{op}$ hence there exists j_1 such that $j \geq j_1$, $\|x_{jp} - x_{op}, z\| < \frac{\varepsilon}{4}$ for all $z \in X$. If $k > k_o$ and $j \geq j_1$ we have

$$\|x_{jk} - x_{ok}, z\| \leq \|x_{jk} - x_{jq}, z\| + \|x_{jq} - x_{oq}, z\| + \|x_{oq} - x_{ok}, z\| < \frac{3\varepsilon}{4} \leq \varepsilon.$$

For $k \leq k_o$ there exists j_2 such that if $j \geq j_2$ then $\|x_{jk} - x_{ok}, z\| < \varepsilon$ for all $z \in X$. Hence for each k, there exists $j_o = max\{j_1, j_2\}$ such that if $j \geq j_o$ then $\|x_{jk} - x_{ok}, z\| < \varepsilon$. So, $\lim_j x_{jk} = x_{ok}$ uniformly on k.

By apply the same argument (b) implies (a). It is clear that (c) implies (a)and (b). It remains to show that(a)and (b) implies (c).

Assume that (a) holds. Let $\varepsilon > 0$, $0 \neq z \in X$, there exists k_o such that if $p, q \geq k_o$ then $\|x_{jp} - x_{jq}, z\| < \frac{\varepsilon}{2}$ for each j. By (b), there exists j_o such that if $p, q \geq j_o$ then $\|x_{pk} - x_{qk}, z\| < \frac{\varepsilon}{2}$ for each k. Let $N = max\{j_o, k_o\}$. If $p, q \geq N$, we have

$$\|x_{NN} - x_{pq}, z\| \leq \|x_{NN} - x_{Nq}, z\| + \|x_{Nq} - x_{pq}, z\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $(x_{jk})_{j,k}$ is Cauchy in Pringsheim's sense. □

Corollary 3.8. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $(x_{jk})_{j,k}$ be a double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$ such that for each j ,$\lim_k x_{jk} = x_{jo}$ and for each k ,$\lim_j x_{jk} = x_{ok}$ and let $\lim_{j,k} x_{jk} = x_o$. We have:

$$\lim_{j\to\infty}(\lim_{k\to\infty} x_{jk}) = \lim_{k\to\infty}(\lim_{j\to\infty} x_{jk}) = \lim_{j,k} x_{jk}.$$

Proof . It is easy to see that $(x_{jo})_j$ is Cauchy. Let $\varepsilon > 0$ and $0 \neq z \in X$. By the proof given in Theorem 3.7, there exists N such that if $m, n \geq N$ then $\|x_{NN} - x_{mn}, z\| < \frac{\varepsilon}{8}$. So if $m, m', n, n' \geq N$ then

$$\|x_{mn} - x_{m'n'}, z\| \leq \|x_{mn} - x_{NN}, z\| + \|x_{NN} - x_{m'n'}, z\| < \frac{\varepsilon}{4}.$$

Now if $n' \to \infty$ we have $\|x_{mn} - x_{m'o}, z\| < \frac{\varepsilon}{4}$, if $n \to \infty$, we deduce that if $m, m' \geq N$, $\|x_{mo} - x_{m'o}, z\| < \frac{\varepsilon}{4}$.

Since $\lim_{j,k} x_{jk} = x_o$, we have for every $\varepsilon > 0$ for each $0 \neq z \in X$ there exists M such that $j, k \geq M$, $\|x_{jk} - x_o, z\| < \frac{\varepsilon}{4}$, since $\lim_k x_{jk} = x_{jo}$, for every $\varepsilon > 0$ for each $0 \neq z \in X$ there exists M' such that $j, k \geq M'$, $\|x_{jk} - x_{jo}, z\| < \frac{\varepsilon}{4}$.

Now consider $K = \max\{N, M, M'\}$. Hence for $j, k \geq K$ we have:

$$\|x_{jo} - x_o, z\| \leq \|x_{jk} - x_{jo}, z\| + \|x_{jk} - x_{j'k}, z\| + \|x_{j'k} - x_o, z\| < \frac{3\varepsilon}{4} < \varepsilon.$$

So $\lim_{j\to\infty} x_{jo} = x_o$.

