



A characterization of multiwavelet packets on general lattices

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Abstract

The objective of this paper is to establish a complete characterization of multiwavelet packets associated with matrix dilation on general lattices Γ in \mathbb{R}^d by virtue of time-frequency analysis, matrix theory and operator theory.

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1. Introduction

The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application. Wavelet packets, due to their nice characteristics have been widely applied to signal processing, coding theory, image compression, fractal theory and solving integral equations and so on. Coifman *et al.*[9] firstly introduced the notion of univariate wavelet packets. Chui and Li [7] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. The introduction of biorthogonal wavelet packets attributes to Cohen and Daubechies [8]. Shen [17] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the orthogonal version of vector-valued wavelet packets [6], the generalized orthogonal multiwavelet packets [19], the orthogonal p -wavelet packets and p -wavelet frame packets related to the Walsh polynomials [13, 15] and the M -band framelet packets [16].

On the other hand, multiwavelets are natural extension and generalization of traditional wavelets. They have received considerable attention from the wavelet research communities both in the theory

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as well as in applications. They can be seen as vector valued-wavelets that satisfy conditions in which matrices are involved rather than scalars as in the wavelet case. Multiwavelets can own symmetry, orthogonality, short support and high order vanishing moments, however traditional wavelets can not possess all these properties at the same time (see [4, 11]). Yang and Cheng [21] generalized the concept of wavelet packets to the case of multiwavelet packets associated with a dilation factor a which were more flexible in applications. Subsequently, Behera [1] extended the results of Yang and Cheng to the multivariate multiwavelet packets associated with a dilation matrix A . He proved lemmas on the so-called splitting trick and several theorems concerning the Fourier transform of the multiwavelet packets and the construction of multiwavelet packets to show that their translates form an orthonormal basis of $L^2(\mathbb{R}^d)$.

The characterization of all multiwavelets associated with general expanding maps of \mathbb{R}^n has been studied in detail by Calogero [5]. In fact, for a given lattice $\Gamma = P \cdot \mathbb{Z}^d$, with $|\det P| > 0$, and dilation matrix A , he characterized all orthonormal multiwavelets $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\}$ in $L^2(\mathbb{R}^d)$ in terms of two basic equations given by

$$\sum_{\ell=1}^L \sum_{j \in \mathbb{Z}} \left| \hat{\psi}^\ell(A^{*j}\xi) \right|^2 = |\det P| \quad \text{for a.e. } \xi \in \mathbb{R}^d,$$

and

$$\sum_{\ell=1}^L \sum_{j=0}^{\infty} \hat{\psi}^\ell(A^{*j}\xi) \overline{\hat{\psi}^\ell(A^{*j}(\xi + \gamma^*))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^d, \gamma^* \in \Gamma^* \setminus A^*\Gamma^*$$

together with the assumptions $\|\psi^\ell\|_2 \geq 1$ for $\ell = 1, 2, \dots, L$. This characterization follows the strategy of the proof of Frazier *et al.* [10] in the classical case. However, there are several differences. For example, in the general case A will not be isotropic, since there may be different eigenvalues. Moreover, A may contain a rotation, so that a neighbourhood U of the origin in general will not be contained in AU . The Calogero's work was extended by Bownik [2], taking into consideration the dilation matrices which preserves the standard lattice \mathbb{Z}^n in terms of affine systems. In the same year, another characterization of multiwavelets was given by Rzeszotnik [12] for expanding dilations that preserves the lattice \mathbb{Z}^n . However, Bownik [3] has presented a new approach to characterize all orthonormal multiwavelets by means of basic equations in the Fourier domain.

Recently, Shah and Ahmad [14] have given the characterization of all multiwavelet packets associated with the dilation matrix A on the standard lattice \mathbb{Z}^d based on the *dual Gramian* approach. Inspired by the above described work, we further investigate the characterization of all multiwavelet packets in the context of a general lattices Γ in \mathbb{R}^d and a strictly expanding matrix A which preserves the lattice Γ , by virtue of time-frequency analysis, matrix theory and operator theory.

2. Notations and Preliminaries

Throughout, this paper, we use the following notations. Let \mathbb{R} and \mathbb{C} be all real and complex numbers, respectively. \mathbb{Z} and \mathbb{Z}^+ denote all integers and all non-negative integers, respectively. \mathbb{Z}^d and \mathbb{R}^d denote the set of all d -tuples integers and d -tuples of reals, respectively. Assume that we have a lattice Γ ($\Gamma = P\mathbb{Z}^d$ for some non-degenerate $d \times d$ matrix P) of \mathbb{R}^d . Let \mathcal{C} be the fundamental domain of Γ given by $\mathcal{C} = P[-1/2, 1/2)^d$; clearly, $\text{Volume}(\mathcal{C}) = |\det P|$. We define dual lattice Γ^* as

the set of points $\gamma^* \in \mathbb{R}^d$ such that $\langle \gamma, \gamma^* \rangle \in 2\pi\mathbb{Z}$ for all $\gamma \in \Gamma$. It is clear that $\Gamma^* = 2\pi(P^*)^{-1}\mathbb{Z}^d$ and that the volume of the fundamental domain \mathcal{C}^* of Γ^* is $(2\pi)^d/|\det P|$. We here want to note that only the origin belongs to $\Gamma^* \cap \mathcal{C}^*$; so we denote by Γ_0^* the set $\Gamma^* \setminus \{0\}$.

Let A denotes a $d \times d$ dilation matrix, whose determinant is $a(a \in \mathbb{Z}, a \geq 2)$. A $d \times d$ matrix A is said to be a dilation matrix for \mathbb{R}^d if

- (i) $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$
- (ii) all eigenvalues λ of A satisfy $|\lambda| > 1$.

Property (i) implies that A has integer entries and hence $|\det A|$ is greater than 1. Let A^* be the dual of A and set $\Lambda = \{1, 2, \dots, L\}$. Moreover, it is easy to see that the number of distinct cosets of $\Gamma \setminus A\Gamma$ is equal to $a = |\det A|$ (see [20]). It is a well known fact that for every $j \geq 1$, we have that

$$\text{card} \left\{ (A^{*j}\mathcal{C}^*) \cap \Gamma^* \right\} \leq |\det A|^j.$$

Now, we define $\mathcal{O}^* = \Gamma^* \setminus A^*\Gamma^*$. Since $\mathcal{O}^* = \Gamma^* \setminus (\cup_{j>0} A^{*j}\Gamma^*)$, for each $\gamma^* \in \mathcal{O}^*$, $\gamma^* \neq 0$, there exists a unique $(j, \tilde{\gamma}^*) \in (\mathbb{Z}^+ \cup \{0\}) \times \mathcal{O}^*$ such that $\gamma^* = A^{*j}\tilde{\gamma}^*$.

