



A remark on boundedness of composition operators between weighted spaces of holomorphic functions on the upper half-plane

Mohammad Ali Ardalani

Department of Mathematics, Faculty of Science, University of Kurdistan, Pasdaran Ave., Postal Code: 66177-175 Sanandaj, Iran.

Abstract

In this paper, we obtain a sufficient condition for boundedness of composition operators between weighted spaces of holomorphic functions on the upper half-plane whenever our weights are standard analytic weights, but they don't necessarily satisfy any growth condition.

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1. Introduction

Looking at Corollary 1.5 of [1], this question arises: is it possible to omit the growth condition (*), on the weight v_1 defined on the upper half-plane G ? In this paper we find a partial answer to this question in a special case. We show at least for analytic weights, condition (*) is not a necessary condition for boundedness of composition operators.

We begin with the preliminaries which are required in continue of this paper.

Definition 1.1. The sets $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $G = \{\omega \in \mathbb{C} : \text{Im } \omega > 0\}$ are the unit disk and upper half-plane respectively.

Definition 1.2. A continuous function $v : G \rightarrow (0, \infty)$ is called a standard weight if $\forall \omega \in G$ $v(\omega) = v(\text{Im } \omega i)$ (i.e v depends only on the imaginary part), $v(si) < v(ti)$ when $0 < s \leq t$ and $\lim_{t \rightarrow 0} v(ti) = 0$.

Example 1.3. Let $\alpha, \beta > 0$. Then functions $v_1(\omega) = (\text{Im } \omega)^\beta$, $v_2(\omega) = (\text{Im } \omega)^\beta \exp(\alpha \text{Im } \omega)$ and $v_3(\omega) = \exp(-\frac{\beta}{\text{Im } \omega})$ are standard and analytic weights.

Email address: M.Ardalani@uok.ac.ir (Mohammad Ali Ardalani)

Definition 1.4. Let v be a standard weight on G . We say v satisfies condition $(*)$ if there are constants $C, \beta > 0$ s.t $\frac{v(ti)}{v(si)} \leq C(\frac{t}{s})^\beta$ whenever $0 < s \leq t$.

It has been proved that v satisfies condition $(*)$ if and only if

$\sup_{k \in \mathbb{Z}} \frac{v(2^{k+1}i)}{v(2^k i)} < \infty$ (see [2]). Indeed, condition $(*)$ states v is increasing but the speed of its growth is under the control and it can't grow too fast. As we commented before, we intend to study the boundedness of composition operators between weighted spaces of holomorphic functions on the upper half-plane in lack of growth condition $(*)$.

Definition 1.5. Let O be an open subset of \mathbb{C} . For a function $f : O \rightarrow \mathbb{C}$ we define the weighted sup-norm

$$\|f\|_v = \sup_{k \in O} |f(z)| v(z)$$

and the space

$$H_v(O) = \{f : G \rightarrow \mathbb{C} : f \text{ is holomorphic and } \|f\|_v < \infty\}$$

From now on, we focus on the cases $O = \mathbb{D}$ or $O = G$.

Remark 1.6. For a weight v defined from \mathbb{D} into $(0, \infty)$, we always assume v is radial (i.e $v(z) = v(|z|)$), continuous and non-increasing weight with respect to $|z|$.

Definition 1.7. Suppose $v : O \rightarrow (0, +\infty)$ is a weight. Corresponding to v , the associated weight \tilde{v} is defined as follows.

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)| : f \in H_v(O), \|f\|_v < 1\}} \quad \forall z \in O$$

Definition 1.8. Let v be standard weight on G . For any $n \in \mathbb{N}$, we define

$$v_n(\omega) = v\left(\frac{4Im \omega}{(|\frac{\omega}{n} + i| + |\frac{\omega}{n} - i|)^2} i\right) \quad \omega \in G$$

and

$$u_n(z) = v\left(n \frac{1 - |z|}{1 + |z|} i\right) \quad z \in \mathbb{D}$$

Definition 1.9. Define $\alpha : \mathbb{D} \rightarrow \mathbb{C}$ by $\alpha(z) = \frac{1+z}{1-z}i$. An easy computation shows that $\alpha(z) = -2\frac{Im z}{|z|^2+1-2Re z} + \frac{1-|z|^2}{|z|^2+1-2Re z}$. Hence $\alpha(\mathbb{D}) \subseteq G$. Put $\beta(\omega) = \frac{\omega-i}{\omega+i} \quad \forall \omega \in G$. Then we have $\alpha \circ \beta = id_G$ and $\beta \circ \alpha = id_{\mathbb{D}}$. Thus $\beta = \alpha^{-1}$ and $\alpha(\mathbb{D}) = G$.

