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A Finite Difference Method for the Smooth Solution of Linear Volterra Integral Equations

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Abstract

The present paper proposes a fast numerical method for the linear Volterra integral equations with regular and weakly singular kernels having smooth solutions. This method is based on the approximation of the kernel, to simplify the integral operator and then discretization of the simplified operator using a forward difference formula. To analyze and verify the accuracy of the method, we examine sample and benchmark problems with known exact solutions.

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1. Introduction and preliminaries

Linear integral equations of Volterra type arise in a variety of science and technology fields [5]. From theoretical point of view, a powerful technique for studying the existence and uniqueness of many initial-boundary value problems is to derive an equivalent system of integral equations, that one can apply an existence and uniqueness theory. For example in [5] some classes of the initial-boundary value problems of heat conduction have been transformed to an equivalent system of integral equations. The resulting systems are usually Volterra type integral equations of the second kind. As a result, the existence and uniqueness of the solutions have become a consequence of the Banach fixed-point theory. A common feature of the integral equations is the smoothness of their solutions. This property is very important when one tries to solve a boundary value problem via the equivalent integral equations. In this paper, we restrict ourselves to the single linear regular and weakly singular Volterra integral equations of the second kind (see 2.1 below).
The purpose of the present paper is to introduce a fast numerical scheme to solve regular and

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weakly singular Volterra integral equations (possibly originated from a boundary value problem) when smoothness of the solution is known. This method is weaker than the previously reported well-known methods such as quadrature method, collocation method and product-integration method. However, flexibility of this numerical scheme is an advantage to increase the order of the truncation error to at least $O(h^2)$ by adding an extrapolation technique (see main theorem below). Besides, the method has an interesting property that is worth mentioning. The contractivity condition (see section 2 below) that plays an important role in the well-posedness of the problem has a direct effect on the stability and accuracy of the numerical scheme. This property is fully illustrated in the numerical example 4.
Nevertheless, in practice there are problems which do not fit in the framework of this method. In fact, they do not satisfy the smoothness condition required for the solution. Fortunately, there are some techniques based on change of function or variable that transforms the integral equations to a similar problem with smooth solutions (see, [3, 6, 7, 9, 12]). A method that is different from the others consists of splitting the solution into regular and singular parts. Using this method, one can transform a single integral equation to a system having smooth solutions. This method is discussed for example, in [9].
This paper is organized as follows: In section two, we briefly state some general results on the solvability and smoothness of the solution. In the third section, we fully describe our numerical scheme. In section four, we analyze the solvability and accuracy of the discretized problem obtained using proposed finite difference method. Section five is devoted to the numerical examples selected from literatures in the context of the Volterra integral equations.

2. Theoretical Considerations

We consider the linear Volterra integral equations of the second kind

$$u(t) = f(t) + \int_0^t (t-s)^{-\nu} k(t,s)u(s)ds, \tag{2.1}$$

where the input data f and k are functions defined on $I = [0, T]$ and

$$D = \{(t, s)|0 \leq s \leq t \leq T\},$$

respectively. Suppose that f and k are given. We distinguish two cases, the regular integral equation, for $\nu \leq 0$ and weakly singular case, for $0 < \nu < 1$.
For the case $\nu \leq 0$, or without loss of generality for $\nu = 0$, the unique solvability is a consequence of contractivity condition

$$\max_{t\in I} \int_0^t (t-s)^{-\nu} |k(t,s)|ds < 1, \tag{2.2}$$

or in a more general approach, without using this property [2].
For the Volterra integral equations with weakly singular kernel, i.e. $0 < \nu < 1$, it is shown that if the input data f and k satisfies a certain regularity conditions then existence, uniqueness and regularity of the solution guaranteed on $I = (0, T]$. For detailed discussion see [4]. It is also shown that under certain additional conditions such as integrability of the derivative of the solution, the unique solution can extended to a regular function on the closed interval $I = [0, T]$ (see Theorem2.1 in [4]).
In this paper, we assume that for the given data f and k, the Volterra integral equation (2.1) has unique smooth solution which makes the following analysis meaningful.