By the same argument, we prove that $(x_{ok})_k$ is Cauchy and $\lim_k x_{ok} = x_o$ thus

$$\lim_{j\to\infty}(\lim_{k\to\infty} x_{jk}) = \lim_{k\to\infty}(\lim_{j\to\infty} x_{jk}) = \lim_{j,k} x_{jk} = x_o.$$

□

4. Strongly uniformly statistically convergence in 2-normed spaces

In this section, we define strongly uniformly statistically convergence to double sequence in 2-normed space and extend this conception.

Definition 4.1. Let $(x_{jk})_{j,k}$ be a double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$ and suppose $(x_{jo})_j$ be a sequence in X. The double sequence $(x_{jk})_{j,k}$ is *strongly uniformly statistically convergent* to $(x_{jo})_j$ if there exists $E \subset \mathbb{N}$ with $d(\mathbb{N} \setminus E) = 0$ such that for every $\varepsilon > 0$, for each $0 \neq z \in X$,

$$d(\{k \in \mathbb{N} : \|x_{jk} - x_{jo}, z\| \geq \varepsilon \qquad foreach j \in E\}) = 0.$$

We write it as *susc*.

Theorem 4.2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $(x_{jk})_{j,k}$ be a double sequence in X such that for each j ,$(x_{jk})_{j,k}$ is statistically convergence and for each k ,$(x_{jk})_{j,k}$ is statistically convergence , Then the following conditions are equivalents:*
(a) For each j , $(x_{jk})_{j,k}$ is susc .
(b) For each k , $(x_{jk})_{j,k}$ is susc .
(c)The double sequence $(x_{jk})_{j,k}$ is strongly statistically Cauchy .

Proof . We prove that (a) implies (b).
By assume, there exists $E \subset \mathbb{N}$ with $d(\mathbb{N} \setminus E) = 0$ such that for every $\varepsilon > 0$ for each $0 \neq z \in X$,$d(\{k : \|x_{jk} - x_{jo}, z\| \geq \varepsilon \quad$ for each $j \in E\}) = 0$. Define for $n \in \mathbb{N}$, for each $0 \neq z \in X$,

$$E_n = \{k \in \mathbb{N} : \|x_{jk} - x_{jo}, z\| < \frac{1}{n} \qquad foreach j \in E\}.$$

It is clear that

$$E_1 \supseteq E_2 \supseteq \ldots \supseteq E_j \supseteq \ldots \qquad , \qquad d(E_j) = 1.$$

Let us choose an arbitrary number $k_1 \in E_1$. Since $d(E_j) = 1$ for $j = 1, 2, ...$, there exists $k_2 \in E_2$ with $k_2 > k_1$ such that if $n \geq k_2$. We have $\frac{E_2(n)}{n} > \frac{1}{2}$, such that $E_2(n) = card(E_2 \cap \{1, 2, ..., n\})$.

Further, Since $d(E_3) = 1$, there exits such a $k_3 > k_2$, $k_3 \in E_3$ that for each $n \geq k_3$, we have $\frac{E_3(n)}{n} > \frac{2}{3}$. Thus we can construct by induction the sequence $k_1 < k_2 < \ldots < k_n < \ldots$ of positive integers such that $k_j \in E_j$, $j = 1, 2, ...$ and

$$\frac{E_j(n)}{n} > 1 - \frac{1}{j} \qquad \forall n \geq k_j.$$

Define
$E_o = \{1, \ldots, k_1\} \cup \{\{k_1, \ldots, k_2\} \cap E_2\} \cup \ldots \cup \{\{k_j, \ldots, k_{j+1}\} \cap E_{j+1}\} \cup \ldots$.
It is easy to check that
$d(E_o) = 1$, $d(\{k \in \mathbb{N} : \|x_{jk} - x_{jo}, z\| \geq \frac{1}{n} \qquad$ for each $j \in E_o\}) = 0$ or

$$\lim_{k \in E_o} x_{jk} = x_{jo}.$$

Since for each k ,$(x_{jk})_{j,k}$ is statistically convergence, by Theorem 2.6 for each k , there exists $A_k \subset \mathbb{N}$ such that $d(A_k) = 1$ and $\lim_j x_{jk} = x_{ok}$. By (I4) there exists $d(A) = 1$ and $card(A \setminus A_k) < \infty$ for all $k \in \mathbb{N}$.