We now recall the notion of higher dimensional multiresolution analysis associated with multiplicity L and orthogonal multiwavelets of $L^2(\mathbb{R}^d)$.

Definition 2.1. A sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ is called a multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A if the following conditions are satisfied:

- (i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- (ii) $\cup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$ and $\cap_{j \in \mathbb{Z}} V_j = \{0\}$;
- (iii) $f \in V_j$ if and only if $f(A \cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
- (iv) there exist L -functions $\varphi_\ell \in V_0, \ell \in \Lambda$, such that the system of functions $\{\varphi_\ell(x - k)\}_{k \in \mathbb{Z}^d, \ell \in \Lambda}$ forms an orthonormal basis for subspace V_0 .

The L -functions whose existence is asserted in (iv) are called *scaling functions* of the given MRA. Given a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$, we define another sequence $\{W_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ by $W_j = V_{j+1} \ominus V_j, j \in \mathbb{Z}$. These subspaces inherit the scaling property of $\{V_j\}$, namely

$$f \in W_j \text{ if and only if } f(A \cdot) \in W_{j+1}. \tag{2.1}$$

Further, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^d) = \bigoplus_{j \in \mathbb{Z}} W_j = V_0 \oplus \left(\bigoplus_{j \geq 0} W_j \right). \tag{2.2}$$

A set of functions $\{\psi_\ell^r : \ell \in \Lambda, 1 \leq r \leq a - 1\}$ in $L^2(\mathbb{R}^d)$ is said to be a set of *basic multiwavelets* associated with the MRA of multiplicity L if the collection

$$\left\{ \psi_\ell^r(\cdot - k) : 1 \leq r \leq a - 1, \ell \in \Lambda, k \in \mathbb{Z}^d \right\}$$

forms an orthonormal basis for W_0 . Now, in view of (2.1) and (2.2), it is clear that if $\{\psi_\ell^r : \ell \in \Lambda, 1 \leq r \leq a-1\}$ is a basic set of multiwavelets, then

$$\left\{ a^{j/2} \psi_\ell^r(A^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell \in \Lambda, 1 \leq r \leq a-1 \right\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$ (see [1, 4]).

For any $n \in \mathbb{Z}^+$, we define the *basic multiwavelet packets* ω_ℓ^n ; $\ell \in \Lambda$ recursively as follows. We denote $\omega_\ell^0 = \varphi_\ell$, $\ell \in \Lambda$, the scaling functions and $\omega_\ell^r = \psi_\ell^r$, $r \in \mathbb{Z}^+$, $\ell \in \Lambda$ as the possible candidates for basic multiwavelets. Then, define

$$\omega_\ell^{s+ar}(x) = \sum_{j \in \Lambda} \sum_{k \in \mathbb{Z}^d} h_{\ell j k}^s a^{1/2} \omega_\ell^r(Ax - k), \quad \ell \in \Lambda, 0 \leq s \leq a-1 \quad (2.3)$$

where $(h_{\ell j k}^s)$ is a unitary matrix (see [1]).

Taking Fourier transform in both sides of (2.3), we obtain

$$(\omega_\ell^{s+ar})^\wedge(\xi) = \sum_{j=1}^L h_{\ell j}^s(B^{-1}\xi) (\omega_\ell^r)^\wedge(B^{-1}\xi). \quad (2.4)$$

Note that (2.3) defines ω_ℓ^n for every non-negative integer n and every ℓ such that $\ell \in \Lambda$. The set of functions $\{\omega_\ell^n : n \in \mathbb{Z}^+, \ell \in \Lambda\}$ as defined above are called the *basic multiwavelet packets* corresponding to the MRA $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ of multiplicity L associated with matrix dilation A .

Definition 2.2. Let $\{\omega_\ell^n : n \in \mathbb{Z}^+, \ell \in \Lambda\}$ be the basic multiwavelet packets. The collection

$$\mathcal{P} = \left\{ |\det A|^{j/2} \omega_\ell^n(A \cdot -k) : \ell \in \Lambda, j \in \mathbb{Z}, k \in \mathbb{Z}^d \right\}$$

is called the *general multiwavelet packets* associated with MRA $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ of multiplicity L over matrix dilation A .

Corresponding to some orthonormal scaling vector $\Phi = \omega_\ell^0$, the family of multiwavelet packets ω_ℓ^n defines a family of subspaces of $L^2(\mathbb{R}^d)$ as follows:

$$U_j^n = \overline{\text{span}} \left\{ a^{j/2} \omega_\ell^n(A^j x - k) : k \in \mathbb{Z}^d, \ell \in \Lambda \right\}; \quad j \in \mathbb{Z}, n \in \mathbb{Z}^+. \quad (2.5)$$

Observe that

$$U_j^0 = V_j, \quad U_j^1 = W_j = \bigoplus_{r=1}^{a-1} U_j^r, \quad j \in \mathbb{Z}$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$, can be written as

$$U_{j+1}^0 = \bigoplus_{r=0}^{a-1} U_j^r. \quad (2.6)$$

A generalization of this result for other values of $n = 1, 2, \dots$ can be written as

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_j^{an+r}, \quad j \in \mathbb{Z}. \tag{2.7}$$

Lemma 2.1. [1]. If $j \geq 0$, then

$$W_j = \bigoplus_{r=0}^{a-1} U_j^r = \bigoplus_{r=a}^{a^2-1} U_{j-1}^r = \dots = \bigoplus_{r=a^t}^{a^{t+1}-1} U_{j-t}^r = \bigoplus_{r=a^j}^{a^{j+1}-1} U_0^r$$

where U_j^n is defined in (2.5). Using this decomposition, we get the multiwavelet packets decomposition of subspaces $W_j, j \geq 0$.

Definition 2.4. The point $x \in \mathbb{R}^d$ is said to be a Lebesgue point of a function f on \mathbb{R}^d if f is integrable in some neighbourhood of x and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{V(B_\varepsilon)} \int_{B_\varepsilon} |f(x) - f(x+y)| dy = 0$$

where B_ε denotes the ball of radius ε about the origin and V denotes volume.