Remark 1.10. Put $\alpha_n(z) = n \frac{1+z}{1-z}i$. Hence $\alpha_n : \mathbb{D} \rightarrow G$. Also α_n as α is an onto and analytic map. Using relation $|1 - z|^2 = 1 + |z|^2 - 2Re z$, it is easy to see that $u_n = v_n \circ \alpha_n$.

Lemma 1.11. Let $n \in \mathbb{N}$ be fixed and arbitrary. Then the map $T : H_{v_n}(G) \rightarrow H_{u_n}(\mathbb{D})$ defined by $(Tf)(z) = f(\alpha_n(z))$ is an onto isometry and $H_{v_n}(G)$ is isometrically isomorphic to $H_{u_n}(\mathbb{D})$.

Proof . See Proposition 1.4 of [2]. \square

Definition 1.12. Put $H(O) := \{f \mid f : O \rightarrow \mathbb{C} \text{ is holomorphic}\}$. For any $f \in H(O)$ the weighted composition operator $C_{\varphi,\psi}(f) = \psi(f \circ \phi)$ where $\varphi, \psi \in H(O)$. If $O = \mathbb{D}$ then we also assume $\varphi(\mathbb{D}) \subseteq \mathbb{D}$.

In special case if $\psi \equiv 1$ then $C_{\varphi,\psi}(f) = C_{\varphi}(f) = f \circ \varphi$ is a composition operator.

Lemma 1.13. Let v and ω be weights on \mathbb{D} . Then the operator $C_{\varphi,\psi} : H_v(\mathbb{D}) \rightarrow H_{\omega}(\mathbb{D})$ is bounded if and only if $\sup_{z \in \mathbb{D}} \frac{|\psi(z)| \omega(z)}{v(\varphi(z))} < \infty$.

Proof . See Proposition 3.1 of [3]. \square

It is easy to see that :

Lemma 1.14. For any $n \in \mathbb{N}$, $\tilde{u}_n = \tilde{v}_n \circ \alpha_n$.

Proof . By definition of associated weight we have

$$\tilde{u}_n(z) = \frac{1}{\sup\{|f(z)| : f : \mathbb{D} \rightarrow \mathbb{C} \text{ is holomorphic}, \|f\|_{u_n} \leq 1\}}$$

and

$$\tilde{v}_n(z) = \frac{1}{\sup\{|f(\alpha_n(z))| : f : G \rightarrow \mathbb{C} \text{ is holomorphic}, \|f\|_{v_n} \leq 1\}}$$

Since $T : H_{v_n}(G) \rightarrow H_{u_n}(\mathbb{D})$ is an onto isometry, $\|f(\alpha_n(z))\|_{u_n} = \|f\|_{v_n}$. This implies that $(\tilde{v}_n \circ \alpha_n)(z) = \tilde{u}_n(z)$. \square

2. Main result

Now, we present main result of this paper:

Theorem 2.1. Let v and v' be standard and analytic weights on G . Also let $\varphi : G \rightarrow G$ be an analytic function. If $\sup_{\omega \in G} \frac{v'(\omega)}{v(\varphi(\omega))} < \infty$, then the composition operator $C_{\varphi} : H_v(G) \rightarrow H_{v'}(G)$ is a bounded operator.