Put $K = E_o \cap A \cap E$ we have that $d(K) = 1$ and for each j, $(x_{jk})_{(j,k) \in K \times K}$ is uniformly convergent to x_{jo}. Hence for each k, $(x_{jk})_{(j,k) \in K \times K}$ is uniformly convergent to x_{ok} by Theorem 3.7. $(x_{jk})_{j,k}$ is Cauchy. So (a) implies (b), (c).

Similar preceding argument (b) implies (a). (c)implies (a) is clear. □

Remark 4.3. By the hypothesis in the preceding theorem and their exists $Sst_2 - \lim_{j,k} x_{jk}$ we have:

$$st - \lim_{j \to \infty} (st - \lim_{k \to \infty} x_{jk}) = st - \lim_{k \to \infty} (st - \lim_{j \to \infty} x_{jk}) = Sst_2 - \lim_{j,k} x_{jk}.$$

Definition 4.4. Let $(x_{jk})_{j,k}$ be a double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$ and suppose $(x_{jo})_j$ be a sequence in X. The double sequence $(x_{jk})_{j,k}$ is *uniformly statistically convergent* to $(x_{jo})_j$ if for every $\varepsilon > 0$, for each $0 \neq z \in X$

$$d_2(\{(j,k) : \|x_{jk} - x_{jo}, z\| \geq \varepsilon\}) = 0.$$

Definition 4.5. Let $(x_{jk})_{j,k}$ be a double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$ and suppose $(x_{ok})_k$ be a sequence in X. The double sequence $(x_{jk})_{j,k}$ is *uniformly statistically convergent* to $(x_{ok})_k$ if for every $\varepsilon > 0$, for each $0 \neq z \in X$

$$d_2(\{(j,k) : \|x_{jk} - x_{ok}, z\| \geq \varepsilon\}) = 0.$$

Theorem 4.6. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $(x_{jk})_{j,k}$ be a double sequence in X such that for each j, $st - \lim_k x_{jk} = x_{jo}$ and for each k , $st - \lim_j x_{jk} = x_{ok}$. Then the following statements are equivalents:*
(a) $(x_{jk})_{j,k}$ is uniformly statistically convergent to $(x_{jo})_j$ for each j ,and $(x_{jo})_j$is statistically convergent to x_o.
(b)$(x_{jk})_{j,k}$ is uniformly statistically convergent to $(x_{ok})_k$ for each k and $(x_{ok})_k$is statistically convergent to x_o.
(c)$st_2 - \lim_{j,k} x_{jk} = x_o$.

Proof . We prove that (a) implies (c) analogously to the proof of Moricz in [11].
Suppose $(n_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $2n_i \leq n_{i+1}$ for $i = 1, 2, ...,$

$$\lim_{n,m} \frac{1}{nm} card(\{(j,k) : j \leq n, k \leq m, \|x_{jk} - x_{jo}, z\| > \frac{1}{2^i}\}) < \frac{1}{2^{2i}} \qquad if n, m \geq n_i.$$

We can define the double sequence $(y_{jk})_{j,k}$ in following way:
If $min\{j,k\} < n_1$, then $y_{jk} := x_{jk}$. In otherwise, if $n_r \leq j < n_{r+1}$,$n_s \leq k < n_{s+1}$, then

$$y_{jk} := \begin{cases} x_{jk} & \text{if } \|x_{jk} - x_o, z\| \leq \frac{1}{2^{min\{r,s\}}} \\ x_{jo} & \text{if } \|x_{jk} - x_o, z\| > \frac{1}{2^{min\{r,s\}}}. \end{cases} \qquad (1)$$