Lemma 2.2. [5]. For any dilation matrix A , there exists a positive integer μ and a constant $\alpha, 0 < \alpha < 1$, such that

$$|A^{*s}\xi| < \alpha^{-s}|\xi|, \quad \text{for every } s \leq -\mu < 0 \text{ and } \xi \in \mathbb{R}^d, \tag{2.8}$$

$$|A^{*s}\xi| > \alpha^{-r}|\xi|, \quad \text{for every } r \geq \mu > 0 \text{ and } \xi \in \mathbb{R}^d. \tag{2.9}$$

Lemma 2.3. [5]. Let $S \subset \mathbb{R}^d$ be a compact set and let $r = r(S)$ be the smallest integer, $r \geq -\mu$, such that $(S - S) \subseteq A^{*r}\mathcal{C}^*$. If

$$\Gamma_j^* = \left\{ \gamma^* \in \Gamma_0^* : A^{*-j}S \cap (A^{*-j}S - \gamma^*) \neq \emptyset \right\},$$

then

$$\text{card} (\Gamma_j^*) \leq |\det A|^{r-j}, \quad \text{for every } j \leq r. \tag{2.10}$$

Lemma 2.4. [5]. Let $S \subset \mathbb{R}^d \setminus \{0\}$ be a compact set. Then, there exists a constant $N = N(S)$ such that for every $\xi \in \mathbb{R}^d$, we have

$$\sum_{j \leq -\mu} \chi_{A^{*-j}S}(\xi) = \sum_{j \leq -\mu} \chi_S(A^{*j}\xi) \leq N < \infty. \tag{2.11}$$

Lemma 2.5. [5]. Let k be the smallest integer such that $2\overline{\mathcal{C}^*} \subseteq A^{*k}\mathcal{C}^*$ and let σ be the largest integer such that, for every $p \leq \sigma$, $A^{*p}\mathcal{C}^* \subseteq \mathcal{C}^*$. Moreover, for every positive integer s and every negative integer p , set

$$\mathcal{O}_{s,p}^* = \left\{ \gamma^* \in \mathcal{O}^* : (A^{*-p}(A^{*-s}\overline{\mathcal{C}^*} + \xi_0)) \cap (A^{*-p}(A^{*-s}\overline{\mathcal{C}^*} + \xi_0) + A^{*-p}\gamma_0^* - \gamma^*) \neq \emptyset \right\}.$$

Then

$$\text{card}(\mathcal{O}_{s,p}^*) \leq |\det A|^{k-s-p-\sigma}, \quad \text{for every } k-p-s > \sigma;$$

$$\text{card}(\mathcal{O}_{s,p}^*) = 0, \quad \text{for every } k-p-s \leq \sigma.$$

We will also consider the set \mathcal{D} as a dense subset of $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{D} = \left\{ f \in L^2(\mathbb{R}^d) : \hat{f} \in L^\infty(\mathbb{R}^d), \hat{f} \text{ has compact support in } \mathbb{R}^d \setminus \{0\} \right\}.$$

3. Main Results

Theorem 3.1. Let $\{\omega_\ell^n : n \in \mathbb{Z}^+, \ell \in \Lambda\}$ be the basic orthonormal multiwavelet packets associated with the scaling functions φ_ℓ . Then

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma} |\langle f, \omega_{\ell,j,\gamma}^n \rangle|^2 = \|f\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}^d) \quad (3.1)$$

if and only if

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} |\hat{\omega}_\ell^n(A^{*j}\xi)|^2 = |\det P|, \quad \text{a.e. } \xi \in \mathbb{R}^d \quad (3.2)$$

$$t_{\gamma^*}(\xi) = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j=0}^{\infty} \hat{\omega}_\ell^n(A^{*j}\xi) \overline{\hat{\omega}_\ell^n(A^{*j}(\xi + \gamma^*))} = 0, \quad \text{a.e. } \xi \in \mathbb{R}^d, \gamma^* \in \mathcal{O}^*. \quad (3.3)$$

Proof . We first show that the function t_{γ^*} given by (3.3) is a well-defined function belonging to $L^1(\mathbb{R}^d)$. We have

$$\begin{aligned} \|t_{\gamma^*}\|_1 &= \int_{\mathbb{R}^d} |t_{\gamma^*}(\xi)| d\xi \leq \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j=0}^{\infty} \int_{\mathbb{R}^d} \left| \hat{\omega}_\ell^n(A^{*j}\xi) \hat{\omega}_\ell^n(A^{*j}(\xi + \gamma^*)) \right| d\xi \\ &= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j=0}^{\infty} |\det A|^{-j} \int_{\mathbb{R}^d} \left| \hat{\omega}_\ell^n(\xi) \hat{\omega}_\ell^n(\xi + A^{*j}\gamma^*) \right| d\xi \\ &\leq \sum_{j=0}^{\infty} |\det A|^{-j} \left(\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \|\hat{\omega}_\ell^n\|_2 \|\hat{\omega}_\ell^n\|_2 \right) < \infty. \end{aligned}$$

We now come back to our proof. We call I the second member of (3.1) and split I in two summands I_1, I_2 . Using Plancherel formula on \mathbb{R}^d , we obtain

$$\begin{aligned}
 (2\pi)^{2d} I &= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma} \left| \langle \hat{f}, \hat{\omega}_{\ell, j, \gamma}^n \rangle \right|^2 \\
 &= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma} \left| \int_{\mathbb{R}^d} |\det A|^{-j/2} \hat{f}(\xi) \overline{\hat{\omega}_{\ell}^n(A^{*-j}\xi)} e^{i\langle \gamma, A^{*-j}\xi \rangle} d\xi \right|^2 \\
 &= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma} |\det A|^j \left| \int_{\mathbb{R}^d} \hat{f}(A^{*j}\xi) \overline{\hat{\omega}_{\ell}^n(\xi)} e^{i\langle \gamma, \xi \rangle} d\xi \right|^2. \tag{3.4}
 \end{aligned}$$

Let $F(\xi) = \hat{f}(A^{*j}\xi) \overline{\hat{\omega}_{\ell}^n(\xi)}$. Then, clearly F has a compact support in $\mathbb{R}^d \setminus \{0\}$ and $F \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ as $f \in \mathcal{D}$. Since $\{\mathcal{C}^* + \gamma^*\}_{\gamma^* \in \Gamma^*}$ is a partition of \mathbb{R}^d , therefore, we have

$$\hat{F}(\gamma) = \int_{\mathcal{C}^*} \left(\sum_{\gamma^* \in \Gamma^*} F(\xi + \gamma^*) \right) e^{-i\langle \gamma, \xi \rangle} d\xi = \frac{(2\pi)^d}{|\det P|} \left(\sum_{\gamma^* \in \Gamma^*} F(\xi + \gamma^*) \right)^\wedge (\gamma) \tag{3.5}$$

where the sum over Γ^* has only finitely many non-zero terms because F is compactly supported. Hence, $\sum_{\gamma^* \in \Gamma^*} F(\xi + \gamma^*) \in L^2(\mathbb{R}^d \setminus \Gamma^*)$.