3. Numerical Scheme

For given positive integer n, let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of I, $t_i = ih$, $f_i = f(t_i)$, $u_i = u(t_i)$, $k_{ij} = k(t_i, t_j)$, for $j = 0, 1, ..., i$ and $i = 0, 1, ..., n$. Before presenting numerical scheme, we recall two well-known results from integration theory.

- The mean value theorem for integrals, discussed in [1].
- The integration by parts, see for example [14].

As a consequence, suppose $Q_{ij}(t) = \int_{t_{j-1}}^{t} (t_i - s)^{-\nu} ds$. Then, on each subinterval $[t_{j-1}, t_j]$, we have

$$\int_{t_{j-1}}^{t_j} (t_i - s)^{-\nu} u(s) ds = Q_{ij}(t_j)u(t_j) - \int_{t_{j-1}}^{t_j} Q_{ij}(s)u'(s) ds, \tag{3.1}$$

for $j = 1, \ldots, i$.
To discretize the integral equation, we collocate equation (2.1) at the grid points $t = t_i$ as follows:

$$u(t_i) = f(t_i) + \int_0^{t_i} (t_i - s)^{-\nu} k(t_i, s)u(s) ds, \tag{3.2}$$

or

$$u_i = f_i + \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} (t_i - s)^{-\nu} k(t_i, s)u(s) ds, \tag{3.3}$$

For simplicity, we derive the numerical scheme in two steps.

- STEP 1. On each subinterval $[t_{j-1}, t_j]$, we approximate $k(t_i, s)$ by $k(t_i, t_j)$, i.e.

$$k(t_i, s) = k_{ij} + \eta_{ij}^{(11)} \quad s \in [t_{j-1}, t_j], \tag{3.4}$$

see for example [11]. Therefore,

$$\begin{aligned} u_i &= f_i + \sum_{j=1}^{i} \left[\int_{t_{j-1}}^{t_j} (k_{ij} + \eta_{ij}^{(11)})(t_i - s)^{-\nu} u(s) \right] ds \\ &= f_i + \sum_{j=1}^{i} k_{ij} \int_{t_{j-1}}^{t_j} (t_i - s)^{-\nu} u(s) ds + \eta_i^{(12)}, \end{aligned} \tag{3.5}$$

where $\eta_{ij}^{(11)}$, $\eta_i^{(12)}$ are truncation errors of approximation on the subintervals $[t_{j-1}, t_j]$ and $[0, t_i]$ respectively. For notation simplicity, let

$$u_i = f_i + \sum_{j=1}^{i} k_{ij}\omega_{ij} + \eta_i^{(12)}, \tag{3.6}$$

where

$$\omega_{ij} = \int_{t_{j-1}}^{t_j} (t_i - s)^{-\nu} u(s) ds. \tag{3.7}$$

- STEP 2. First by using the identity (3.1) we find

$$\omega_{ij} = Q_{ij}(t_j)u(t_j) - \int_{t_{j-1}}^{t_j} Q_{ij}(s)u'(s)ds. \tag{3.8}$$

Next we assume $u \in C^2[0,T]$ and apply the finite difference formula

$$\int_{t_{j-1}}^{t_j} Q_{ij}(s)u'(s)ds = \frac{u_j - u_{j-1}}{h} \int_{t_{j-1}}^{t_j} Q_{ij}(s)ds + \eta_{ij}^{(21)}, \tag{3.9}$$

to the integral operator on the right hand side of (3.8), for details see [10]. This gives

$$\omega_{ij} = Q_{ij}(t_j)u_j - \frac{u_j - u_{j-1}}{h} \int_{t_{j-1}}^{t_j} Q_{ij}(s)ds - \eta_{ij}^{(21)}, \tag{3.10}$$

where $\eta_{ij}^{(21)}$ is the local truncation error on the subinterval $[t_{j-1}, t_j]$.

