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Free and constrained equilibrium states in a variational problem on a surface

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Abstract

We study the equilibrium states for an energy functional with a parametric force field on a region of a surface. Consideration of free equilibrium states is based on Lyusternik - Schnirelman's and Skrypnik's variational methods. Consideration of equilibrium states under a constraint of geometrical character is based on an analog of Skrypnik's method, described in [P. Vyridis, *Bifurcation in a Variational Problem on a Surface with a Constraint*, Int. J. Nonlinear Anal. Appl. 2 (1) (2011), 1-10]. In local coordinates, equilibrium points satisfy an elliptic boundary value problem.

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1. Introduction

Let M be a smooth surface in $\mathbb{R}^3$ and $\vec{\eta}(x)\,,\ x \in \mathbb{R}^3$ a continuously differentiable vector field identified to the normal vector field for every $x \in M$. Let $U \subset \mathbb{R}^3$ an open set with $\operatorname{diam} U < \delta$, where $\delta > 0$ small enough and $S = M \cap U$ an open and connected set in M, with boundary ∂S consisting of two non-intersecting sufficiently smooth components Γ and Γ_1. We denote by $\vec{\nu}(x)$ a differentiable vector field in $\mathbb{R}^3$, which is the normal vector field of the one - dimensional curve ∂S for every $x \in \partial S$, located in the tangent plane $T_xM \subset \mathbb{R}^3$. We also denote by $\vec{\tau}(x)$ for $x \in \mathbb{R}^3$ a continuously differentiable vector field vector field identified for each $x \in \partial S$ to the unitary tangent vector field of

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the curve ∂S, and belonging to the tangent plane T_xM for each $x \in \partial S$. We assume that the mean curvature H of surface M does not vanish:

$$H \neq 0\,. \tag{1.1}$$

Let a vector field $\vec{u} \in H_0(S,\, T_xM)$, where

$$H_0(S,\, T_xM) = \left\{ \vec{u} \in W_2^1(S,\, T_xM)\,,\ \vec{u}|_\Gamma \in W_2^2(\Gamma,\, T_xM)\,,\ \vec{u}|_{\Gamma_1} = \vec{0} \right\}\,.$$

We denote by $W_2^1(S, T_xM)$ and $W_2^2(\Gamma, T_xM)$ the Sobolev spaces of functions defined on S and Γ with values in $T_xM \subset \mathbb{R}^3$, respectively. For every specific $\vec{u} \in H_0(S,\, T_xM)$, we introduce the following functionals

$$F[\,\vec{u}\,] = \frac{1}{2}\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{u})\,\xi_{kl}(\vec{u})\,dS + \frac{1}{2}\int_\Gamma |\delta_i\delta_i\,\vec{u}|^2 ds \tag{1.2}$$

$$G[\,\vec{u}\,] = \int_\Gamma q(\vec{u}, x)\,ds \tag{1.3}$$

$$I[\,\vec{u}, \lambda\,] = F[\,\vec{u}\,] - \lambda\, G[\,\vec{u}\,]\,, \quad \lambda \in \mathbb{R}. \tag{1.4}$$

The coefficients $a_{ijkl} \in L_\infty(S)$ satisfy the symmetry properties $a_{ijkl}(x) = a_{klij}(x)$, and they are positive definite, i.e.

$$a_{ijkl}(x)\,\xi^{ij}\xi^{kl} \geq \Lambda\,\xi^{ij}\,\xi^{ij}\,, \qquad \Lambda > 0\,. \tag{1.5}$$

The tensor $\xi_{ij}(\vec{u})$ is defined as:

$$\xi_{ij}(\vec{u}) = \frac{1}{2}\,(\nabla_i u^j + \nabla_j u^i)\,, \tag{1.6}$$

where ∇_i is the i-th component of the tangent differentiation with respect to the surface M [2]:

$$\nabla_i = \frac{\partial}{\partial x^i} - \eta^i(x)\,\eta^j(x)\,\frac{\partial}{\partial x^j}\,, \quad i = 1, 2, 3\,, \quad x \in M \tag{1.7}$$

and δ_i is the i-th component of the tangent directional differentiation along the curve ∂S:

$$\delta_i = \tau^i(x)\,\frac{d}{ds} = \tau^i(x)\,\tau^j(x)\,\frac{\partial}{\partial x^j}\,, \quad i = 1, 2, 3\,, \quad x \in \partial S\,. \tag{1.8}$$

For the above differential operators the following formulae of integration by parts hold on $S \subset M$ [5]:

$$\int_S u\nabla_i v\,dS = \int_{\partial S} u\,v\,\nu^{\,i}\,ds - \int_S H n^i u\,v\,dS - \int_S v\,\nabla_i u\,dS\,, \tag{1.9}$$

and on a closed curve ∂S, located in the surface M [8]:

$$\int_{\partial S} u\,\delta_i\,v\,ds = -\int_{\partial S} (K\nu^{\,i} + R\eta^i)\,u\,v\,ds - \int_{\partial S} v\,\delta_i\,u\,ds\,, \tag{1.10}$$

where

$$H = -\nabla_i \eta^i \tag{1.11}$$

is the mean curvature of surface M [2], K is the geodesic curvature, and R is the normal curvature of curve ∂S, located in the surface M [1]. These geometric characteristics of a curve, located in a surface, are subjected to the Darboux frame [1] :

$$\frac{d\vec{\tau}}{ds} = K\,\vec{\nu} + R\,\vec{n}\,, \quad \frac{d\vec{\nu}}{ds} = -K\,\vec{\tau} + k\,\vec{n}\,, \quad \frac{d\vec{n}}{ds} = -R\,\vec{\tau} - k\,\vec{\nu}\,, \tag{1.12}$$

where k is the geodesic torsion of curve ∂S, and s is the natural parameter. Finally, we assume that function q is three times differentiable with the following properties

$$q(\vec{0},x)=0\,,\quad q_{u^i}(\vec{0},x)=0\,,\quad x\in\Gamma\,,\quad i=1,2,3. \tag{1.13}$$

Hence, the first term of (1.2) represents the elastic energy of the medium, stored by the deformation. The second term of denotes the work, done by the outer forces due to the deformation of the shell Γ. Finally, the (1.3) denotes the stored potential energy of the shell. The medium is fixed up to a part Γ_1 of the boundary ∂S.

A critical point for functional (1.4) is a vector field $\vec{u}\in H_0(S,\,T_xM)$ such that

$$I'[\,\vec{u},\lambda\,]\,\vec{v}=F'[\,\vec{u}\,]\,\vec{v}-\lambda\,G'[\,\vec{u}\,]\,\vec{v}=0\,, \tag{1.14}$$

or equivalently

$$\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{u})\,\xi_{kl}(\vec{v})\,dS+\int_\Gamma \delta_i\delta_i\,\vec{u}\,\delta_j\delta_j\,\vec{v}\,ds-\lambda\int_\Gamma q_{u^i}(\vec{u},x)\,v^i ds=0 \tag{1.15}$$

for all $\vec{v}\in H_0(S,\,T_xM)$. We note that $\vec{u}=\vec{0}$ is a solution for equation (1.15) due to the second relation of (1.13), therefore a critical point for (1.4) for all $\lambda\in\mathbb{R}$.