Proof . For any arbitrary and fixed $n \in \mathbb{N}$ consider the following diagram.

$$\begin{array}{ccc} C_{\varphi} : & H_v(G) & \longrightarrow & H_{v'}(G) \\ & \downarrow T' & & \downarrow T'' \\ C_{\varphi,\psi} : & H_{v_n}(G) & \longrightarrow & H_{v'_n}(G) \end{array}$$

where $(T'f)(\omega) = f(\omega) \frac{v(\omega)}{v_n(\omega)}$, $(T''g)(\omega) = g(\omega) \frac{v'(\omega)}{v'_n(\omega)}$ and $C_{\varphi,\psi}$ is defined by $C_{\varphi,\psi} := T'' \circ C_{\varphi} \circ (T')^{-1}$. Evidently, T', T'' are onto isometries. Hence C_{φ} is a bounded operator if and only if the weighted composition operator $C_{\varphi,\psi}$ is a bounded operator. Now, consider the following diagram.

$$\begin{array}{ccc} C_{\varphi,\psi} : & H_{v_n}(G) & \longrightarrow & H_{v'_n}(G) \\ & \downarrow T & & \downarrow T \\ C_{\varphi_1,\psi_1} : & H_{u_n}(\mathbb{D}) & \longrightarrow & H_{u'_n}(\mathbb{D}) \end{array}$$

where T is defined as in 1.11. Also C_{φ_1, ψ_1} is defined by $C_{\varphi_1, \psi_1} := T \circ C_{\varphi, \psi} \circ T^{-1}$ where $\varphi_1 := \alpha_n^{-1} \circ \varphi \circ \alpha_n : \mathbb{D} \rightarrow \mathbb{D}$ and $\psi_1 := \psi \circ \alpha_n : \mathbb{D} \rightarrow \mathbb{C}$. T is an onto isometry (see Lemma 1.11). Thus, $C_{\varphi, \psi}$ is a bounded operator if and only if C_{φ_1, ψ_1} is a bounded operator. Hence, C_φ is a bounded operator if and only if the weighted composition operator C_{φ_1, ψ_1} is a bounded operator. Now, using Lemma 1.13 C_φ is a bounded operator if and only if

$$\sup_{z \in \mathbb{D}} \frac{|\psi_1(z)| |u'_n(z)|}{\tilde{u}_n(\varphi_1(z))} < \infty$$

But,

$$\begin{aligned} \sup_{z \in \mathbb{D}} \frac{|\psi_1(z)| |u'_n(z)|}{\tilde{u}_n(\varphi_1(z))} &< \infty \\ \Leftrightarrow \sup_{z \in \mathbb{D}} \frac{|\psi(\alpha_n(z))| |v'_n(\alpha_n(z))|}{\tilde{u}_n((\alpha_n^{-1} \circ \varphi \circ \alpha_n)(z))} &< \infty \\ \Leftrightarrow \sup_{z \in \mathbb{D}} \frac{\frac{v'_n(\alpha_n(z)) v_n(\varphi(\alpha_n(z))) v'_n(\alpha_n(z))}{v'_n(\alpha_n(z)) v(\varphi(\alpha_n(z)))}}{\tilde{v}_n \circ \alpha_n(\alpha_n^{-1} \circ \varphi \circ \alpha_n(z))} &< \infty \\ \Leftrightarrow \sup_{z \in \mathbb{D}} \frac{\frac{v'(\alpha_n(z)) v_n(\varphi(\alpha_n(z)))}{v(\varphi(\alpha_n(z)))}}{\tilde{v}_n(\varphi(\alpha_n(z)))} &< \infty \\ \Leftrightarrow \sup_{z \in \mathbb{D}} \frac{v'(\alpha_n(z)) v_n(\varphi(\alpha_n(z)))}{v(\varphi(\alpha_n(z))) \tilde{v}_n(\varphi(\alpha_n(z)))} &< \infty. \end{aligned}$$

It is well-known that $v_n \leq \tilde{v}_n$ (see [3]). Therefore,

$$\sup_{z \in \mathbb{D}} \frac{v'(\alpha_n(z)) v_n(\varphi(\alpha_n(z)))}{v(\varphi(\alpha_n(z))) \tilde{v}_n(\varphi(\alpha_n(z)))} < \sup_{z \in \mathbb{D}} \frac{v'(\alpha_n(z))}{v(\varphi(\alpha_n(z)))}.$$

Now, surjectivity of α_n completes the proof. \square

References

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