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Periodic solution for a delay nonlinear population equation with feedback control and periodic external source

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Abstract

In this paper, sufficient conditions are investigated for the existence of periodic (not necessarily positive) solutions for nonlinear several time delay population system with feedback control. Nonlinear system affected by an periodic external source is studied. Existence of a control variable provides the extension of some previous results obtained in other studies. We give a illustrative example in order to indicate the validity of the assumptions.

Keywords: Schauder's fixed-point theorem; Periodic solution; Population equation; Feedback control.
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1. Introduction and preliminaries

The analysis of periodic systems has long been a topic of interest. In particular, in the last few years the problem of the existence of periodic solutions for the nonlinear delay differential equations has received considerable attention (see, for example [10, 17, 11, 5, 6, 7, 3]). In this direction, an important question, which has been studied extensively by a number or authors is whether nonlinear equations can support periodic solutions or not. For example, in theoretical aspects, knowledge of periodic solutions is important for understanding the phase portrait of the nonlinear equations and specially the qualitative behavior of solutions (see, for example [13, 12, 4, 15, 8]). On the applied side, in the problem of periodic optimization, arising for instance in design of solar heating systems where

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the ambient temperature represents a periodic input, there occurs the need to compute the periodic solutions of a differential equation with periodic coefficients. As a matter of fact, the idea of running processes in a periodic way is not at all a new one. Many applications can be found in literature, particularly in the field of chemical engineering and mathematical biology. Indeed, the performance of many processes, even of industrial size, can be considerably improved by the implementation of a periodic control [1]. An important class of nonlinear equations can be represented by the following nonlinear delay population equation:

$$\frac{dx}{dt} = x(t)[\rho(x) - a(x)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i)], \tag{1.1}$$

where $\alpha, \beta_i > 0$, and ρ, a, b_i, $(i = 1, 2, \ldots, n)$ are continuous functions. Eq. (1.1) may be regarded as a nonlinear prototypical model for variation of the population of an organism, when there is density dependent growth which depends not only on the population at time t, but also on the population at previous times. Considering the biological and environment periodicity it is reasonable to study system with periodic coefficients. Thus, ρ, a, b_i, $(i = 1, 2, \ldots, n)$ are continuous T-periodic functions.

On the other hand, in some situation, one may wish to alter the position of $x(t)$, but to keep its stability. This is of significance in the control procedure of ecology balance. One of the techniques to achieve this aim is to alter system (1.1) structurally by introducing "indirect" control variables. In this regards, Eq. (1.1) can be extended to the following nonlinear delay population equation with control variable:

$$\frac{dx}{dt} = x(t)[\rho(x) - a(x)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) - c(t)u(t)], \tag{1.2}$$

$$\frac{du}{dt} = -\eta(t)u(t) + \sum_{i=1}^{n} g(t)x^{\beta_i}(t-\sigma_i), \tag{1.3}$$

where $\alpha, \beta_i > 0$, a, b_i, g, c, ρ and η, $(i = 1, 2, \ldots, n)$ are continuous, T-periodic functions with $\int_0^T \eta \neq 0$ and $\int_0^T \rho \neq 0$. During the last decade, many scholars has paid their attention to Eq. (1.2) and to other nonlinear population equations as special cases of (1.2) (see, for instance [16, 10] and the references therein). In this paper, we deal with system (1.2). In the light of above discussion, it seems reasonable to consider (1.2) and asks when this system has a periodic solution. In addition, one may take into account the nonlinear system (1.2) while the time variation of the population density, namely, dx/dt is directly affected by an periodic external source, denoted by $\mathcal{S}(t)$. In such a case, the population system (1.2) should be recast in the following form

$$\frac{dx}{dt} = x(t)[\rho(x) - a(x)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) - c(t)u(t)] - \mathcal{S}(t) \tag{1.4}$$

$$\frac{du}{dt} = -\eta(t)u(t) + \sum_{i=1}^{n} g(t)x^{\beta_i}(t-\sigma_i), \tag{1.5}$$

where $\mathcal{S}(t)$ is continuous, periodic function.