The first aim of this work is the investigation of the critical points for functional (1.4). One approach is to treat (1.15) as a bifurcation problem, based on Skrypnik's variational method [7]. According to this theory, every $\lambda\in\mathbb{R}$, which corresponds to a non zero critical point $\vec{u}\in H_0(S,\,T_xM)$ of (1.4), is a bifurcation point for (1.15). A second approach is to follow the Lyusternik - Schnirelman's variational method [3], which allows to prove the existence of countable different solutions for equation (1.15).

The second aim is the investigation of the critical points for functional (1.4), under the existence of a constraint with the property of leaving the length of curve Γ invariant on the surface M. The constraint restricts the domain of (1.15) to a smaller subspace X_1 of $H_0(S,\,T_xM)$. A generalized Skrypnik's method [8] approaches (1.15) as a bifurcation problem on subspace X_1.

We also show that under additional smoothness of boundary ∂S, coefficients a_{ijkl} and function q, the integral equation (1.15) can be written in the equivalent form of an elliptic boundary value problem.

2. Functional spaces on curve and surface

Let Γ a smooth closed curve in $\mathbb{R}^3$, parametrized by natural parameter s. Then $\vec{u}\in W_2^2(\Gamma,\mathbb{R}^3)$ means that $\vec{u}\in W_2^2\left((0,L),\mathbb{R}^3\right)$ where L is the length of the closed curve Γ and

$$\vec{u}(0)=\vec{u}(L)\,,\quad \vec{u}'(0)=\vec{u}'(L).$$

The norms on spaces $L_2(\Gamma,\mathbb{R}^3)$ and $W_2^2(\Gamma,\mathbb{R}^3)$ are defined, respectively, as:

$$\|\vec{u}\|_{L_2(\Gamma,\mathbb{R}^3)}=\left[\int_0^L|\vec{u}(s)|^2ds\right]^{1/2}, \tag{2.1}$$

$$\|\vec{u}\|_{W_2^2(\Gamma,\mathbb{R}^3)}=\left[\int_0^L\left(|\vec{u}''(s)|^2+|\vec{u}(s)|^2\right)ds\right]^{1/2}. \tag{2.2}$$

After a straight calculation we see that

$$\|\vec{u}\| = \left[\int_\Gamma \left(|\delta_i \delta_i \vec{u}|^2 + |\vec{u}|^2 \right) \, ds \right]^{1/2} \tag{2.3}$$

defines a norm in $W_2^2(\Gamma, \mathbb{R}^3)$ equivalent to (2.2).

Let S a domain on the surface $M \subset \mathbb{R}^3$. Then there exists a cover of open sets $U_a \subset \mathbb{R}^3$, $a = 1, 2, ..., N$, such that $S \cap U_a$ is a graph of a two times differentiable function f_a defined on bounded domain $V_a \subset \mathbb{R}^2$. The graph of each function f_a is located on a coordinate system which is transformed from the initial one by an appropriate composition of a translation and rotation. Then

$$S \cap U_a = \left\{ (x^1, x^2, f_a(x^1, x^2)), \; (x^1, x^2) \in V_a \right\} .$$

On such a coordinate system on $\mathbb{R}^3$ we pick up the axes x^1, x^2 from the tangent plane of the surface M at the point $x_a \in S \cap U_a$, while the axis x^3 comes along the normal vector $\vec{\eta}$ of the surface M at the point x_a. In this specific system of local coordinates, the components of the normal vector of M at x_a satisfy the relations:

$$\eta^3 = \frac{1}{\sqrt{1 + |\mathrm{grad} f_a|^2}} \,, \quad \eta^j = -\eta^3 \frac{\partial f_a}{\partial x^j} \,, \quad j = 1, 2 \,, \tag{2.4}$$

the area element is given by

$$dS = \sqrt{1 + |\mathrm{grad} f_a|^2} \, dx^1 dx^2 = \frac{1}{n^3} \, dx^1 \, dx^2 \,, \tag{2.5}$$

and the components of the tangential differentiation (1.7) are written as:

$$\nabla_i = (\delta_{ik} - n^i n^k) \frac{\partial}{\partial x^k} \,, \quad i = 1, 2, 3 \,, \quad k = 1, 2 \,, \tag{2.6}$$

where δ_{ik} stands for the Kronecker symbol.

We consider the partition of unity on the surface $S \subset M$ which corresponds to the cover $\{U_a\}$, $a = 1, ..., N$

$$\mathrm{supp} \psi_a \subset S \cap U_a \,, \quad \psi_a \in C_0^\infty(S \cap U_a) \,, \quad \psi_a(x) \geq 0, \quad \sum_{a=1}^N \psi_a(x) = 1 \,.$$

Then $\vec{u} \in L_2(S, \mathbb{R}^3)$ and $\vec{u} \in W_2^1(S, \mathbb{R}^3)$ mean, respectively, that $\psi_a \vec{u} \circ f_a^{-1} \in L_2(V_a, \mathbb{R}^3)$ and $\psi_a \vec{u} \circ f_a^{-1} \in W_2^1(V_a, \mathbb{R}^3)$. The norms on spaces $L_2(S, \mathbb{R}^3)$ and $W_2^1(S, \mathbb{R}^3)$ are chosen, respectively, as:

$$\|\vec{u}\|_{L_2(S,\mathbb{R}^3)} = \left[\sum_{a=1}^N \int_{V_a} |\psi_a \vec{u} \circ f_a^{-1}|^2 dx^1 dx^2 \right]^{1/2} , \tag{2.7}$$

$$\|\vec{u}\|_{W_2^1(S,\mathbb{R}^3)} = \left[\sum_{a=1}^N \int_{V_a} \left(\left|\frac{\partial(\psi_a \vec{u} \circ f_a^{-1})}{\partial x^1}\right|^2 + \left|\frac{\partial(\psi_a \vec{u} \circ f_a^{-1})}{\partial x^2}\right|^2 \right) dx^1 \, dx^2 \right]^{1/2} . \tag{2.8}$$

The symbol $\circ f_a^{-1}$ will be omitted in the rest of this paper.

Proposition 2.1. *The expression*

$$\|u\| = \left[\int_S \xi_{ij}(\vec{u}) \, \xi_{ij}(\vec{u}) \, dS + \int_S |\vec{u}|^2 \, dS \right]^{1/2} \tag{2.9}$$

defines a norm on $W_2^1(S, \mathbb{R}^3)$ *equivalent to* (2.8).