Using Plancherel formula on $L^2(\mathbb{R}^d \setminus \Gamma^*)$ and by virtue of (3.5), we obtain

$$\begin{aligned}
 \frac{|\det P|}{(2\pi)^d} \sum_{\gamma \in \Gamma} |\hat{F}(\gamma)|^2 &= \frac{(2\pi)^d}{|\det P|} \sum_{\gamma \in \Gamma} \left| \left(\sum_{\gamma^* \in \Gamma^*} F(\xi + \gamma^*) \right)^\wedge (\gamma) \right|^2 \\
 &= \int_{\mathcal{C}^*} \left| \sum_{\gamma^* \in \Gamma^*} F(\xi + \gamma^*) \right|^2 d\xi \\
 &= \int_{\mathbb{R}^d} \sum_{\gamma^* \in \Gamma^*} F(\xi + \gamma^*) \overline{F(\xi)} d\xi, \tag{3.6}
 \end{aligned}$$

where the interchange of summation and integration is justified by the compact support of F . Also, by (3.6) and (3.4), we have

$$\begin{aligned}
 |\det P| (2\pi)^d I &= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} |\det A|^j \int_{\mathbb{R}^d} \left| \hat{f}(A^{*j}\xi) \right|^2 \left| \hat{\omega}_{\ell}^n(\xi) \right|^2 d\xi \\
 &\quad + \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} |\det A|^j \int_{\mathbb{R}^d} \overline{\hat{f}(A^{*j}\xi)} \hat{\omega}_{\ell}^n(\xi) \\
 &\quad \times \left(\sum_{\gamma^* \neq 0} \hat{f}(A^{*j}(\xi + \gamma^*)) \overline{\hat{\omega}_{\ell}^n(\xi + \gamma^*)} \right) d\xi. \tag{3.7}
 \end{aligned}$$

We write I_1 for the first sum of the terms in (3.7) and I_2 for the second sum of the terms in the same expansion. Thus

$$|\det P| (2\pi)^d I = I_1 + I_2, \quad (3.8)$$

where we can rewrite I_1 as

$$I_1 = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\hat{\omega}_\ell^n(A^{*j}\xi)|^2 d\xi. \quad (3.9)$$

□

Lemma 3.2. *For every $f \in \mathcal{D}$ and $\omega_\ell^n \in L^2(\mathbb{R}^d)$, we have*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} |\det A|^j \sum_{\gamma^* \in \Gamma_0^*} \int_{\mathbb{R}^d} \left| \hat{f}(A^{*j}\xi) \hat{f}(A^*(\xi + \gamma^*)) \hat{\omega}_\ell^n(\xi) \hat{\omega}_\ell^n(\xi + \gamma^*) \right| d\xi < \infty.$$

Proof . Let $\text{supp}(f) \subset S = \{\xi \in \mathbb{R}^d : a < |\xi| < b\}$ with $0 < a < b$. Then, by (2.9), there exists a positive integer m such that for every $j \geq m$, we have that $|A^{*j}\gamma^*| > 2b$, for every $\gamma^* \in \Gamma_0^*$. Hence, for such j and γ^* , we have

$$\hat{f}(A^{*j}\xi) \hat{f}(A^*(\xi + \gamma^*)) = 0$$

and

$$\begin{aligned} & \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} |\det A|^j \sum_{\gamma^* \in \Gamma_0^*} \int_{\mathbb{R}^d} \left| \hat{f}(A^{*j}\xi) \hat{f}(A^*(\xi + \gamma^*)) \hat{\omega}_\ell^n(\xi) \hat{\omega}_\ell^n(\xi + \gamma^*) \right| d\xi \\ &= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j < m} |\det A|^j \sum_{\gamma^* \in \Gamma_0^*} \\ & \quad \times \int_{(A^{*-j}S) \cap (A^{*-j}S - \gamma^*)} \left| \hat{f}(A^{*j}\xi) \hat{f}(A^*(\xi + \gamma^*)) \hat{\omega}_\ell^n(\xi) \hat{\omega}_\ell^n(\xi + \gamma^*) \right| d\xi \\ &\leq \frac{1}{2} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j < m} |\det A|^j \sum_{\gamma^* \in \Gamma_0^*} \left[\int_{(A^{*-j}S) \cap (A^{*-j}S - \gamma^*)} \left| \hat{f}(A^{*j}\xi) \hat{\omega}_\ell^n(\xi) \right|^2 d\xi \right. \\ & \quad \left. + \int_{(A^{*-j}S) \cap (A^{*-j}S - \gamma^*)} \left| \hat{f}(A^{*j}(\xi + \gamma^*)) \hat{\omega}_\ell^n(\xi + \gamma^*) \right|^2 d\xi \right] \\ &\leq \frac{1}{2} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j < m} |\det A|^j \sum_{\gamma^* \in \Gamma_0^*} \left[\int_{(A^{*-j}S) \cap (A^{*-j}S - \gamma^*)} \left| \hat{f}(A^{*j}\xi) \hat{\omega}_\ell^n(\xi) \right|^2 d\xi \right. \\ & \quad \left. + \int_{(A^{*-j}S) \cap (A^{*-j}S + \gamma^*)} \left| \hat{f}(A^{*j}\xi) \hat{\omega}_\ell^n(\xi) \right|^2 d\xi \right] \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{-\mu < j < m} |det A|^j \sum_{\gamma^* \in \Gamma_0^*} \int_{(A^{*-j}S) \cap (A^{*-j}S + \gamma^*)} \left| \hat{f}(A^{*j}\xi) \hat{\omega}_\ell^n(\xi) \right|^2 d\xi \\ &+ \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j < -\mu} |det A|^j \sum_{\gamma^* \in \Gamma_0^*} \int_{(A^{*-j}S) \cap (A^{*-j}S + \gamma^*)} \left| \hat{f}(A^{*j}\xi) \hat{\omega}_\ell^n(\xi) \right|^2 d\xi \\ &\equiv \sigma_1 + \sigma_2. \end{aligned}$$