The rest of this paper is organized as follows. In section 2, we give certain conditions to guarantee the existence of at least one periodic solution for (1.2). The proof hinges on methods for finding Green's function and Schauder's fixed point theorem applied to integral operator (2.3) which is a reformulation of (2.2). In addition, the existence problem of periodic solutions of system (1.2) is equivalent to that of periodic solutions of (2.2). In section 3, we deal with the nonlinear population system with source. Similarly, proof is based on Schauders fixed point theorem, applied to integral equation (3.4) which is a reformulation of (3.1).

Theorem 1.1. (Schauder [2]) Let X be a Banach space and Λ be a closed, bounded and convex subset of X. If $\Gamma : \Lambda \mapsto \Lambda$ is a compact operator, then Γ has at least one fixed point on Λ.

Besides, we also invoke the following weak version of Arzela-Ascoli theorem [9].

Theorem 1.2. (Arzela-Ascoli) Let $\{\xi_n(t)\}$ be a sequence of real functions on $[0, T]$ which is uniformly bounded and equicontinuous. Then $\{\xi_n(t)\}$ has a uniformly convergent subsequence.

2. Main results

In this section, we shall study the existence of periodic solutions of system (1.2). To do this, we transform this system of couple equations into one integral equation. In this way, we introduce the following integral operator Ξ on the Banach space $(\mathcal{C}_T, \|.\|)$,

$$\Xi : \mathcal{C}_T \to \mathcal{C}_T,$$

$$(\Xi x)(t) = \int_0^T G(t,s)\{\sum_{i=1}^{n} g(s)x^{\beta_i}(s-\sigma_i)\}ds, \tag{2.1}$$

where, $\mathcal{C}_T = \{\xi \mid \xi$ is a continuous T-perodic function on R $\}$ and for $\xi \in \mathcal{C}_T$ we define $\|\xi\| = \sup_{t\in[0,T]} |\xi(t)|$. Clearly, $(\mathcal{C}_T, \|.\|)$is a Banach space. The kernel of the integral operator (2.1) is given by:

$$G(t,s) = \begin{cases} \frac{\exp(\int_0^T \eta(\theta)d\theta)}{\exp(\int_0^T \eta(\theta)d\theta)-1} \exp(\int_t^s \eta(\theta)d\theta) & 0 \le s \le t \le T, \\ \\ \frac{1}{\exp(\int_0^T \eta(\theta)d\theta)-1} \exp(\int_t^s \eta(\theta)d\theta) & 0 \le t \le s \le T, \end{cases}$$

where $\int_0^T \eta \neq 0$. The following lemma is useful for introducing the integral operator (2.3).

Lemma 2.1. *Let g and η are belong to $\mathcal{C}_T$ as well as $\int_0^T \eta \neq 0$. Suppose that u is a continuous real function such that for some $\xi \in \mathcal{C}_T$, $\Xi\,\xi = u$. Then u is a T-periodic solution of the second equation in* (1.2).

Proof . By assumption,

$$\begin{aligned} u = (\Xi\,\xi)(t) &= \int_0^T G(t,s)\{\sum_{i=1}^{n} g(s)\xi^{\beta_i}(s-\sigma_i)\}ds \\ &= \int_0^t G(t,s)\{\sum_{i=1}^{n} g(s)\xi^{\beta_i}(s-\sigma_i)\}ds + \int_t^T G(t,s)\{\sum_{i=1}^{n} g(s)\xi^{\beta_i}(s-\sigma_i)\}ds \\ &= \frac{\exp(\int_0^T \eta(\theta)d\theta)}{\exp(\int_0^T \eta(\theta)d\theta)-1} \int_0^t \exp(\int_t^s \eta(\theta)d\theta)\{\sum_{i=1}^{n} g(s)\xi^{\beta_i}(s-\sigma_i)\}ds \\ &+ \frac{1}{\exp(\int_0^T \eta(\theta)d\theta)-1} \int_t^T \exp(\int_t^s \eta(\theta)d\theta)\{\sum_{i=1}^{n} g(s)\xi^{\beta_i}(s-\sigma_i)\}ds \end{aligned}$$