Proof . It is obvious that the corresponding inner product $(\vec{u},\vec{v})$ to (2.9) is bilinear and symmetric, while $(\vec{u},\vec{u}) = 0$ implies $\vec{u} = 0$ for each $\vec{u} \in W_2^1(S,\mathbb{R}^3)$. We prove the equivalence of (2.9) to the standard norm (2.8). We consider the family of functions $\{\varphi_a\}$ corresponding to the cover $\{U_a\}$, $a = 1, 2, ..., N$, such that

$$\varphi_a \in C_0^\infty(U_k)\,, \quad \varphi_a(x) \geq 0\,, \quad \sum_{k=1}^{N} \varphi_a^2(x) = 1\,, \quad x \in U_a\,.$$

Now for every $\varepsilon > 0$ there exists $\delta > 0$ such for $\operatorname{diam} U_a < \delta$ the inequalities

$$|n^i(x)| < \varepsilon\,, \quad i = 1, 2\,, \quad 1 \leq \frac{1}{n^3(x)} \leq \frac{1}{\sqrt{1-2\,\varepsilon^2}} \quad x \in U_a\,. \tag{2.10}$$

hold. This means

$$|n^i(x)\, n^j(x)| \to 0\,, \quad i,j = 1,2\,, \quad x \in U_a\,. \tag{2.11}$$

In this chosen system of local coordinates, using (1.6), (2.5) and (2.6) the expression (2.9) can be written in the form:

$$\|u\|^2 = \sum_{a=1}^{n} \left(I_{1,a}(\vec{u}) + I_{2,a}(\vec{u}) + I_{3,a}(\vec{u}) \right), \tag{2.12}$$

where

$$I_{1,a}(\vec{u}) = \frac{1}{2} \int_{V_a} \varphi_a^2(x)\,(\delta_{ik} - n^i n^k)\,(\delta_{il} - n^i n^l)\, \frac{\partial u^j}{\partial x^k}\, \frac{\partial u^j}{\partial x^l}\, \frac{1}{n^3}\, dx^1 dx^2\,, \tag{2.13}$$

$$I_{2,a}(\vec{u}) = \frac{1}{2} \int_{V_a} \varphi_a^2(x)\,(\delta_{ik} - n^i n^k)\,(\delta_{jl} - n^j n^l)\, \frac{\partial u^j}{\partial x^k}\, \frac{\partial u^i}{\partial x^l}\, \frac{1}{n^3}\, dx^1 dx^2\,, \tag{2.14}$$

$$I_{3,a}(\vec{u}) = \int_{V_a} \varphi_a^2(x)\, |\vec{u}|^2\, \frac{1}{n^3}\, dx^1 dx^2\,. \tag{2.15}$$

We estimate the integral $I_{1,a}(\vec{u})$. Using the inequalities (2.10) we obtain:

$$\frac{(1-\varepsilon^2)^2}{2} \int_{V_a} \varphi_a^2\, \frac{\partial u^j}{\partial x^k}\, \frac{\partial u^j}{\partial x^k}\, dx^1 dx^2 \leq I_{1,a} \leq \frac{(1+\varepsilon^2)^2}{2\sqrt{1-2\,\varepsilon^2}} \int_{V_a} \varphi_a^2\, \frac{\partial u^j}{\partial x^k}\, \frac{\partial u^j}{\partial x^k}\, dx^1 dx^2\,. \tag{2.16}$$

Using the identity

$$\varphi_a\, \frac{\partial u^j}{\partial x^k} = \frac{\partial}{\partial x^k}(\varphi_a u^j) - u^j\, \frac{\partial \varphi_a}{\partial x^k}$$

we find that

$$\begin{aligned}
&\int_{V_a} \varphi_a^2\, \frac{\partial u^j}{\partial x^k}\, \frac{\partial u^j}{\partial x^k}\, dx^1 dx^2 \\
&\quad = \int_{V_a} \left[\frac{\partial}{\partial x^k}(\varphi_a u^j) - u^j\, \frac{\partial \varphi_a}{\partial x^k} \right] \left[\frac{\partial}{\partial x^k}(\varphi_a u^j) - u^j\, \frac{\partial \varphi_a}{\partial x^k} \right] dx^1 dx^2 \\
&\quad = \int_{V_a} \frac{\partial(\varphi_a u^j)}{\partial x^k}\, \frac{\partial(\varphi_a u^j)}{\partial x^k}\, dx^1 dx^2 + \int_{V_a} \big(\frac{\partial \varphi_a}{\partial x^k}\big)^2 |\vec{u}|^2 dx^1 dx^2 \\
&\quad - 2 \int_{V_a} \frac{\partial(\varphi_a u^j)}{\partial x^k}\, u^j\, \frac{\partial \varphi_a}{\partial x^k}\, dx^1 dx^2.
\end{aligned}$$

Since $\operatorname{supp}(\varphi_a \circ f_a^{-1}) \subset V_a$, we set:

$$m = \min_{V_a} \left|\frac{\partial \varphi_a}{\partial x^k}\right|, \quad M = \max_{V_a} \left|\frac{\partial \varphi_a}{\partial x^k}\right|.$$

Using the inequality

$$2\left|\frac{\partial(\varphi_a u^j)}{\partial x^k}\, u^j\, \frac{\partial \varphi_a}{\partial x^k}\right| \le \varepsilon \left|\frac{\partial(\varphi_a u^j)}{\partial x^k}\right|^2 + \frac{1}{\varepsilon}\, |\vec{u}|^2 \left|\frac{\partial \varphi_a}{\partial x^k}\right|^2$$

for $\varepsilon > 0$, we conclude that

$$\begin{aligned}\frac{(1-\varepsilon^2)^2}{2}\left[(1-\varepsilon)\int_{V_a}\left|\frac{\partial(\varphi_a u^j)}{\partial x^k}\right|^2 dx^1dx^2 + (m - \frac{M}{\varepsilon})\int_{V_a}|\vec{u}|^2\, dx^1dx^2\right] \le \\ \le\ I_{1,a}(\vec{u})\ \le \\ \le \frac{(1+\varepsilon^2)^2}{2\sqrt{1-2\,\varepsilon^2}}\left[(1+\varepsilon)\int_{V_a}\left|\frac{\partial(\varphi_a u^j)}{\partial x^k}\right|^2 dx^1dx^2 + M(1+\frac{1}{\varepsilon})\int_{V_a}|\vec{u}|^2\, dx^1dx^2\right].\end{aligned} \tag{2.17}$$

For the integral $I_{2,a}(\vec{u})$, we obtain a similar estimation to (2.17), since

$$-\frac{\partial u^i}{\partial x^k}\,\frac{\partial u^i}{\partial x^k} \le -(\frac{\partial u^1}{\partial x^2})^2 - (\frac{\partial u^1}{\partial x^2})^2 \le \frac{\partial u^i}{\partial x^k}\,\frac{\partial u^k}{\partial x^i} \le \frac{\partial u^i}{\partial x^k}\,\frac{\partial u^i}{\partial x^k}.$$

This means that

$$\begin{aligned}&\frac{(1-\varepsilon^2)^2}{2}\left[(1-\varepsilon)\int_{V_a}\left|\frac{\partial(\varphi_a u^j)}{\partial x^k}\right|^2 dx^1dx^2 + (m - \frac{M}{\varepsilon})\int_{V_a}|\vec{u}|^2\, dx^1dx^2\right] \\ &\quad \le\ I_{1,a}(\vec{u}) + I_{2,a}(\vec{u}) \\ &\quad \le \frac{(1+\varepsilon^2)^2}{\sqrt{1-2\,\varepsilon^2}}\left[(1+\varepsilon)\int_{V_a}\left|\frac{\partial(\varphi_a u^j)}{\partial x^k}\right|^2 dx^1dx^2 \right. \\ &\quad \left. + M(1+\frac{1}{\varepsilon})\int_{V_a}|\vec{u}|^2\, dx^1dx^2\right].\end{aligned} \tag{2.18}$$

Let $C(\varepsilon)$ and K_ε be positive constants, possibly with additional indexes, with the properties

$$C(\varepsilon) \to C > 0\,, \quad K_\varepsilon \to +\infty\,, \qquad \varepsilon \to 0\,.$$