The evaluation of σ_1 is very easy:

$$\begin{aligned} \sigma_1 &\leq (m + \mu - 1) \left\| \hat{f} \right\|_\infty^2 \left\| \hat{\omega}_\ell^n \right\|_2^2 |det A|^m \cdot \max_{-\mu < j < m} \left\{ \text{card} \left\{ \gamma^* \in \Gamma_0^* : |A^{*j}\gamma^*| \leq 2b \right\} \right\} \\ &\leq \text{const} \left\| \hat{f} \right\|_\infty^2 \left\| \hat{\omega}_\ell^n \right\|_2^2 < \infty. \end{aligned} \tag{3.10}$$

Evaluation of σ_2 is more complicated and to do this we must use Lemma 2.6, and Lemma 2.7. Let Γ_j^* , r , and N be as in these lemmas. Then, we have

$$\begin{aligned} \sigma_2 &\leq \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j < -\mu} |det A|^j \sum_{\gamma^* \in \Gamma_j^*} \int_{A^{*-j}S} \left| \hat{f}(A^{*j}\xi) \hat{\omega}_\ell^n(\xi) \right|^2 d\xi \\ &\leq \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j < -\mu} |det A|^j |det A|^{r-j} \left\| \hat{f} \right\|_\infty^2 \int_{A^{*-j}S} \left| \hat{\omega}_\ell^n(\xi) \right|^2 d\xi \\ &\leq N |det A|^r \left\| \hat{f} \right\|_\infty^2 \left\| \hat{\omega}_\ell^n \right\|_2^2 < \infty. \end{aligned} \tag{3.11}$$

Therefore, from (3.10) and (3.11) we get the required result. \square

An immediate consequence of Lemma 3.2 is the following:

Corollary 3.3. *If*

$$I = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma} \left| \langle f, \omega_{\ell,j,\gamma}^n \rangle \right|^2 < \infty, \text{ for all } f \in \mathcal{D}.$$

Then the function

$$\tau(\xi) = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \left| \hat{\omega}_\ell^n(A^{*j}\xi) \right|^2 \tag{3.12}$$

is locally integrable in $\mathbb{R}^d \setminus \{0\}$.

Proof . From Lemma 3.2 and (3.8), we know that $I < \infty$ if and only if $I_1 < \infty$. Choosing $\hat{f} = \chi_S$, where S is a compact set in $\mathbb{R}^d \setminus \{0\}$, we have the corollary. \square

We manipulate the expression of I_2 to introduce t_{γ^*} as in (3.3). We know, by definition of \mathcal{O}^* , that for every $\gamma^* \in \Gamma_0^*$ there exist a unique $\gamma^{*'} \in \mathcal{O}^*$ and a unique non-negative integer q such that $\gamma^* = A^{*q}\gamma^{*'}$. So, we have

$$\begin{aligned}
I_2 &= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \hat{\omega}_\ell^n(A^{*-j}\xi) \sum_{\gamma^* \in \Gamma_0^*} \hat{f}(\xi + A^{*j}\gamma^*) \overline{\hat{\omega}_\ell^n(A^{*-j}\xi + \gamma^*)} d\xi \\
&= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \sum_{\gamma^* \in \mathcal{O}^*} \sum_{q \geq 0} \sum_{j \in \mathbb{Z}} \hat{\omega}_\ell^n(A^{*-j}\xi) \hat{f}(\xi + A^{*j+q}\gamma^*) \overline{\hat{\omega}_\ell^n(A^{*-j}\xi + A^{*q}\gamma^*)} d\xi \\
&= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \sum_{\gamma^* \in \mathcal{O}^*} \sum_{q \geq 0} \sum_{p \in \mathbb{Z}} \hat{\omega}_\ell^n(A^{*q-p}\xi) \hat{f}(\xi + A^{*p}\gamma^*) \overline{\hat{\omega}_\ell^n(A^{*q}(A^{*-p}\xi + \gamma^*))} d\xi \\
&= \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \sum_{\gamma^* \in \mathcal{O}^*} \sum_{p \in \mathbb{Z}} \hat{f}(\xi + A^{*p}\gamma^*) t_{\gamma^*}(A^{*-p}\xi) d\xi.
\end{aligned}$$

Thus, we have rewritten I_2 as

$$I_2 = \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \sum_{\gamma^* \in \mathcal{O}^*} \sum_{p \in \mathbb{Z}} \hat{f}(\xi + A^{*p}\gamma^*) t_{\gamma^*}(A^{*-p}\xi) d\xi. \quad (3.13)$$

Sufficient part: Assume that (3.2) and (3.3) are satisfied, then by (3.7) we have that $I_1 = |\det P| \|\hat{f}\|_2^2$ and by (3.13) we have that $I_2 = 0$. Therefore, by (3.8) and Plancherel formula on \mathbb{R}^d , we obtain

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma} |\langle f, \omega_{\ell,j,\gamma}^n \rangle|^2 = I = \frac{1}{|\det P|(2\pi)^d} (I_1 + I_2) = \frac{1}{(2\pi)^d} \|\hat{f}\|_2^2 = \|f\|_2^2.$$

Necessary part: Suppose that (3.1) holds. Let τ be the function defined by (3.12). Note that the family of functions $\{g_s\}_{s \geq 1}$, defined by

$$g_s(\xi) = \frac{1}{\text{meas}(A^{*-s}\mathcal{C}^*)} \chi_{(A^{*-s}\mathcal{C}^*)}(\xi),$$

is an approximation of the identity unit (see [18]). Let $B = B(x_0, \rho)$ be the open ball with center $x_0 \in \mathbb{R}^d$ and radius $\rho > 0$. We have, by Corollary 3.3, that $\tau \cdot \chi_B \in L^1(\mathbb{R}^d)$. Thus, $\|g_s * (\tau \cdot \chi_B) - \tau \cdot \chi_B\|_{L^1} \rightarrow 0$, so that there exists a subsequence $\{s_j\}_{j \geq 1}$ such that

$$\lim_{j \rightarrow +\infty} \frac{1}{\text{meas}(A^{*-s_j}\mathcal{C}^*)} \int_{(A^{*-s_j}\mathcal{C}^* + \xi_0)} \tau(\xi) \cdot \chi_B(\xi) d\xi = \tau(\xi_0) \cdot \chi_B(\xi_0),$$

for *a.e.* $\xi_0 \in \mathbb{R}^d$. Therefore, it suffices to choose $\xi_0 \in B$ to have, for *a.e.* $\xi_0 \in \mathbb{R}^d$,

$$\lim_{j \rightarrow +\infty} \frac{1}{\text{meas}(A^{*-sj}\mathcal{C}^*)} \int_{(A^{*-sj}\mathcal{C}^* + \xi_0)} \tau(\xi) d\xi = \tau(\xi_0). \tag{3.14}$$