and so

$$\begin{aligned} u &= \frac{\exp(\int_0^T \eta(\theta)d\theta)\exp(-\int_0^t \eta(\theta)d\theta)}{\exp(\int_0^T \eta(\theta)d\theta)-1}\int_0^t \exp(\int_0^s \eta(\theta)d\theta)\{\sum_{i=1}^n g(s)\xi^{\beta_i}(s-\sigma_i)\}ds \\ &+ \frac{\exp(-\int_0^t \eta(\theta)d\theta)}{\exp(\int_0^T \eta(\theta)d\theta)-1}\int_t^T \exp(\int_0^s \eta(\theta)d\theta)\{\sum_{i=1}^n g(s)\xi^{\beta_i}(s-\sigma_i)\}ds. \end{aligned}$$

So, we obtains

$$\begin{aligned} \frac{du}{dt} &= \frac{-\eta(t)\exp(\int_0^T \eta(\theta)d\theta)\exp(-\int_0^t \eta(\theta)d\theta)}{\exp(\int_0^T \eta(\theta)d\theta)-1}\int_0^t \exp(\int_0^s \eta(\theta)d\theta)\{\sum_{i=1}^n g(s)\xi^{\beta_i}(s-\sigma_i)\}ds \\ &+ \frac{\exp(\int_0^T \eta(\theta)d\theta)}{\exp(\int_0^T \eta(\theta)d\theta)-1}\{\sum_{i=1}^n g(t)\xi^{\beta_i}(t-\sigma_i)\} \\ &+ \frac{-\eta(t)\exp(-\int_0^t \eta(\theta)d\theta)}{\exp(\int_0^T \eta(\theta)d\theta)-1}\int_t^T \exp(\int_0^s \eta(\theta)d\theta)\{\sum_{i=1}^n g(s)\xi^{\beta_i}(s-\sigma_i)\}ds \\ &- \frac{1}{\exp(\int_0^T \eta(\theta)d\theta)-1}\{\sum_{i=1}^n g(t)\xi^{\beta_i}(t-\sigma_i)\} \\ &= -\eta(t)u(t)+\sum_{i=1}^n g(t)\xi^{\beta_i}(t-\sigma_i). \end{aligned}$$

Which shows that u is a solution of the second equation in (1.2) with $x=\xi$. Clearly, u is a continuous T-periodic function.

Note that $G(t,s)$ is, in fact, the Green's function of the second equation in (1.2). Therefore, by using methods for Green's function, we may find the kernel of the integral operator (2.1). However, our approach is different, but for going through the details of finding Greens function we refer the reader to [14].

According to Lemma 2.1, it may be deduced that existence problem of T-periodic solution of the system (1.2) is equivalent to that of T-periodic solution of the following equation

$$\frac{dx}{dt} = x(t)[\rho(t)-a(t)x^{\alpha}(t)-\sum_{i=1}^n b_i(t)x^{\beta_i}(t-\sigma_i)-c(t)(\Xi x)(t)]. \tag{2.2}$$

We introduce the following integral operator on $\mathcal{C}_T$,

$$(\Gamma x)(t) = \int_0^T H(t,s)x(s)\{a(s)x^{\alpha}(s)+\sum_{i=1}^n b_i(s)x^{\beta_i}(s-\sigma_i)+c(s)(\Xi x)(s)\}ds, \tag{2.3}$$

where, the kernel is given by

$$H(t,s) = \begin{cases} \frac{1}{\exp(\int_0^T \rho(\theta)d\theta)-1}\exp(-\int_t^s \rho(\theta)d\theta) & 0\le s\le t\le T, \\ \\ \frac{\int_0^T \rho(\theta)d\theta}{\exp(\int_0^T \rho(\theta)d\theta)-1}\exp(-\int_t^s \rho(\theta)d\theta) & 0\le t\le s\le T. \end{cases}$$

□

Now, similar to the proof of Lemma 2.1 we can consider the following lemma.