Then (**??**) becomes

$$\begin{aligned}C_1(\varepsilon)\int_{V_a}\left|\frac{\partial(\varphi_a u^j)}{\partial x^k}\right|^2 dx^1dx^2 - K_{1,\varepsilon}\int_{V_a}|\vec{u}|^2\, dx^1dx^2 \le \\ \le\ I_{1,a}(\vec{u}) + I_{2,a}(\vec{u})\ \le \\ \le C_2(\varepsilon)\int_{V_a}\left|\frac{\partial(\varphi_a u^j)}{\partial x^k}\right|^2 dx^1dx^2 + K_{2,\varepsilon}\int_{V_a}|\vec{u}|^2\, dx^1dx^2\,.\end{aligned} \tag{2.19}$$

We introduce the functions:

$$\psi_a(x) = \frac{\varphi_a(x)}{\varphi(x)}, \quad a = 1, 2..., N, \quad \varphi(x) = \sum_{a=1}^{N}\varphi_a(x) \ge 0\,, \quad x \in V_a\,.$$

The functions ψ_a are a partition of unity corresponding to the cover $\{U_a\}$, $a = 1, 2, ..., N$. Then inequality (2.19) reduces to

$$\begin{aligned} C_1(\varepsilon) \int_{V_a} &|\frac{\partial(\varphi\,\psi_a u^j)}{\partial x^k}|^2\, dx^1 dx^2 - K_{1,\varepsilon} \int_{V_a} |\vec{u}|^2\, dx^1 dx^2 \\ &\le\ I_{1,a}(\vec{u}) + I_{2,a}(\vec{u}) \qquad (2.20) \\ &\le C_2(\varepsilon) \int_{V_a} |\frac{\partial(\varphi\,\psi_a u^j)}{\partial x^k}|^2\, dx^1 dx^2 + K_{2,\varepsilon} \int_{V_a} |\vec{u}|^2\, dx^1 dx^2. \end{aligned}$$

Using the identity

$$|\frac{\partial(\varphi\,\psi_a u^j)}{\partial x^k}|^2 = |\frac{\partial \varphi}{\partial x^k}|^2 |\psi_a u^j|^2 + |\varphi|^2 |\frac{\partial(\psi_a u^j)}{\partial x^k}|^2 + 2\,\frac{\partial \varphi}{\partial x^k}\,\psi_a u^j \varphi\,\frac{\partial(\psi_a u^j)}{\partial x^k}\,,$$

and estimating every term as above we derive:

$$\begin{aligned} C_1(\varepsilon) \int_{V_a} |\frac{\partial(\psi_a u^j)}{\partial x^k}|^2\, dx^1 dx^2 - K_{0,\varepsilon} \int_{V_a} |\psi_a u^j|^2 dx^1 dx^2 - \\ -K_{1,\varepsilon} \int_{V_a} |\vec{u}|^2\, dx^1 dx^2 + \int_{V_a} \varphi_a^2 |\vec{u}|^2 dx^1 dx^2 \le \\ \le\ I_{1,a}(\vec{u}) + I_{2,a}(\vec{u}) + I_{3,a}(\vec{u})\ \le \qquad (2.21) \\ \le C_2(\varepsilon) \int_{V_a} |\frac{\partial(\psi_a u^j)}{\partial x^k}|^2\, dx^1 dx^2 + K_{3,\varepsilon} \int_{V_a} |\psi_a u^j|^2 dx^1 dx^2 + \\ +K_{2,\varepsilon} \int_{V_a} |\vec{u}|^2\, dx^1 dx^2 + K \int_{V_a} \varphi_a^2 |\vec{u}|^2 dx^1 dx^2\,. \end{aligned}$$

Fixing $\varepsilon > 0$ and summing over all $a = 1, 2, ..., N$ from (2.21) we get

$$C_1 \|\vec{u}\|^2_{W^1_2(S,\mathbb{R}^3)} - K_1 \|\vec{u}\|^2_{L_2(S,\mathbb{R}^3)} \le \|u\|^2 \le C_2 \|\vec{u}\|^2_{W^1_2(S,\mathbb{R}^3)} + K_2 \|\vec{u}\|^2_{L_2(S,\mathbb{R}^3)}\,.$$

The Sobolev embedding of $W^1_2(S, \mathbb{R}^3)$ into $L_2(S, \mathbb{R}^3)$ implies that there exists constant $C > 0$ such that

$$C_1 \|\vec{u}\|^2_{W^1_2(S,\mathbb{R}^3)} - K_1 \|\vec{u}\|^2_{L_2(S,\mathbb{R}^3)} \le \|\vec{u}\|^2 \le C \|\vec{u}\|^2_{W^1_2(S,\mathbb{R}^3)}\,.$$

Therefore

$$\frac{C_1}{1+K_1}\, \|\vec{u}\|^2_{W^1_2(S,\mathbb{R}^3)} \le \|\vec{u}\|^2 \le C\, \|\vec{u}\|^2_{W^1_2(S,\mathbb{R}^3)}\,.$$

□

We consider the following space

$$H(S, \mathbb{R}^3) = \big\{\vec{u} \in W^1_2(S, \mathbb{R}^3) : \vec{u}|_{\partial S} \in W^2_2(\partial S, \mathbb{R}^3)\big\} = W^1_2(S, \mathbb{R}^3) \cap W^2_2(\partial S, \mathbb{R}^3)\,.$$

Proposition 2.2. *$H(S, \mathbb{R}^3)$ is a Hilbert space endowed with the norm*

$$\|\vec{u}\|_{H(S,\mathbb{R}^3)} = \big(\, \|\vec{u}\|^2_{W^1_2(S,\mathbb{R}^3)} + \|\vec{u}\|^2_{W^2_2(\partial S,\mathbb{R}^3)}\big)^{1/2}\,. \qquad (2.22)$$

Proof . We are going to show that $H(S,\mathbb{R}^3)$ is a complete space. Let $\{\vec{u}_n\}$ be a Cauchy sequence in $H(S,\mathbb{R}^3)$. Then due to the completeness of spaces $W_2^1(S,\mathbb{R}^3)$ and $W_2^2(\partial S,\mathbb{R}^3)$, we have $\vec{u}_n \to \vec{u}$ in $W_2^1(S,\mathbb{R}^3)$ and $\vec{u}_n|_{\partial S} \to \vec{v}$ in $W_2^2(\partial S,\mathbb{R}^3)$. Hence, the Sobolev imbedding theorem implies that $\vec{u}_n|_{\partial S} \to \vec{u}|_{\partial S}$ in $L_2(\partial S)$. Therefore, the uniqueness of the limit in $L_2(\partial S)$ implies that $\vec{u}|_{\partial S} = \vec{v}$, which means that $\vec{u}|_{\partial S} \in W_2^2(\partial S,\mathbb{R}^3)$. $\square$

Finally, we introduce the space

$$H_0(S,T_xM) = \left\{ \vec{u} \in W_2^1(S,T_xM) : \ \vec{u}|_\Gamma \in W_2^2(\Gamma,T_xM), \ \ \vec{u}|_{\Gamma_1} = \vec{0} \right\} .$$

From proposition (2.2), $H_0(S,\mathbb{R}^3)$ is a Hilbert space with respect to norm (2.22).