Since ω_{ij} in (3.8) is a linear combinations of u_j and u_{j-1}, we can rearrange and write equation (3.10) as

$$\omega_{ij} = \alpha_{ij}u_j + \beta_{ij}u_{j-1} - \eta_{ij}^{(21)}, \tag{3.11}$$

where

$$\alpha_{ij} = Q_{ij}(t_j) - \frac{1}{h}\int_{t_{j-1}}^{t_j} Q_{ij}(s)ds, \quad \beta_{ij} = \frac{1}{h}\int_{t_{j-1}}^{t_j} Q_{ij}(s)ds. \tag{3.12}$$

and

$$Q_{ij}(s) = \frac{(t_i - t_{j-1})^{1-\nu} - (t_i - s)^{1-\nu}}{1-\nu}, \tag{3.13}$$

$$\int_{t_{j-1}}^{t_j} Q_{ij}(s)ds = \frac{(t_j - t_{j-1})(t_i - t_{j-1})^{1-\nu}}{(1-\nu)} + \frac{(t_i - t_j)^{2-\nu} - (t_i - t_{j-1})^{2-\nu}}{(1-\nu)(2-\nu)}. \tag{3.14}$$

Inserting ω_{ij} from (3.11) in (3.6) give

$$\begin{aligned} u_i &= f_i + \sum_{j=1}^{i} k_{ij}\left(\alpha_{ij}u_j + \beta_{ij}u_{j-1} - \eta_{ij}^{(21)}\right) \\ &= f_i + \sum_{j=1}^{i} k_{ij}\left(\alpha_{ij}u_j + \beta_{ij}u_{j-1}\right) + \eta_i^{(22)}, \end{aligned} \tag{3.15}$$

where $\eta_i^{(22)}$ is the error which is defined in terms of the truncation errors $\eta_{ij}^{(11)}$, $\eta_i^{(12)}$ generated at the STEP 1 and $\eta_{ij}^{(21)}$ at the STEP 2.
Equation (3.15) together with the initial condition $u_0 = f_0$ characterize the following finite difference method.

Numerical Scheme 3.1. *The recursive relation*

$$\begin{cases} u_i^h = f_i + \sum_{j=1}^{i} k_{ij}(\alpha_{ij}u_j^h + \beta_{ij}u_{j-1}^h), \quad i = 1, \ldots, n, \\ u_0^h = f_0, \end{cases} \tag{3.16}$$

generates an approximate solution for Volterra integral equation (2.1), where u_i^h is an approximation to $u(t_i)$, provided that α_{ij} and β_{ij} satisfy (3.12).

Using equations (3.13), (3.14) one can express α_{ij} and β_{ij} explicitly in terms of i, j and the step size h.

Remark 3.2. *The numerical scheme 3.1 is equivalent to the linear system*

$$A^h U^h = F^h, \tag{3.17}$$

where A^h is a lower triangular $(n+1)\times(n+1)$ square matrix, $U^h = [u_0^h, u_1^h, \ldots, u_n^h]^T$ is the unknown vector, $F^h = [f_0^h, f_1^h, \ldots, f_n^h]^T$ is a known vector and T is the matrix-vector transpose operator.

Remark 3.3. *Using back substitution algorithm, the linear system presented in Remark* 3.2 *can be solved without storing the coefficient matrix completely. This property makes implementation of the method more faster than the other methods.*

4. Main Results

The following two lemmas is on the well-posedness of the system (3.17).

Lemma 4.1. *Suppose that the input data f and k are sufficiently smooth functions. Then*

$$\alpha_{ij} = O(h^{1-\nu}), \quad \beta_{ij} = O(h^{1-\nu}), \quad as \quad h \to 0, \tag{4.1}$$

hence A^h is non-singular for sufficiently small h.

Proof . The proof of this result is a direct consequence of the definition α_{ij} and β_{ij} in (3.12). □

Lemma 4.2. *Consider the Volterra integral equation*

$$u(t) = f(t) + \mu \int_0^t (t-s)^{-\nu} k(t,s)u(s)ds. \tag{4.2}$$

Then for $\nu \leq 0$ (the regular case) and for sufficiently small μ, the system in (3.17) is diagonally dominant.