Lemma 2.2. Let a, b_i, g, μ, ρ, η, c are belong to $\mathcal{C}_T$ as well as $\int_0^T \rho \neq 0$. Suppose that x is a continuous T-periodic function, then $(\Gamma x)(t)$ is T-periodic function and satisfies the following differential equation:

$$(\Gamma x)'(t) = x(t)[\rho(t) - a(t)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) - c(t)(\Xi x)(t)]. \tag{2.4}$$

Corollary 2.3. Let a, b_i, g, μ, ρ, η, c are belong to $\mathcal{C}_T$ as well as $\int_0^T \rho \neq 0$. Suppose that x is a continuous T-periodic function and $x \in \mathcal{C}_T$ be a fixed point of the operator Γ, i.e., $\Gamma x = x$, then x is a T-periodic solution of equation (2.2).

Lemma 2.2 and Corollary 2.3 are useful for proving the following main theorem. We set,
$Q = \sup_{t\in[0,T]} |G(s,t)|$, $\quad K = \sup_{t\in[0,T]} |H(s,t)|$,
$W = \sup_{t\in[0,T]}\{|a(t)| + \sum_{i=1}^{n} |b_i(t)| + nTQ|c(t)|\,|g(t)|\}$,
$\mathbf{M}(A) = \max\{A^{\alpha}, A^{\beta_1}, \ldots, A^{\beta_n}\}, \quad A \in \mathbb{R}$.

Theorem 2.4. *Let B_R be the closed sphere in $\mathcal{C}_T$ and let*

$$\int_0^T \{|a(s)| + \sum_{i=1}^{n} |b_i(s)| + n|c(s)| \int_0^T |G(\theta,s)g(\theta)|d\theta\}ds \leq \frac{1}{K\boldsymbol{M}(R)}. \tag{2.5}$$

Then, the integral operator Γ, is defined by (2.3), maps B_R into B_R and has at least one fixed point.

Proof . First, we indicate that Γ maps B_R into B_R. To do this, let ξ be an arbitrary periodic function belong to B_R. For any $z \in \{\alpha, \beta_1, \ldots, \beta_n\}$ we have,

$$\|\xi\|^z \leq R^z \leq \mathbf{M}(R),$$

therefore,

$$\begin{aligned}
|(\Gamma\xi)(t)| &= |\int_0^T H(t,s)x(s)\{a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) + c(s)(\Xi x)(s)\}ds| \\
&\leq \int_0^T |H(t,s)|\|x(s)\|\{|a(s)|\|x(s)\|^{\alpha} + \sum_{i=1}^{n} |b_i(s)|\|x\|^{\beta_i} \\
&+ |c(s)| \int_0^T |G(\theta,s)|\{\sum_{i=1}^{n} |g(\theta)|\|x\|^{\beta_i}\}d\theta\}ds \\
&\leq \int_0^T |H(t,s)|\, R\,\{|a(s)|R^{\alpha} + \sum_{i=1}^{n} |b_i(s)|R^{\beta_i} + |c(s)| \int_0^T |G(\theta,s)|\{\sum_{i=1}^{n} |g(\theta)|R^{\beta_i}\}d\theta\}ds \\
&\leq R\,K\,\mathbf{M}(R) \int_0^T \{|a(s)| + \sum_{i=1}^{n} |b_i(s)| + |c(s)| \int_0^T |G(\theta,s)|\{\sum_{i=1}^{n} |g(\theta)|\}d\theta\}ds \\
&\leq R.
\end{aligned}$$