Proposition 2.3. *The expression*

$$\|\vec{u}\| = \left[\int_S a_{ijkl}(x)\, \xi_{ij}(\vec{u})\, \xi_{kl}(\vec{u})\, dS + \int_\Gamma |\delta_i\delta_i\vec{u}|^2\, ds \right]^{1/2} \tag{2.23}$$

defines a norm on $H_0(S,T_xM)$ equivalent to (2.22).

Proof . Since $a_{ijkl} \in L_\infty(S)$, using propositions (2.1) and (2.2) we obtain:

$$\|\vec{u}\|^2 \le c \int_S \xi_{ij}(\vec{u})\, \xi_{ij}(\vec{u})\, dS + \int_\Gamma |\delta_i\delta_i\vec{u}|^2 ds \le c_1 \|\vec{u}\|^2_{H_0(S,T_xM)} , \tag{2.24}$$

where c and c_1 are positive constants. It is enough to show that there exists constant $C > 0$ such that

$$\|\vec{u}\|^2_{L_2(S,T_xM)} + \|\vec{u}\|^2_{L_2(\Gamma,T_xM)} \le C\, \|\vec{u}\|^2 . \tag{2.25}$$

Suppose that the inequality (2.25) is not valid. Then there exists a sequence $\{\vec{u}_n\} \subset H_0(S,T_xM)$ such that

$$\|\vec{u}_n\|^2 < \frac{1}{n} \left(\|\vec{u}_n\|^2_{L_2(S,T_xM)} + \|\vec{u}_n\|^2_{L_2(\Gamma,T_xM)} \right) , \quad n \in \mathbb{N} . \tag{2.26}$$

We introduce the sequence:

$$\vec{v}_n = \frac{\vec{u}_n}{\sqrt{\|\vec{u}_n\|^2_{L_2(S,T_xM)} + \|\vec{u}_n\|^2_{L_2(\Gamma,T_xM)}}} , \quad n \in \mathbb{N} . \tag{2.27}$$

Then $\vec{v}_n \in H_0(S,T_xM)$,

$$\|\vec{v}_n\|^2_{L_2(S,T_xM)} + \|\vec{v}_n\|^2_{L_2(\Gamma,T_xM)} = 1 , \quad \|\vec{v}_n\| < \frac{1}{n} ,$$

and

$$\|\vec{v}_n\|^2_{H(S,T_xM)} = \|\vec{v}_n\|^2_{W_2^1(S,T_xM)} + \|\vec{v}_n\|^2_{W_2^2(\Gamma,T_xM)} = 1 + c\, \|\vec{v}_n\| \ \le 1 + \frac{c}{n} ,$$

where c is a positive constant. This means that sequence $\{\vec{v}_n\}$ is bounded in space $H_0(S,T_xM)$ and that bound is independent of n. Hence, there exists a subsequence of $\vec{v}_n$ (we keep the same index n), which weakly converges to a $\vec{v} \in H_0(S,T_xM)$. The compactness of Sobolev embedding of $H_0(S,T_xM)$ into the spaces $L_2(S,T_xM)$ and $L_2(\Gamma,T_xM)$ implies that

$$\|\vec{v}\| = 0 , \quad \|\vec{v}\,\|^2_{L_2(S,T_xM)} + \|\vec{v}\,\|^2_{L_2(\Gamma,T_xM)} = 1 . \tag{2.28}$$

From the first relation of (2.28) we get

$$\xi_{ij}(\vec{v}) = 0$$

or

$$\nabla_i v^j + \nabla_j v^i = 0 \tag{2.29}$$

for all i, j. Since $n^i \nabla_i = 0$, we multiply (2.29) by n^i and integrate the result over S to obtain:

$$\int_S n^i \nabla_j v^i dS = 0$$

for all j. Using formula (1.9), and considering that $n^i(x)v^i(x) = 0$, since $\vec{v}(x) \in T_x M$, we get:

$$\int_S v^i \nabla_j n^i dS = 0$$

for all j. Suppose that $\vec{v} \neq \vec{0}$. Then necessarily $\nabla_i n^j = 0$ holds for all i, j. Consequently, we have that $\nabla_i n^i = 0$ for $i = j$. From (1.11), we conclude that the mean curvature H of surface M vanishes. This contradicts to the initial hypothesis (1.1). Thus $\vec{v} = \vec{0}$, which also contradicts to the second relation of (2.28). □

3. Free equilibrium states

Under the additional assumption of smoothness

$$\partial S \in C^\infty, \quad a_{ijkl} \in C^\infty(\overline{S}), \quad q \in C^\infty(T_x M, \partial S) \tag{3.1}$$

the integral equation (1.15) can be written in the classical form of a boundary value problem

$$\begin{aligned} &H\eta^l b_{ijkl}(x)\,\xi_{ij}(\vec{u}) + \nabla_l\left[\,b_{ijkl}(x)\,\xi_{ij}(\vec{u})\right] = 0, && x \in S \\ &b_{ijkl}(x)\,\xi_{ij}(\vec{u})\nu^l + (K^2 + R^2 - K - R)\,Du^k + \\ &\qquad + D^2 u^k - \lambda q_{u^k}(\vec{u}, x) = 0, && x \in \Gamma \\ &\vec{v} = \vec{0}, && x \in \Gamma_1 \end{aligned} \tag{3.2}$$

for all k. Here $D = \delta_i \delta_i$, and

$$b_{ijkl} = a_{ijkl} + a_{ijlk}. \tag{3.3}$$

Equation (3.2) is derived from (1.15) using the formulae (1.9), (1.10) and (1.12). Equation (3.2) is called equilibrium condition and describes the balance between the outer forces and the stress forces.

Theorem 3.1. *The number* λ_0 *is a bifurcation point for problem* (1.15)*, if and only if*

$$\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{u})\,\xi_{kl}(\vec{v})\,dS + \int_\Gamma \delta_i \delta_i\,\vec{u}\,\delta_j \delta_j\,\vec{v}\,ds - \lambda_0 \int_\Gamma q_{u^i u^k}(\vec{0}, x)\,v^i u^k ds = 0 \tag{3.4}$$

has a nonzero solution $\vec{u} \in H_0(S, T_x M)$ *for all* $\vec{v} \in H_0(S, T_x M)$.

Proof . First we note that the functional (1.3) is differentiable due to the smoothness of function q. The compactness of Sobolev embedding of $W_2^2(\Gamma, \mathbb{R}^3)$ into $C(\Gamma, \mathbb{R}^3)$ implies that the functional (1.3) is weakly continuous and its differential $G'[\,\vec{u}\,]$ satisfies the local Lipschitz continuous with

$$G'[\,\vec{u}\,] = A\,\vec{u} + N(\vec{u})\,,$$

where operator $A : H_0(S, T_xM) \longrightarrow H_0(S, T_xM)$ is defined as

$$(A\,\vec{u}, \vec{v})_{H_0(S,T_xM)} = \int_\Gamma q_{u^i u^j}(\vec{0}, x)\, v^i u^j\, ds\,,$$

and

$$(N(\vec{u}), \vec{v})_{H_0(S,T_xM)} = \int_\Gamma \left[q_{u^i}(\vec{u}, x) - q_{u^i u^j}(\vec{0}, x)\, u^j \right]\, v^i\, ds$$

for all $\vec{v} \in H_0(S, T_xM)$. Obviously, operator A is linear and symmetric. The above embedding implies that operator A is compact and there exists a positive constant $C > 0$ such that

$$\|N(\vec{u})\|_{H_0(S,T_xM)} \leq C\, \|\vec{u}\|^2_{H_0(S,T_xM)}\,.$$

Based on proposition (2.3), the functional (1.2) can be written in the equivalent form

$$F[\,\vec{u}\,] = \frac{1}{2}\, \|\vec{u}\,\|^2_{H_0(S,T_xM)} = \frac{1}{2}\, (\vec{u},\, \vec{u})_{H_0(S,T_xM)}\,,$$

where $(\ ,\ )$ denotes the inner product of space $H_0(S, T_xM)$. Thus, the integral equation (3.4) can be represented as

$$(\vec{u},\, \vec{v}) -\, \lambda_0\, (A\,\vec{u},\, \vec{v}) = 0\,. \tag{3.5}$$