Let $\xi_0 \neq 0$ be a fixed point such that (3.14) holds and define the function \hat{f}_s as

$$\hat{f}_s(\xi) = \sqrt{\frac{|\det A|^s}{(2\pi)^d}} \chi_{(A^{*-s}\mathcal{C}^* + \xi_0)}(\xi).$$

We note that for s sufficiently large we have $0 \notin A^{*-s}\overline{\mathcal{C}^*} + \xi_0$. Clearly, $f_s \in \mathcal{D}$ and $\text{meas}(A^{*-s}\mathcal{C}^*) = (2\pi)^d |\det A|^{-s} |\det P|^{-1}$ imply that

$$\|\hat{f}_s\|_2^2 = \frac{1}{|\det P|}.$$

Using the same notation as in (3.8), and adding the superscript s to denote the dependence on this choice of f_s , we have

$$|\det P|(2\pi)^d I^s = I_1^s + I_2^s. \tag{3.15}$$

Since (3.1) holds, we have $I^s = \|f_s\|_2^2 = (2\pi)^{-d} \|\hat{f}_s\|_2^2 = (2\pi)^{-d} |\det P|^{-1}$. Using the definition of I_1^s in (3.9), (3.15) becomes

$$1 = \frac{1}{|\det P| \cdot \text{meas}(A^{*-s}\mathcal{C}^*)} \int_{(A^{*-s}\mathcal{C}^* + \xi_0)} \tau(\xi) d\xi + I_2^s.$$

In order to prove (3.2), it is necessary to show that $I_2^s \rightarrow 0$, for $s \rightarrow +\infty$, which in turn imply that $I_1^s \rightarrow 1$ and, by (3.14), we have that $\tau(\xi_0) = |\det P|$. Hence, we have only to prove that $\lim_{s \rightarrow +\infty} I_2^s = 0$.

Setting

$$\mathcal{I}_2^s = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} |\det A|^j \sum_{\gamma^* \in \Gamma_0^*} \int_{\mathbb{R}^d} \left| \hat{f}_s(A^{*j}\xi) \hat{f}_s(A^{*j}(\xi + \gamma^*)) \hat{\omega}_\ell^n(\xi) \hat{\omega}_\ell^n(\xi + \gamma^*) \right| d\xi.$$

Clearly, it suffices to show that for every $\omega_\ell^n \in L^2(\mathbb{R}^d)$, we have

$$\lim_{s \rightarrow +\infty} I_2^s = 0. \tag{3.16}$$

Set $S = A^{*-s}\overline{\mathcal{C}^*} + \xi_0$ and write \mathcal{I}_2^s as

$$\mathcal{I}_2^s = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \in \mathbb{Z}} \frac{|\det A|^{j+s}}{(2\pi)^d} \sum_{\gamma^* \in \Gamma_0^*} \int_{(A^{*-j}S) \cap (A^{*-j}S - \gamma^*)} \left| \hat{\omega}_\ell^n(\xi) \hat{\omega}_\ell^n(\xi + \gamma^*) \right| d\xi.$$

If the integral in I_2^s is different from zero, then the set of integration is not empty, so that we have necessarily

$$\text{diam} (A^{*-j}(A^{*-s}\mathcal{C}^* + \xi_0)) = \text{diam} (A^{*-j-s}\mathcal{C}^*) \geq \min \{|\gamma^*| : \gamma^* \in \Gamma_0^*\}.$$

This is possible only if $-j - s \geq j_0$, where j_0 is a fixed integer depending on the lattice and on the expanding map. Since for $n = 1, 2, \dots$, $\ell \in \Lambda$, we have

$$2 |\hat{\omega}_\ell^n(\xi) \hat{\omega}_\ell^n(\xi + \gamma^*)| \leq |\hat{\omega}_\ell^n(\xi)|^2 + |\hat{\omega}_\ell^n(\xi + \gamma^*)|^2.$$

By changing the variables as in the proof of Lemma 3.2, we obtain

$$\mathcal{I}_2^s = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j=-\infty}^{-j_0-s} \frac{|\det A|^{j+s}}{(2\pi)^d} \sum_{\gamma^* \in \Gamma_0^*} \int_{(A^{*-j}S) \cap (A^{*-j}S - \gamma^*)} |\hat{\omega}_\ell^n(\xi)|^2 d\xi. \quad (3.17)$$

Since $\overline{\mathcal{C}^*} = -\overline{\mathcal{C}^*}$, we have that

$$(S - S) = \left((A^{*-s}\overline{\mathcal{C}^*} + \xi_0) - (A^{*-s}\overline{\mathcal{C}^*} + \xi_0) \right) = A^{*-s}(\overline{\mathcal{C}^*} - \overline{\mathcal{C}^*}) = A^{*-s}(2\overline{\mathcal{C}^*}).$$

Let k be the positive integer such that $2\overline{\mathcal{C}^*} \subseteq A^{*k}\mathcal{C}^*$. Then, $(S - S) \subseteq A^{*k-s}\mathcal{C}^*$. By applying Lemma 2.3 with $S = A^{*-s}\overline{\mathcal{C}^*} + \xi_0$ and $r = k - s$, we obtain

$$\text{card} \left(\{ \gamma^* \in \Gamma_0^* : A^{*-j}S \cap (A^{*-j}S - \gamma^*) \neq \emptyset \} \right) \leq |\det A|^{k-j-s}, \quad (3.18)$$

for every $j \leq k - s$. Now, if s is sufficiently large, then by (3.17) and (3.18), we obtain

$$\mathcal{I}_2^s \leq \frac{|\det A|^k}{(2\pi)^d} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j=-\infty}^{-j_0-s} \int_{(A^{*-j}(A^{*-s}\mathcal{C}^* + \xi_0))} |\hat{\omega}_\ell^n(\xi)|^2 d\xi. \quad (3.19)$$

Now, we choose s_0 (sufficiently large) and a compact set $K \subset \mathbb{R}^d \setminus \{0\}$ such that, for every $s \geq s_0$, $(A^{*-s}\mathcal{C}^* + \xi_0) \subseteq K$ and if N is as in Lemma 2.7, we have from (3.19)

$$\begin{aligned} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j=-\infty}^{-j_0-s_0} \int_{(A^{*-j}(A^{*-s}\mathcal{C}^* + \xi_0))} |\hat{\omega}_\ell^n(\xi)|^2 d\xi &\leq \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j=-\infty}^{-j_0-s_0} \int_{A^{*-j}K} |\hat{\omega}_\ell^n(\xi)|^2 d\xi \\ &\leq \int_{\mathbb{R}^d} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \left(\sum_{j=-\infty}^{-j_0-s_0} \chi_{A^{*-j}K}(\xi) \right) |\hat{\omega}_\ell^n(\xi)|^2 d\xi \\ &\leq N \|\hat{\omega}_\ell^n\|^2. \end{aligned}$$

Since I_2^s is the tail of a convergent series, we obtain (3.16) and consequently (3.2).