Thus $(\Gamma\xi)(t)$ is belong to B_R. Since ξ is an arbitrary periodic function belong to B_R, the integral operator Γ maps B_R into B_R. In the sequel, we show that Γ is a compact operator on the Banach space $\mathcal{C}_T$. To do this, suppose that $\{\xi_m\}$ is an arbitrary sequences on B_R, that is bounded and for all

$m \in \mathbb{N}$ and $t \in [0,T]$, we have $|\xi_m(t)| \le R$. According to Lemma 2.2, for any $m \in \mathbb{N}$ and $t \in [0,T]$, we have

$$(\Gamma\xi_m)'(t) = \xi_m(t)[\rho(t) - a(t)\xi_m^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)\xi_m^{\beta_i}(t-\sigma_i) - c(t)(\Xi\xi_m)(t)].$$

Therefore,

$$\begin{aligned}
&|(\Gamma\xi_m)'(t)| \\
\le\ & |\xi_m(t)|[|\rho(t)| + |a(t)\xi_m^{\alpha}(t)| + \sum_{i=1}^{n} |b_i(t)\xi_m^{\beta_i}(t-\sigma_i)| + |c(t)(\Xi\xi_m)(t)|] \\
\le\ & R\,[|\rho(t)| + \mathbf{M}(R)\{|a(t)| + \sum_{i=1}^{n} |b_i(t)| + n|c(t)| \int_0^T |G(\theta,t)g(\theta)|d\theta\}] \\
\le\ & R\,[\|\rho\| + \mathbf{M}(R)\{\|a\| + \sum_{i=1}^{n} \|b_i\| + nTQ\|c\|\|g\|\}] \\
\le\ & R\,[\|\rho\| + \mathbf{M}(R)W].
\end{aligned}$$

Thus, for any $t,s \in [0,T]$, one obtains

$$|(\Gamma\xi_m)(t) - (\Gamma\xi_m)(s)| \le R\,[\|\rho\| + \mathbf{M}(R)W]|t-s|.$$

In this way, for given $\varepsilon > 0$, if we consider $\delta = \frac{\varepsilon}{R\,[\|\rho\| + \mathbf{M}(R)W]}$, then

$$|(\Gamma\xi_m)(t) - (\Gamma\xi_m)(s)| \le \varepsilon \qquad \textit{for all } m \in \mathbb{N} \textit{ and } |t-s| \le \delta.$$

Thus, $\{(F\xi_m)(t)\}$ as a sequence of functions on $[0,\omega]$ is equicontinuous. Therefore, based on Arzela-Ascoli theorem there exist a subsequent of $\{(F\xi_m)(t)\}$, denoted by $\{(F\xi_{m_i})(t)\}$, which is uniformly convergence on $[0,T]$. This means that $\{(F\xi_{m_i})(t)\}$ is convergent on B_R and consequently, Γ is a compact bounded operator. Therefore, Theorem 1.1 implies that the integral operator Γ has at least a fixed point on B_R, in which, by Corollary 2.3, is a T-periodic solution of the equation (2.2) or, equivalently, T-periodic solution of the nonlinear population system (1.2). This completes the proof of theorem. □

3. Nonlinear population system with source

In such a case, the population system (1.2) transforms into the system (1.4). With due attention to the population source term, namely $\mathcal{S}(t)$ in (1.4), Eq. (2.2) and the operator defined in 2.3 are converted into the following forms, respectively:

$$\frac{dx}{dt} = x(t)[\rho(t) - a(t)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) - c(t)(\Xi x)(t)] - \mathcal{S}(t), \tag{3.1}$$

and

$$(\Gamma_{\mathcal{S}}x)(t) = (\Gamma x)(t) + \mathcal{S}_H(t). \tag{3.2}$$

Wherein, $\mathcal{S}_H(t) = \int_0^T H(t,\theta)\mathcal{S}(\theta)d\theta$.