Under the above notations, the variational method of I. V. Skrypnik [7] provides that $\lambda_0 \in \mathbb{R}$, corresponding to a non zero critical point $\vec{u}$ of the functional (1.4), is a bifurcation point for equation (1.14), if and only if

$$I''[\,\vec{0}, \lambda_0](\vec{u}, \vec{v}) = (I''[\,\vec{0}, \lambda_0]\,\vec{u},\, \vec{v}) = 0 \tag{3.6}$$

is satisfied by a non zero solution for all $\vec{v} \in H_0(S, T_xM)$. Since (3.6) is equivalent to (3.4) or (3.5) we proved our assertion. □

We define a closed subspace

$$H_1(S, T_xM) = \{\vec{u} \in H_0(S, T_xM) :\ (\vec{u}, \vec{v}) = 0\,,\ \vec{v} \in \overset{\circ}{W}{}_1^2\,(S, T_xM)\,\}$$

of $H_0(S, T_xM)$, where $(\ ,\ )$ is the inner product with respect to norm (2.23), and $\vec{v} \in \overset{\circ}{W}{}_2^1\,(S, T_xM)$ means that $\vec{v} \in W_2^1(S, T_xM)$ with $\vec{v} = \vec{0}$ on ∂S. Thus, $H_1(S, T_xM)$ is the orthogonal complement of $\overset{\circ}{W}{}_2^1\,(S, T_xM)$, and

$$H_0(S, T_xM) = H_1(S, T_xM) \oplus \overset{\circ}{W}{}_2^1\,(S, T_xM)\,.$$

Proposition 3.2. *A vector field $\vec{u}$ is a solution of equation* (1.15)*, if and only if it is a critical point of functional* (1.4)*, restricted in $H_1(S, T_xM)$.*

Proof . Let $\vec{u} \in H_0(S, T_xM)$ be a solution of equation (1.15). Then

$$F'[\,\vec{u}\,]\,\vec{v} - \lambda\,G'[\,\vec{u}\,]\,\vec{v} = \big(F'[\,\vec{u}\,] - \lambda\,G'[\,\vec{u}\,]\,,\,\vec{v}\big) = 0$$

for all $\vec{v} \in H_0(S, T_xM)$. Since

$$\vec{v} = \vec{v}_1 + \vec{v}_2\,, \quad \vec{v}_1 \in H_1(S, T_xM)\,, \quad \vec{v}_2 \in \overset{\circ}{W}{}^1_2\,(S, T_xM)$$

from proposition (2.3) we derive

$$0 = \big(F'[\,\vec{u}\,]\,,\,\vec{v}_2\big) = \int_S a_{ijkl}(x)\,\xi_{ij}(\vec{u})\,\xi_{kl}(\vec{v}_2)\,dS + \int_\Gamma \delta_i\delta_i\,\vec{u}\,\delta_j\delta_j\,\vec{v}_2\,ds = (\vec{u}, \vec{v}_2)$$

for $\vec{v}_2 \in \overset{\circ}{W}{}^1_2\,(S, T_xM)$, which means that $\vec{u} \in H_1(S, T_xM)$. The reverse assertion is obvious. □

We assume that the function q satisfies, in addition to (1.13), the following conditions

$$q(\vec{u}, x) > 0\,,\ q_{u^i}(\vec{u}, x)u^i > 0\,,\ q(-\vec{u}, x) = q(\vec{u}, x)\,,\ q_{u^i}(c\,\vec{u}, x) = c^{p+1}q_{u^i}(\vec{u}, x) \tag{3.7}$$

for all $\vec{u} \in H_0(S, T_xM)$, $x \in \partial S$, $c \in \mathbb{R}$ and $p > 0$.

Theorem 3.3. *For every $\lambda > 0$ equation* (1.15) *admits a countable set of non zero solutions.*

Proof . Since the functionals (1.2) and (1.3) have the properties described in the proof of theorem 3.1, we can apply the Lyusternik - Schnirelman's variational method [3]. Thus, for $\lambda > 0$ and $\alpha > 0$ there exists a countable set $\vec{w}_n$, μ_n of different solutions for the problem

$$F'[\,\vec{w}_n\,] - \mu_n\lambda\,G'[\,\vec{w}_n\,] = 0\,, \quad F[\,\vec{w}_n\,] = \alpha\,,$$

where $\vec{w}_n \in H_1(S, T_xM)$, or equivalently

$$\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{w}_n)\,\xi_{kl}(\vec{v})\,dS + \int_\Gamma \delta_i\delta_i\,\vec{w}_n\,\delta_j\delta_j\,\vec{v}\,ds - \mu_n\lambda\int_\Gamma q_{u^i}(\vec{w}_n, x)\,v^i ds = 0\,, \tag{3.8}$$

$$\|\vec{w}_n\|^2_{H_0(S,T_xM)} = \alpha$$

for all $\vec{v} \in H_1(S, T_xM)$. We set $\vec{w}_n = c_n\vec{u}_n$, where $c_n \in \mathbb{R}$. Then from (3) we derive that $\mu_n > 0$ and

$$\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{u}_n)\,\xi_{kl}(\vec{v})\,dS + \int_\Gamma \delta_i\delta_i\,\vec{u}_n\,\delta_j\delta_j\,\vec{v}\,ds - \mu_n c_n^p\lambda\int_\Gamma q_{u^i}(\vec{u}_n, x)\,v^i ds = 0\,,$$

$$\|\vec{u}_n\|^2_{H_0(S,T_xM)} = \frac{\alpha}{c_n^2}\,.$$

Choosing $c_n = \mu_n^{-1/p}$, we see that $\vec{u}_n$ is a solution for (1.15). □

Note that, under the assumptions of smoothness (3.1), the integral equation (3.4) is equivalent to the boundary value problem:

$$\begin{aligned} &H\eta^l b_{ijkl}(x)\,\xi_{ij}(\vec{u}) + \nabla_l\,[\,b_{ijkl}(x)\,\xi_{ij}(\vec{u})] = 0, && x \in S \\ &b_{ijkl}(x)\,\xi_{ij}(\vec{u})\nu^l + (K^2 + R^2 - K - R)\,Du^k + \\ &\quad +D^2u^k - \lambda_0 q_{u^iu^k}(\vec{0}, x)u^i = 0, && x \in \Gamma \\ &\vec{v} = \vec{0}, && x \in \Gamma_1 \end{aligned} \tag{3.9}$$

for all k.