Proof . The proof of this lemma is based on definitions (3.12) and asymptotic behavior of α_{ij} and β_{ij}.
□ In sequence we will verify and illustrate this result in the numerical example 5.4.
The following lemma gives the order of truncation error corresponding to the STEP 1.

Lemma 4.3. *If $u : [0,T] \to \mathbb{R}$ and $k : D \to \mathbb{R}$ are sufficiently smooth functions, then*

$$\int_{t_{j-1}}^{t_j} k(t_i,s)(t_i-s)^{-\nu}u(s)ds = k_{i,j}\int_{t_{j-1}}^{t_j}(t_i-s)^{-\nu}u(s)ds + O(h^2). \tag{4.3}$$

Proof . The proof is a direct consequence of the Taylor expansion theorem with integral reminder term (see theorem 9.29 in [1]). □

Remark 4.4. *In the first step, instead of using approximations*

$$k(t_i, s) \approx k(t_i, t_j), \qquad k(t_i, s) \approx k(t_i, t_{j-1}), \quad s \in [t_{j-1}, t_j],$$

other more accurate interpolation schemes may be used to obtain a method with higher order truncation error. However, in this work we preferred a weaker approximation technique to avoid increasing run-time.

The following lemma gives the order of truncation error corresponding to the approximation in STEP 2 using (3.9).

Lemma 4.5. *Suppose that $u \in C^2[0, T]$. Then*

$$\int_{t_{j-1}}^{t_j} Q_{ij}(s)u'(s)ds = \frac{u_j - u_{j-1}}{h} \int_{t_{j-1}}^{t_j} Q_{ij}(s)ds + c_1 h^{4-\nu} + O(h^{5-\nu}), \tag{4.4}$$

where c_1 is a constant not depending on h.

Proof . Suppose

$$\varepsilon_{ij} = \int_{t_{j-1}}^{t_j} Q_{ij}(s)u'(s)ds - \frac{u_j - u_{j-1}}{h} \int_{t_{j-1}}^{t_j} Q_{ij}(s)ds \tag{4.5}$$

then using Taylor's theorem with integral remainder we have

$$\varepsilon_{ij} = \int_{t_{j-1}}^{t_j} \frac{1}{h}[\int_{t_{j-1}}^{t} u''(s)(s - t_{j-1})ds - \int_{t}^{t_j} u''(s)(t_j - s)ds](t_i - t)^{-\nu+1}dt \tag{4.6}$$

Change of the order of integration yields

$$\varepsilon_{ij} = \frac{1}{2-\nu} \int_{t_{j-1}}^{t_j} u''(s)[(t_i - s)^{-\nu+2} - (\frac{t_j - s}{h}(t_i - t_{j-1})^{-\nu+2} + \frac{s - t_{j-1}}{h}(t_i - t_j)^{-\nu+2})]ds \tag{4.7}$$

The terms in the bracket under the sign of integral is the error of the linear Lagrange's interpolation of the function $(t_i - s)^{-\nu+2}$ at the points t_{j-1} and t_j. Applying the order of linear interpolation i.e. $O(h^2)$ gives the required order $O(h^{4-\nu})$ (see [16] Lemma 1 page 264). □

Theorem 4.6 (Main Theorem). *Consider the linear operator*

$$K_i u = \int_0^{t_i} (t_i - s)^{-\nu} k(t_i, s)u(s)ds. \tag{4.8}$$

on $C^2[0, T]$ and K_i^h the corresponding discrete operator defined by

$$K_i^h u = \sum_{j=1}^{i} k_{ij}(\alpha_{ij}u_j + \beta_{ij}u_{j-1}). \tag{4.9}$$

Then the truncation error satisfy an asymptotic expansion of the form

$$(K_i - K_i^h)u = \begin{cases} c_1 h + c_2 h^2 + O(h^3), & \text{for} \quad \nu \leq 0, \\ c_1 h + c_2 h^2 + c_3 h^{3-\nu} + O(h^3), & \text{for} \quad 0 < \nu < 1, \end{cases} \tag{4.10}$$

where c_1, c_2 and c_3 are constants independent of h.