Theorem 3.1. *Let*

$$\Lambda = B(\mathcal{S}_H, \|\mathcal{S}_H\|) = \{\xi \in \mathcal{C}_T : \ \|\xi - \mathcal{S}_H\| \le \|\mathcal{S}_H\|\},$$

be the close sphere of radius $\|\mathcal{S}_H\|$ with center $\mathcal{S}_H$ in $\mathcal{C}_T$.

Suppose that

$$\int_0^T \{|a(s)| + \sum_{i=1}^n |b_i(s)| + n|c(s)| \int_0^T |G(\theta, s)g(\theta)|d\theta\}ds \le \frac{1}{2K\boldsymbol{M}(2\|\mathcal{S}_H\|)}. \qquad (3.3)$$

Then, the integral operator $\Gamma_{\mathcal{S}}$ maps Λ into Λ and has at least one fixed point.

Proof . Let $\xi \in \Lambda$, then

$$|\xi(t) - \mathcal{S}_H(t)| \le \|\mathcal{S}_H\| \quad for\ any\ t \in [0.T],$$

thus

$$\|\xi(t)\| \le 2\|\mathcal{S}_H\|.$$

This shows that, for any $z \in \{\alpha, \beta_1, \dots, \beta_n\}$ and all $\xi \in \Lambda$ we have,

$$\|\xi\|^z \le (2\|\mathcal{S}_H\|)^z \le \mathbf{M}(2\|\mathcal{S}_H\|).$$

Consequently, for arbitrary $\xi \in \Lambda$, as proceed in the proof of Theorem 2.4, we obtain

$$\begin{aligned} |(\Gamma_{\mathcal{S}}x)(t) - \mathcal{S}_H(t)| &= |(\Gamma x)(t)| \\ &\le 2\|\mathcal{S}_H\|K\,\mathbf{M}(2\|\mathcal{S}_H\|) \int_0^T \{|a(s)| + \sum_{i=1}^n |b_i(s)| \\ &+ n|c(s)| \int_0^T |G(\theta, s)g(\theta)|d\theta\}ds \\ &\le \|\mathcal{S}_H\|. \end{aligned}$$

Therefore, the operator $\Gamma_{\mathcal{S}}$ maps Λ into Λ. Compactness of the $\Gamma_{\mathcal{S}}$, with due attention to the following inequality

$$|(\Gamma_{\mathcal{S}}\xi_m)'(t)| \le 2\,\|\mathcal{S}_H\|[\|\rho\| + \mathbf{M}(2\|\mathcal{S}_H\|)W] + \|\mathcal{S}\|,$$

is proofed similar to the Theorem 2.4. Therefore, Theorem 1.1 implies that the integral operator $\Gamma_{\mathcal{S}}$ has at least a fixed point on $B(\mathcal{S}_H, \|\mathcal{S}_H\|)$. Consequently, system (1.4) has at least a T-periodic solution on $B(\mathcal{S}_H, \|\mathcal{S}_H\|)$, since fixed points of the integral operator $\Gamma_{\mathcal{S}}$ are the T-periodic solutions of system equation (1.4). This completes the proof of theorem. □

4. Illustrative example

Consider the following system of neutral population dynamics with delay and feedback control

$$\begin{aligned} \frac{dN}{dt} &= N(t)[\frac{10 - \sin(2\pi t)}{2 + \cos(2\pi t)} - \frac{\sin(4\pi t) - \cos(4\pi t)}{36 - 4\sin(4\pi t) - \cos(2\pi t)}N^{\frac{1}{3}}(t) - \frac{1 - 2\sin(4\pi t)}{32 + \sin(2\pi t) - \cos(2\pi t)}N^{\frac{5}{3}}(t - \sigma_1) \\ &- \frac{1 + 2\cos(4\pi t)}{32 + \sin(2\pi t) - \cos(2\pi t)}N^4(t - \sigma_2) - \frac{\cos(8\pi t)}{104 - 7\sin(2\pi t) + \cos(2\pi t)}u(t)] \\ \frac{du}{dt} &= (\frac{-4 - \sin(2\pi t) + 2\cos(2\pi t)}{10 + 2\sin(2\pi t) - 3\cos(2\pi t)})u(t) + \frac{\sin(8\pi t)}{28 - \cos(8\pi t)}(N^{\frac{5}{3}}(t - \sigma_1) + N^4(t - \sigma_2)), \end{aligned}$$