Put $K = \{(j,k) : y_{jk} \neq x_{jk}\}$, we show that $d_2(K) = 0$. If $min\{j,k\} < n_1$ then $y_{jk} = x_{jk}$, we may assume that for some $r, s \geq 1$

$$n_r \leq n < n_{r+1} \qquad n_s \leq m < n_{s+1}$$

Suppose $p := min\{r,s\}$. By (1), we have
$\{(j,k) : j \leq n, k \leq m, y_{jk} \neq x_{jk}\} =$
$\{(j,k) : n_p \leq j \leq n, n_p \leq k \leq m, \|x_{jk} - x_o, z\| > \frac{1}{2^p}\}$
$\cup\bigcup_{q=1}^{p-1}[\{(j,k) : n_q \leq j \leq n, n_q \leq k < n_{q+1}, \|x_{jk} - x_o, z\| > \frac{1}{2^q}\}\cup$
$\{(j,k) : n_q \leq j < n_{q+1}, n_q \leq k \leq m, \|x_{jk} - x_o, z\| > \frac{1}{2^q}\}]$.
By definition of sequence, we obtain
$\frac{1}{nm} card(\{(j,k) : j \leq n, k \leq m, y_{jk} \neq x_{jk}\}$
$\leq \frac{1}{2^p} + \sum_{q=1}^{p-1}(\frac{n_{q+1}}{m}.\frac{1}{2^q} + \frac{n_{q+1}}{n}.\frac{1}{2^q})$
$\leq \frac{1}{2^p} + (\frac{n_p}{m} + \frac{n_p}{n})\sum_{q=1}^{p-1} \frac{1}{2^{2q-(p-1-q)}} < \frac{1}{2^{2p}} + \frac{1}{2^{p-1}} \to 0.$
Since $p := min\{r,s\} \to \infty$. Hence $d_2(K) = 0$. So, for $(x_{jk})_{(j,k)\in K\times K}$, we consider $\varepsilon > 0$ and $0 \neq z \in X$, there exists n_0 such that if $j, k \geq n_0$ then $\|x_{jk} - x_{jo}, z\| < \varepsilon$. By hypothesis, since for each j , x_{jo} is statistically convergent to x_o by Theorem 2.12 there exists $K' \subset \mathbb{N}$ with $d(K') = 1$ and $\lim_{j\in K'} x_{jo} = x_o$.

Suppose that $K_o = \{(j,k) \in K^c : j \in K'\}$; we have $d_2(K_o) = 1$. Consider $\varepsilon > 0$ and $0 \neq z \in X$, there exists n_0 such that if $(j,k) \in K_o$, $j, k \geq n_0$ then $\|x_{jk} - x_{jo}, z\| < \frac{\varepsilon}{2}, \|x_{jo} - x_o, z\| < \frac{\varepsilon}{2}$, hence for all $\varepsilon > 0$ and $0 \neq z \in X$ there exists n_0 such that $(j,k) \in K_o$, $j, k \geq n_0$ then $\|x_{jk} - x_o, z\| < \varepsilon$. Therefore $st_2 - \lim_{j,k} x_{jk} = x_o$.

Now, we prove that (c) implies (a). Since $st_2 - \lim_{j,k} x_{jk} = x_o$, we have for all $\varepsilon > 0$ and $0 \neq z \in X$, there exists n_0 and $K \subset \mathbb{N}$ such that $d_2(K) = 1$ and $(j,k) \in K$, $j,k \geq n_0$ then $\|x_{jk} - x_o, z\| < \frac{\varepsilon}{2}$.

Let $H = \{j \in \mathbb{N} : d(\{k : (j,k) \notin K\}) \neq 0\}$ and $K_o = \{(j,k) \in K^c, j \in H\}$. So we have $d(H) = 1$ and $d_2(K_o) = 1$. Consider $j \in H$ be fix with $j \geq n_0$. If $k \geq n_0$ and $(j,k) \in K_o$ then $\|x_{jk} - x_o, z\| < \frac{\varepsilon}{2}$.

Now, if $k \to \infty$, we have $\|x_{jo} - x_o, z\| < \frac{\varepsilon}{2}$, if $j \geq n_0$. So, if $(j,k) \in K_o$, $j,k \geq n_0$ $\|x_{jk} - x_{jo}, z\| < \varepsilon$ then $(x_{jk})_{j,k}$ is uniformly statistically convergent to $(x_{jo})_j$. The equivalence (b) and (c) is similar to above. □

Remark 4.7. Observe that by the hypothesis in the preceding theorem we deduce that

$$st - \lim_{j\to\infty}(st - \lim_{k\to\infty} x_{jk}) = st - \lim_{k\to\infty}(st - \lim_{j\to\infty} x_{jk}) = st_2 - \lim_{j,k} x_{jk}.$$

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