Proof .. The proof is a direct consequence of the Taylor expansion theorem and the result of Lemma 4.5. □.

Conclusion 4.7. *Using the results of the main theorem, we can apply the Richardson's extrapolation technique to increase the order of the finite difference method (see chapter* 10*,* [11]*).*

Definition 4.8. *Let*

$$U = [u_0, u_1, \ldots, u_n]^T, \quad U^h = [u_0^h, u_1^h, \ldots, u_n^h]^T, \quad \hat{U}^h = [\hat{u}_0^h, \hat{u}_1^h, \ldots, \hat{u}_n^h]^T,$$

be the vector of the exact solution u, the approximate solution u^h, and extrapolated approximations of u generated using u_i^h and $u_i^{h/2}$, i.e.

$$\hat{u}_i^h = \frac{1}{h}(2u_i^{h/2} - u_i^h).$$

We define the errors of approximation and extrapolated approximation as follows:

$$\|U - U^h\| = \max_{0\le i\le n} |u_i - u_i^h|, \quad \|U - \hat{U}^h\| = \max_{0\le i\le n} |u_i - \hat{u}_i^h|,$$

where u_i^h is generated by using numerical scheme (3.16) or (3.17).

5. Numerical Examples

In this section, we examine four examples of the regular and weakly singular Volterra integral equations with known exact solutions.

Example 5.1. *Consider the regular Volterra integral equation*

$$u(t) = e^{-t} + \int_0^t e^{-(t-s)} \sin(t-s)u(s)ds, \quad 0 < t < 1,$$

with the exact solution $u(t) = e^{-t}(1 + \frac{1}{2}t^2)$, [13]. We solve this example for $n = 128$ and $n = 256$ using numerical scheme given in 3.16. Numerical results for this example are presented in the Table 1.

h	$Order$	$\|U - U^h\|$	$\|U - \hat{U}^h\|$	$Run - Time/sec$
$1/128$	$O(h)$	$9.76E-4$	$---$	3
$1/256$	$---$	$4.85E-4$	$1.32E-6$	3

Table 1. Results of Example 5.1

Example 5.2. *To examine the accuracy of the numerical scheme* 3.1*, we consider the integral equation of the form*

$$u(t) = f(t) + \int_0^t (t-s)^{-\frac{1}{2}} u(s)ds, 0 \le t \le 1$$

with $f(t) = t - \frac{4}{3}t^{\frac{3}{2}}$ and the exact solution $u(t) = t$, [8]. For $h = 1/128$, $\|U - U^h\| \approx 1.2E-15$. This example corresponds to the super convergence case for the proposed finite difference numerical scheme. Note that in [8] this example is solved by a global minimization procedure and using a quite few multiquadric radial basis function. The global error of $2.37E-10$ is reported by authors (See [8] for more details).

Example 5.3. *Consider the integral equation*

$$u(t) = f(t) + \int_0^t (t-s)^{-\frac{1}{2}} k(t,s)u(s)ds, 0 \leq t \leq 1$$

corresponding to the following data

$$f(t) = (1+t)^{-\frac{1}{2}} + \frac{\pi}{8} - \frac{1}{4}\arcsin(\frac{1-t}{1+t}) \qquad k(t,s) = -\frac{1}{4}$$

and the exact solution $u(t) = (1+t)^{-\frac{1}{2}}$*, [11]. We solve this example using numerical scheme* 3.1 *and extrapolation technique. Numerical results are illustrated in the Table* 2*. For comparison, we note that the error of approximation using product integration method with step size* 0.05 *is about* $1.0E-7$ *(See [11] for more details).*

h	$Order$	$\|U-U^h\|$	$\|U-\hat{U}^h\|$	$Run-Time/sec$
$1/128$	$O(h^2)$	$5.32E-7$	$---$	3
$1/256$	$---$	$1.33E-7$	$4.84E-10$	3

Table 2. Results of Example 5.3

Example 5.4. *To analyze the accuracy of the numerical scheme* 3.1 *(without extrapolation) with respect to parameter* μ*, consider the class of the integral equations*