We now prove (3.1) \Rightarrow (3.3). By taking (3.9) into account, we can write (3.8) as

$$\|f\|_2^2 = I = \frac{1}{(2\pi)^d |\det P|} (I_1 + I_2) = \|f\|_2^2 + \frac{I_2}{(2\pi)^d |\det P|}, \quad \text{for every } f \in \mathcal{D},$$

with I_2 as in (3.13). An application of the polarization identity then gives us

$$\int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \sum_{\gamma^* \in \mathcal{O}^*} \sum_{p \in \mathbb{Z}} \hat{g}(\xi + A^{*p} \gamma^*) t_{\gamma^*}(A^{*-p} \xi) d\xi = 0, \quad \text{for every } f, g \in \mathcal{D}. \tag{3.20}$$

Let $\gamma^* \in \mathcal{O}^*$ be fixed. Then, we have $t_{\gamma^*} \in L^1(\mathbb{R}^d)$. Let ξ_0 be a Lebesgue point for t_{γ^*} such that $\xi_0 \neq 0$ and $\xi_0 + \gamma_0^* \neq 0$. Let f_s and g_s be such that

$$\hat{f}_s(\xi) = \sqrt{\frac{|\det A|^s}{(2\pi)^d}} \chi_{(A^{*-s} \mathcal{C}^* + \xi_0)}(\xi) \quad \text{and} \quad \hat{g}_s(\xi) = \sqrt{\frac{|\det A|^s}{(2\pi)^d}} \chi_{(A^{*-s} \mathcal{C}^* + \xi_0 + \gamma_0^*)}(\xi).$$

For sufficiently large s , we have

$$0 \notin (A^{*-s} \overline{\mathcal{C}^*} + \xi_0) \cap (A^{*-s} \overline{\mathcal{C}^*} + \xi_0 + \gamma_0^*),$$

so that $f_s, g_s \in \mathcal{D}$. With this choice of f and g the expression in (3.20) becomes

$$\begin{aligned} 0 &= \frac{|\det A|^s}{(2\pi)^d} \int_{(A^{*-s} \mathcal{C}^* + \xi_0)} t_{\gamma_0^*}(\xi) d\xi + \sum_{\substack{(p, \gamma^*) \in \mathbb{Z} \times \mathcal{O}^* \\ (p, \gamma^*) \neq (0, \gamma_0^*)}} \int_{\mathbb{R}^d} \hat{f}_s(\xi) \hat{g}_s(\xi + A^{*p} \gamma^*) t_{\gamma^*}(A^{*-p} \xi) d\xi \\ &= \frac{1}{|\det P| \cdot \text{meas}(A^{*-s} \mathcal{C}^*)} \int_{(A^{*-s} \mathcal{C}^* + \xi_0)} t_{\gamma_0^*}(\xi) d\xi + J^s. \end{aligned}$$

Now, we show that $J^s \rightarrow 0$, as $s \rightarrow +\infty$, which in turn implies that

$$0 = \lim_{s \rightarrow +\infty} \frac{1}{\text{meas}(A^{*-s} \mathcal{C}^*)} \int_{(A^{*-s} \mathcal{C}^* + \xi_0)} t_{\gamma_0^*}(\xi) d\xi = t_{\gamma_0^*}(\xi_0),$$

for *a.e.* $\xi_0 \in \mathbb{R}^d$, and as a result, we obtain (3.3).

We now compute separately the summation for $p \geq 0$ and $p < 0$ in J^s . To do so, set

$$J^s = J_+^s + J_-^s = \sum_{\substack{p \geq 0, \gamma^* \in \mathcal{O}^* \\ (p, \gamma^*) \neq (0, \gamma_0^*)}} \int_{\mathbb{R}^d} \dots d\xi + \sum_{p < 0; \gamma^* \in \mathcal{O}^*} \int_{\mathbb{R}^d} \dots d\xi. \tag{3.21}$$

We start with J_+^s . If there exists ξ such that $\hat{f}_s(\xi) \hat{g}_s(\xi + A^{*p} \gamma^*) \neq 0$, then $\xi \in (A^{*-s} \overline{\mathcal{C}^*} + \xi_0) \cap (A^{*-s} \overline{\mathcal{C}^*} + \xi_0 - A^{*p} \gamma^* + \gamma_0^*)$ so that in this case, we have

$$\text{diam}(A^{*-s} \mathcal{C}^* + \xi_0) \geq \min \left\{ |\gamma_0^* - A^{*p} \gamma^*| : p \geq 0; \gamma^* \in \mathcal{O}^*; (p, \gamma^*) \neq (0, \gamma_0^*) \right\}. \tag{3.22}$$

Since $\gamma_0^* \in \mathcal{O}^*$ and $(p, \gamma^*) \neq (0, \gamma_0^*)$, we have $A^{*p} \gamma^* \neq \gamma_0^*$ for every $p \geq 0$. Then, there exists a positive constant δ such that

$$\begin{aligned} \min \left\{ |\gamma_0^* - A^{*p}\gamma^*| : p \geq 0; \gamma^* \in \mathcal{O}^*; (p, \gamma^*) \neq (0, \gamma^*) \right\} \\ \geq \min \left\{ |\gamma^* - \gamma_1^*| : \gamma^*, \gamma_1^* \in \Gamma^*; \gamma^* \neq \gamma_1^* \right\} = \delta. \end{aligned} \quad (3.23)$$

Hence, by (3.22) and (3.23), $\text{diam}(A^{*-s}\mathcal{C}^* + \xi_0) \geq \delta$ which is absurd for $s \rightarrow +\infty$. Thus, for sufficiently large s , we have $\hat{f}_s(\xi)\hat{g}_s(\xi + A^{*p}\gamma^*) = 0$ for all ξ , whence

$$\lim_{s \rightarrow +\infty} J_+^s = 0. \quad (3.24)$$

We consider J_-^s and we have

$$\begin{aligned} |t_{\gamma^*}(\xi)| &\leq \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \geq 0} |\hat{\omega}_\ell^n(A^{*j}\xi) \hat{\omega}_\ell^n(A^{*j}(\xi + \gamma^*))| \\ &\leq \frac{1}{2} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \geq 0} \left(|\hat{\omega}_\ell^n(A^{*j}\xi)|^2 + |\hat{\omega}_\ell^n(A^{*j}(\xi + \gamma^*))|^2 \right). \end{aligned}$$