Theorem 3.4. *If the conditions of smoothness* (3.1) *hold, then every solution of boundary value problem (4.17) is C^∞ differentiable.*

Proof . From equation (1.15) it follows that $\vec{u}$ is a weak solution for the differential equation

$$H\eta^l b_{ijkl}(x)\,\xi_{ij}(\vec{u}) + \nabla_l\,[\,b_{ijkl}(x)\,\xi_{ij}(\vec{u})] = 0\,. \tag{3.10}$$

In the introduced system of local coordinates on $x \in S$, using (2.6), the higher derivative term of (3.10)

$$(L\vec{u})^k = \nabla_l\,[\,b_{ijkl}(x)\,\xi_{ij}(\vec{u})]$$

is represented as

$$(L\vec{u})^k = (\,b_{ijkl} + b_{jikl}\,)\Big(\frac{\partial^2 u^j}{\partial x^l \partial x^i} + n^i n^s n^l n^r \frac{\partial^2 u^j}{\partial x^r \partial x^s} - n^i n^s \frac{\partial^2 u^j}{\partial x^l \partial x^s} - n^l n^r \frac{\partial^2 u^j}{\partial x^r \partial x^i}\,\Big)\,,$$

where $j,k = 1,2,3,$ and $i,l,s,r = 1,2$. Thus in a small enough neighborhood of $x \in S$, and considering (2.11) this term is defined as

$$(L_0\vec{u})^k = (\,b_{ijkl} + b_{jikl}\,)\,\frac{\partial^2 u^j}{\partial x^l \partial x^i}\,.$$

Let vectors $\vec{\zeta}$, $\vec{\theta} \in T_x M$. From (3.3) and (1.5), we can verify that

$$(\,b_{ijkl} + b_{jikl})\,\zeta^i\theta^j\zeta^k\theta^l \geq 2\,\Lambda|\vec{\zeta}|^2|\vec{\theta}|^2\,,$$

which means that L is an elliptic operator. Let $\vec{u}$ a solution for (3.4). Now according to [6], since $\vec{u}|_\Gamma \in W_2^2(\Gamma, T_x M)$, the solution of (3.10) belongs to space $W_2^{2+1/2}(S, T_x M)$. Thus, equation (3.4) is equivalent to

$$\int_\Gamma \big[\,b_{ijkl}(x)\,\xi_{ij}(\vec{u})\nu^l + (K^2 + R^2 - K - R)\,Du^k - \lambda\,q_{u^i u^k}(\vec{0}, x)\,u^i) + D^2 u^k\,\big]\,v^k\,ds = 0\,, \tag{3.11}$$

where

$$b_{ijkl}(x)\,\xi_{ij}(\vec{u})\nu^l + (K^2 + R^2 - K - R)\,Du^k - \lambda\,q_{u^i u^k}(\vec{0}, x)\,u^i \in L_2(\Gamma).$$

Equation (3.11) implies that $\vec{u}|_\Gamma \in W_2^4(\Gamma, T_x M)$, and consequently $\vec{u} \in W_2^{4+1/2}(S, T_x M)$. Iterating this argument, we find that $\vec{u} \in C^\infty$. $\square$

4. Equilibrium states under a constraint

The mapping

$$y : \partial S \longrightarrow M \qquad y(x) = x + \vec{u}(x)\,, \quad \vec{u} \in H_0(S,\, T_x M)\,, \tag{4.1}$$

for small values of $\|\vec{u}\|$, leaves invariant the length $l(\Gamma)$ of the curve Γ on surface M if

$$l\big(y(\Gamma)\big) = l(\Gamma)\,. \tag{4.2}$$

As in section 2, the domain S in M can be considered locally as a graph of a smooth function $f(x^1, x^2)$ on a bounded domain $V \subset \mathbb{R}^2$ with $\partial V \in C^1$, that is:

$$S \cap U = \big\{(x^1, x^2, f(x^1, x^2))\,, \quad (x^1, x^2) \in V \subset \mathbb{R}^2\big\}\,,$$

where $U \subset \mathbb{R}^3$ with small enough diameter. In order to describe the point $(x^1, x^2, f(x^1, x^2))$ when $(x^1, x^2) \in \partial V$, we introduce a local coordinate system in $\mathbb{R}^2$ such that the axis x^1 lies in the tangential direction of ∂V at point (x^1, x^2) and axis x^2 comes along the normal vector of the curve ∂V at the same point. Thus, vectors

$$\vec{\tau} = (1, 0, -\frac{n^1}{n^3}) = (1, 0, \frac{\partial f}{\partial x^1}) \tag{4.3}$$

and $\vec{n}$ at point $(x^1, x^2, f(x^1, x^2))$ form a basis for $\mathbb{R}^2$. In this coordinate system, curve ∂V can be defined locally in a small neighborhood of point (x^1, x^2) as

$$x^2 = h(x^1)\,,$$

where h is a differentiable function on a small neighborhood $(-\varepsilon, \varepsilon)$ of $0 \in \mathbb{R}$, with

$$h(0) = 0\,, \quad h'(0) = 0\,. \tag{4.4}$$

Thus, curve ∂S in the same system of local coordinates can be defined locally as:

$$\partial S \cap U = \big\{ \big(x^1, h(x^1), f(x^1, h(x^1))\big)\,, \quad x^1 \in (-\varepsilon, \varepsilon) \big\}\,.$$

Using the above coordinates in $\mathbb{R}^3$, we denote the parametric representation of curve Γ by

$$\Gamma(x^1) = \big(x^1, h(x^1), f(x^1, h(x^1))\big)\,, \quad x^1 \in [-a, a]\,,$$

with $\Gamma(a) = \Gamma(-a)$ and $\Gamma'(a) = \Gamma'(-a)$. Since $\Gamma'(x^1) \neq \vec{0}$, we can choose the arc length $t \in [0, L]$ instead of x^1 as a parametrization of curve Γ, where L is the length of curve Γ. In this case

$$|\dot{\Gamma}(t)| = \big|\frac{d\Gamma(t)}{dt}\big| = 1 \tag{4.5}$$

holds. Thus, according to (4.1) curve Γ transforms to the curve

$$\gamma(t) = \Gamma(t) + \vec{u}\big(\Gamma(t)\big)\,, \quad t \in [0, L]$$

or

$$\gamma^1(t) = t + u^1\,, \quad \gamma^2(t) = h(t) + u^2\,, \quad \gamma^3(t) = f(t + u^1,\, h(t) + u^2)\,. \tag{4.6}$$

Consequently, constraint (4.2) holds if

$$\int_0^L \sqrt{g_{ij}\big(\gamma(t)\big)\, \dot{\gamma}^i(t)\, \dot{\gamma}^j(t)}\, dt = l(\Gamma) = L\,, \tag{4.7}$$

where $g_{ij}(x)$ are the components of metric tensor at $x \in M$. We define the functional

$$\Phi[\,\vec{u}\,] = \int_0^L \big[\, g_{ij}\big(\Gamma(t) + \vec{u}\big) \frac{d}{dt}\big(\Gamma^i(t) + u^i\big)\, \frac{d}{dt}\big(\Gamma^j(t) + u^j\big)\big]^{1/2} dt - L\,. \tag{4.8}$$

Obviously, (4.2) holds if

$$\Phi[\,\vec{u}\,] = 0. \tag{4.9}$$

The mapping $\Phi\,:\, H_0(S,\, T_xM) \longrightarrow \mathbb{R}$ is continuously differentiable in a small neighborhood of $\vec{0} \in H_0(S,\, T_xM)$.