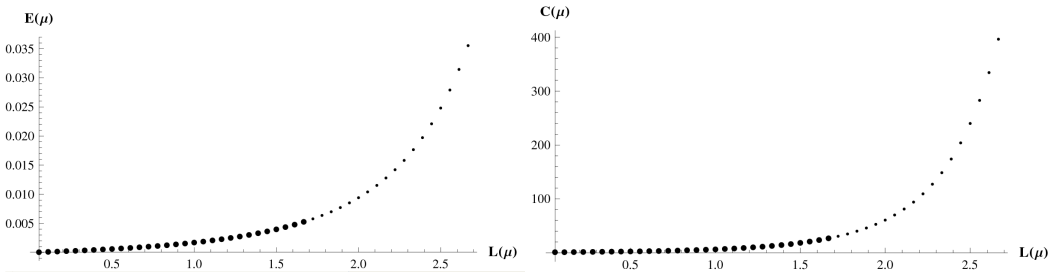
$$u(t) = f(t) + \mu \int_0^t (t-s)^{-\frac{1}{2}}(2+s+t+st)u(s)ds, 0 \leq t \leq 1,$$

where f *is chosen such that* $u(t) = t^2$*. We define the error* $E(\mu) = \|U-U^h\|$*, the Lipschitz constant of the operator (*4.6*) as*

$$L(\mu) = |\mu| \max_{0\leq t\leq 1} \int_0^t (t-s)^{-\frac{1}{2}}|k(t,s)|ds = \frac{26}{3}|\mu|,$$

and the condition number $C(\mu)$ *of the matrix* A^h*.*
Using the finite difference method defined by the numerical scheme 3.1 *we show that if* μ *varies in the interval* $(1/156, 4/13)$*, then the function values* $L(\mu)$*,* $E(\mu)$ *and* $C(\mu)$ *increasingly vary in the intervals* $(0.05546, 2.6667)$*,* $(0.00005, 0.03553)$ *and* $(1.1, 396.)$ *respectively. Therefore the growth of the error and the condition number are related to the growth of the Lipschitz constant.*
For this example, the restriction $L(\mu) < 1$*, which guaranties the efficiency of the method is very similar to the contractivity condition investigated by the authors for example* [2] *and* [4] *in the regular and singular cases respectively.*
Figure 1 illustrate that the error (left) and condition number (right) are increasing functions with respect to Lipschitz factor L*. Finally, in the figure 1, the points with larger point size indicate that the corresponding matrix* A^h *in numerical scheme presented by the Remark* 3.2 *is either row-wise or column-wise diagonally dominant.*

Results for the numerical example 4 *corresponding to the extreme values of* μ *are shown in the Table 3. The run-time in each case is approximately six seconds.*



Left Figure: Graph of $L(\mu), E(\mu)$ Right Figure: Graph of $L(\mu), C(\mu)$

Figure 1: Graph of the errors and condition numbers.

μ	h	$L(\mu)$	$C(\mu)$	$\|U - U^h\|$	$\|U - U^{h/2}\|$	$\|U - \hat{U}^{h/2}\|$
$1/156$	$1/128$	$1/18$	1.1	$5.1E-5$	$2.6E-5$	$8.2E-7$
$4/13$	$1/128$	$8/3$	396	$3.5E-2$	$1.7E-2$	$1.9E-4$

Table 3. Results for Numerical Example 5.4

Numerical examples in this paper are solved using codes in Mathematica version 5.2 and Matlab version 9.

6. Conclusion

In this paper, we presented a fast numerical scheme for generating the smooth solutions of Volterra integral equations having regular or weakly singular (non-logarithmic) kernels. It is shown that applying this method to an integral equation leads to a linear system that can be solved without storing the coefficient matrix. Extension of this method for integral equations with logarithmic kernels will be considered in future works. It is apparent that the order of the truncation is $O(h)$ which is lower than the previously reported methods. However, it is shown that the numerical results can be improved by applying the extrapolation technique to increase the order of the truncation error to at least $O(h^2)$.

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