Thus

$$\begin{aligned} |J_-^s| &\leq \frac{1}{2} \sum_{\gamma^* \in \mathcal{O}^*} \sum_{p < 0} |\det A|^p \int_{\mathbb{R}^d} \hat{f}_s(A^{*p}\xi) \hat{g}_s(A^{*p}(\xi + \gamma^*)) \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \geq 0} |\hat{\omega}_\ell^n(A^{*j}\xi)|^2 d\xi \\ &\quad + \frac{1}{2} \sum_{\gamma^* \in \mathcal{O}^*} \sum_{p < 0} |\det A|^p \int_{\mathbb{R}^d} \hat{f}_s(A^{*p}\xi) \hat{g}_s(A^{*p}(\xi + \gamma^*)) \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \geq 0} |\hat{\omega}_\ell^n(A^{*j}(\xi + \gamma^*))|^2 d\xi. \end{aligned} \quad (3.25)$$

We note that the first integral, after the change of variables $\xi' = \xi + \gamma^*$, is essentially the same as the second integral, with the role of f_s and g_s interchanged. Hence, it suffices to show that the first summand in (3.25) goes to zero when $s \rightarrow +\infty$.

Define an operator

$$T(\xi) = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \geq 0} |\hat{\omega}_\ell^n(A^{*j}\xi)|^2.$$

An easy computation shows that $T \in L^1(\mathbb{R}^d)$:

$$\begin{aligned} \|T\|_1 &= \int_{\mathbb{R}^d} |T(\xi)| d\xi = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \geq 0} |\det A|^{-j} \int_{\mathbb{R}^d} |\hat{\omega}_\ell^n(A^{*j}\xi)|^2 d\xi \\ &= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell \in \Lambda} \sum_{j \geq 0} |\det A|^{-j} \|\hat{\omega}_\ell^n\|_2^2 < \infty. \end{aligned}$$

Set

$$\mathcal{J}_s = \sum_{\gamma^* \in \mathcal{O}^*} \sum_{p < 0} |\det A|^p \int_{\mathbb{R}^d} \hat{f}_s(A^{*p}\xi) \hat{g}_s(A^{*p}(\xi + \gamma^*)) T(\xi) d\xi.$$

Therefore, in order to show that $\mathcal{J}_s \rightarrow 0$ for $s \rightarrow +\infty$, we have to prove that $\lim_{s \rightarrow +\infty} \mathcal{J}_s = 0$.

Using Lemma 2.8, we have that

$$\begin{aligned} \mathcal{J}_s &= \sum_{p < 0} |\det A|^p \int_{\mathbb{R}^d} \sum_{\gamma^* \in \mathcal{O}_{s,p}^*} \hat{f}_s(A^{*p}\xi) \hat{g}_s(A^{*p}(\xi + \gamma^*)) T(\xi) d\xi \\ &\leq \sum_{p < k-s-\sigma} \frac{|\det A|^{p+s}}{(2\pi)^d} \text{card}(\mathcal{O}_{s,p}^*) \int_{(A^{*-p}(\text{supp } \hat{f}_s))} T(\xi) d\xi \\ &\leq \frac{|\det A|^{k-\sigma}}{(2\pi)^d} \sum_{p < k-s-\sigma} \int_{(A^{*-p}(\text{supp } \hat{f}_s))} T(\xi) d\xi. \end{aligned} \tag{3.26}$$

Let s_0 be such that $0 \notin A^{*-s_0}\overline{\mathcal{C}^*} + \xi_0$. Since $\overline{\mathcal{C}^*}$ is compact, therefore, we have

$$\delta' = \text{dist}(0, A^{*-s_0}\mathcal{C}^* + \xi_0) > 0.$$

Further, by the definition of σ , we have

$$(A^{*-s_0}\mathcal{C}^* + \xi_0) = (A^{*-s_0}(A^{*-s_0-s}\mathcal{C}^*) + \xi_0) \subseteq (A^{*-s_0}\mathcal{C}^* + \xi_0)$$

for every $s \geq s_0 - \sigma$. These inclusions and (2.9) imply that, for every $s \geq s_0 - \sigma$ and for every $-p \geq \mu$,

$$\text{dist}(0, A^{*-p}(A^{*-s}\mathcal{C}^* + \xi_0)) \geq \alpha^p \text{dist}(0, A^{*-s}\mathcal{C}^* + \xi_0) \geq \alpha^p \delta', \tag{3.27}$$

where $\alpha < 1$ as in Lemma 2.2, is smaller than 1. Hence from (3.26), (3.27), and Lemma 2.7, we have that, for s sufficiently large,

$$\begin{aligned} \mathcal{J}_s &\leq \frac{|\det A|^{k-\sigma}}{(2\pi)^d} \sum_{p < k-s-\sigma} \int_{(A^{*-p}(A^{*-s}\overline{\mathcal{C}^*} + \xi_0))} T(\xi) \cdot \chi_{(A^{*-p}(A^{*-s}\overline{\mathcal{C}^*} + \xi_0))}(\xi) d\xi \\ &\leq \frac{|\det A|^{k-\sigma}}{(2\pi)^d} \sum_{p < k-s-\sigma} \int_{|\xi| \geq \alpha^p \delta'} T(\xi) \cdot \chi_{(A^{*-p}(A^{*-s_0}\overline{\mathcal{C}^*} + \xi_0))}(\xi) d\xi \\ &\leq \frac{|\det A|^{k-\sigma}}{(2\pi)^d} \int_{|\xi| \geq \alpha^{k-s-\sigma} \delta'} T(\xi) \sum_{p < k-s-\sigma} \chi_{(A^{*-p}(A^{*-s_0}\overline{\mathcal{C}^*} + \xi_0))}(\xi) d\xi \\ &\leq \frac{N|\det A|^{k-\sigma}}{(2\pi)^d} \int_{|\xi| \geq \alpha^{k-s-\sigma} \delta'} T(\xi) d\xi. \end{aligned}$$

Since $\alpha < 1$, k and σ are fixed, this implies that $\lim_{s \rightarrow +\infty} \mathcal{J}_s = 0$. Hence, (3.3) is true for every $\gamma_0^* \in \mathcal{O}^*$ and for every Lebesgue point ξ_0 of $\tau_{\gamma_0^*}$. This completes the proof of the theorem.

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