which is an example of nonlinear population dynamics system (1.2) with

$$Q = \sup_{t\in[0,T]} |G(s,t)| = 3.1329, \quad K = \sup_{t\in[0,T]} |H(s,t)| = 1.0031, \quad \rho(t) = \frac{10-\sin(2\pi t)}{2+\cos(2\pi t)},$$

$$G(t,s) = \begin{cases} 3.1329 \exp(\int_t^s \eta(\theta)d\theta) & 0 \le s \le t \le T, \\ 2.1329 \exp(\int_t^s \eta(\theta)d\theta) & 0 \le t \le s \le T, \end{cases}$$

$$\eta(t) = \frac{4+\sin(2\pi t) - 2\cos(2\pi t)}{10 + 2\sin(2\pi t) - 3\cos(2\pi t)},$$

$$H(t,s) = \begin{cases} 0.0031 \exp(\int_s^t \rho(\theta)d\theta) & 0 \le s \le t \le T, \\ 1.0031 \exp(\int_s^t \rho(\theta)d\theta) & 0 \le t \le s \le T, \end{cases}$$

$$n = 2, \quad T = 1, \quad R = 1,$$

$$\mathbf{M}(R) = 1, \quad \|\mathcal{S}_H\| = 0.0017, \quad \mathbf{M}(2\|\mathcal{S}_H\|) = 0.1518,$$

$$a(t) = \frac{\sin(4\pi t) - \cos(4\pi t)}{36 - 4\sin(4\pi t) - \cos(2\pi t)}, \qquad c(t) = \frac{\cos(8\pi t)}{104 - 7\sin(2\pi t) + \cos(2\pi t)},$$

$$g(t) = \frac{\sin(8\pi t)}{28 - \cos(8\pi t)}, \qquad \mathcal{S}(t) = \frac{1-\sin(2\pi t)}{2+\sin(4\pi t)},$$

$$b_1(t) = \frac{1 - 2\sin(4\pi t)}{32 + \sin(2\pi t) - \cos(2\pi t)}, \qquad b_2(t) = \frac{1+2\cos(4\pi t)}{32+\sin(2\pi t) - \cos(2\pi t)},$$

$$\alpha = \frac{1}{3}\ , \beta_1 = \frac{5}{3},\ \beta_2 = 4.$$

With due attention to the data above, we have

$$\int_0^T \{|a(s)| + \sum_{i=1}^n |b_i(s)| + n|c(s)| \int_0^T |G(\theta,s)g(\theta)|d\theta\}ds$$

$$\le \int_0^T \{|a(s)| + \sum_{i=1}^n |b_i(s)| + nTQ|c(s)| \int_0^T |g(\theta)|d\theta\}ds$$

$$= 0.1137 \le \frac{1}{K\mathbf{M}(R)} = 0.9969.$$

Also, for the system with external source, $\mathcal{S}(t) = \frac{1-\sin(2\pi t)}{2+\sin(4\pi t)}$, we have

$$\int_0^T \{|a(s)| + \sum_{i=1}^n |b_i(s)| + nTQ|c(s)| \int_0^T |g(\theta)|d\theta\}ds = 0.1137 \le \frac{1}{2K\mathbf{M}(2\|\mathcal{S}_H\|)} = 3.2836.$$

Therefore, the Inequality 2.5 in Theorem 2.4 and Inequality 3.3 for the nonlinear system with source in Theorem 3.1 are valid for our examples.

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