On a fixed point $x \in \Gamma \subset M$ in this system of local coordinates we have $g_{ij}(x) = \delta_{ij}$ and

$$g_{ij}(y) = \frac{\partial\, y^k}{\partial\, x^i}\,\frac{\partial\, y^k}{\partial\, x^j}\,, \quad k = 1,2,3\,, \quad i,j = 1\,.$$

Using the coordinate transformation (4.6), we obtain:

$$g_{11}(y) = (1 + u^1_{x^1} + u^2_{x^2}h')^2 + (h' + u^2_{x^2}h' + u^2_{x^1})^2 + \left[\, f_{y^1}(1 + u^1_{x^1} + u^2_{x^2}h') + f_{y^2}(h' + u^2_{x^2}h' + u^2_{x^1})\,\right]^2 .$$

From (1.8), we obtain

$$\frac{d}{dt}\,u^i(\Gamma^i(t)) = \frac{\partial u^i}{\partial x^j}\,\dot\Gamma^j(t) = \tau^j\delta_j u^i\,.$$

Considering the estimate (2.11) for a small enough neighborhood of $x \in \Gamma$, and relations (4.4), for $\vec u = \vec 0$ we derive that

$$\Phi'[\,\vec 0\,]\,\vec v = \int_0^L (v^1_{x^1} + \tau^k\delta_k v^1)\,dt\,.$$

Finally, from (4.5), (4.3), (1.8), (1.10), and (1.12) we derive that

$$\Phi'[\,\vec 0\,]\,\vec v = \int_\Gamma (\delta_1 v^1 + \tau^k\delta_k v^1)\,ds = \int_\Gamma \delta_1 v^1 ds = -\int_\Gamma (K\,\nu^1 + R\,n^1)\,v^1 ds\,. \tag{4.10}$$

Proposition 4.1. *There exists decomposition of space $H_0(S, T_xM)$ in direct sum*

$$H_0(S,T_xM) = X_1 \oplus X_2\,,$$

where

$$X_1 = \left\{\vec v \in H_0(S,T_xM) : \quad \int_\Gamma (K\nu^1 + Rn^1)\,v^1 ds = 0\right\},$$

$$X_2 = \left\{\vec v \in H_0(S,T_xM) : \quad v^1|_\Gamma = \frac{C\,(K\nu^1 + Rn^1)}{\|K\nu^1 + Rn^1\|_{L_2(\Gamma)}}\,,\ C \neq 0\right\}$$

and a differentiable mapping r from a neighborhood of $\vec 0 \in X_1$ to a neighborhood of $\vec 0 \in X_2$, such that the solutions of equation (4.9) can be expressed as

$$\vec u = \vec v + r[\,\vec v\,]\,, \quad \vec v \in X_1 \tag{4.11}$$

with

$$r[\,\vec 0\,] = \vec 0\,, \quad r'[\,\vec 0\,] = 0\,. \tag{4.12}$$

Proof . This conclusion comes directly from Lyapunov - Schmidt decomposition and the implicit function theorem [4]. From (4.8) it is obvious that $\Phi[\,\vec 0\,] = 0$. Thus, we set $X_1 = \mathrm{Ker}\Phi'[\,\vec 0\,]$ and $X_2 = X_1^\perp$. □

Now a critical point for the functional (1.4) under the constraint (4.9), for a given $\lambda \in \mathbb{R}$, is the vector field $\vec v \in X_1$, which satisfies the relation

$$I'[\,\vec v, \lambda\,]\,\vec w = 0 \tag{4.13}$$

for each $\vec{w} \in X_1$. Assuming (4.11), equation (4.13) can be written equivalently as

$$\begin{aligned} &\int_S a_{ijkl}(x)\, \xi_{ij}\left(\vec{w} + r'[\,\vec{v}\,]\vec{w}\right)\, \xi_{kl}\left(\vec{v} + r[\,\vec{v}\,]\right) dS + \\ &+ \int_\Gamma \delta_i\delta_i\, \left(\vec{v} + r[\,\vec{v}\,]\right) \delta_j\delta_j \left(\vec{w} + r'[\,\vec{v}\,]\vec{w}\right) ds - \\ &-\lambda \int_\Gamma q_{u^i}\left(\vec{v} + r[\,\vec{v}\,], x\right)\left(w^i + (r'\,[\,\vec{v}\,]\vec{w})^i\right) ds = 0. \end{aligned} \tag{4.14}$$

Note that the vector field $\vec{v} = \vec{0}$ is a critical point for the functional (1.4) under the constraint (4.9), due to (4.12). The linearised equation, which corresponds to (4.13), is

$$I''[\,\vec{0}, \lambda\,]\,(\vec{v}, \vec{w}) = 0\,, \quad \vec{v}, \vec{w} \in X_1\,, \tag{4.15}$$

or equivalently

$$\int_S a_{ijkl}(x)\, \xi_{ij}(\vec{v})\, \xi_{kl}(\vec{w})\, dS + \int_\Gamma \delta_i\delta_i\vec{v}\, \delta_j\delta_j\vec{w}\, ds - \lambda \int_\Gamma q_{u_iu_j}(\vec{0}, x)\, v^i w^j\, ds = 0\,. \tag{4.16}$$

Theorem 4.2. *The number λ_0 is a bifurcation point for problem* (4.13)*, if and only if equation (4.16) has a nonzero solution for all $\vec{w} \in X_1$.*

Proof . The properties of functional Φ, described in proposition (4.1) and functionals F and G, described in the proof of proposition (3.1), allow us to apply a generalized variant of Skrypnik's method, demonstrated in [8], which states the existence of bifurcation points for equation (1.4) under constraint (4.9). Note that, because of proposition (2.3), the integral equation (3.6) can be written as

$$(\vec{v},\, \vec{w}) -\, \lambda\, (A\,\vec{v},\, \vec{w}) = 0$$

for all $\vec{w} \in X_1$. □

Under the additional assumptions of smoothness (3.1), using formulae (1.9), (1.10), and proposition (4.1), the integral equation (4.16) in local coordinates reduces to the equivalent boundary value problem:

$$\begin{aligned} &H\eta^l b_{ijkl}(x)\, \xi_{ij}(\vec{u}) + \nabla_l\, [\, b_{ijkl}(x)\, \xi_{ij}(\vec{u})] = 0, && x \in S \\ &b_{ij1l}(x)\, \xi_{ij}(\vec{u})\nu^l + (K^2 + R^2 - K - R)\, Du^1 + \\ &\quad + D^2u^1 - \lambda_0 q_{u^iu^1}(\vec{0}, x)u^i = K\nu^1 + Rn^1, && x \in \Gamma \\ &b_{ij2l}(x)\, \xi_{ij}(\vec{u})\nu^l + (K^2 + R^2 - K - R)\, Du^2 + \\ &\quad + D^2u^2 - \lambda_0 q_{u^iu^2}(\vec{0}, x)u^i = 0, && x \in \Gamma \\ &\vec{v} = \vec{0}, && x \in \Gamma_1\,. \end{aligned} \tag{4.17}$$

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