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Coupled coincidence point theorems for maps under a new invariant set in ordered cone metric spaces

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Abstract

In this paper, we prove some coupled coincidence point theorems for mappings satisfying generalized contractive conditions under a new invariant set in ordered cone metric spaces. In fact, we obtain sufficient conditions for existence of coupled coincidence points in the setting of cone metric spaces. Some examples are provided to verify the effectiveness and applicability of our results.

Keywords: ψ-map; φ-map; coupled coincidence point; strongly (F, g)-invariant set.
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1. Introduction

Since the concept of a cone metric was introduced by Huang and Zhang [5], many fixed point theorems have been proved by some authors. The existence of fixed points for certain mappings in ordered metric spaces has been studied by Ran and Reurings [16]. Afterwards, Nieto and López [11] extended the result of Ran and Reurings [16] for nondecreasing mappings and applied their results to obtain a unique solution for a first order differential equation. In 2006, Bhaskar and Laksmikantham [3] first studied the existence of coupled fixed points in partially ordered metric spaces. So far, many mathematicians have studied coupled fixed point results for mappings under various contractive conditions in different metric spaces. Recently, Sintunavarat et.al.[19] established coupled fixed points for weak contraction mappings by using the concept of F-invariant set. In this paper we introduce the concept of strongly (F, g)-invariant set and obtain sufficient conditions for existence

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of coupled coincidence points for mappings satisfying generalized contractive conditions related to ψ and φ-maps under strongly (F, g)-invariant set in ordered cone metric spaces. Finally, we supply some examples to illustrate our obtained results.

2. Preliminaries

In this section we present some basic notations, definitions, and necessary results from existing literature.

Definition 2.1. [3] Let $(X, \sqsubseteq)$ be a partially ordered set and $F : X \times X \to X$ be a self-map. One can say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for all $x_1, x_2 \in X$, $x_1 \sqsubseteq x_2$ implies $F(x_1, y) \sqsubseteq F(x_2, y)$ for any $y \in X$, and for all $y_1, y_2 \in X$, $y_1 \sqsupseteq y_2$ implies $F(x, y_1) \sqsubseteq F(x, y_2)$ for any $x \in X$.

Definition 2.2. [4] Let $(X, \sqsubseteq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two self-mappings. F has the mixed g-monotone property if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is, for all $x_1, x_2 \in X$, $gx_1 \sqsubseteq gx_2$ implies $F(x_1, y) \sqsubseteq F(x_2, y)$ for any $y \in X$, and for all $y_1, y_2 \in X$, $gy_1 \sqsubseteq gy_2$ implies $F(x, y_1) \sqsupseteq F(x, y_2)$ for any $x \in X$.

Definition 2.3. [3] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 2.4. [8] An element $(x, y) \in X \times X$ is called

(i) a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$,

(ii) a common coupled fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 2.5. [4] Let X be a nonempty set. One can say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $g(F(x, y)) = F(gx, gy)$, for all $x, y \in X$.

Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \;\Rightarrow\; ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

For any cone $P \subseteq E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \textit{ implies } \|x\| \leq k\,\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. Rezapour and Hamlbarani [13] proved that there are no normal cones with normal constant $k < 1$.

Definition 2.6. [2] Let P be a cone. A nondecreasing mapping $\varphi : P \to P$ is called a φ-map if

(φ_1) $\varphi(\theta) = \theta$ *and* $\theta \prec \varphi(w) \prec w$ *for* $w \in P \setminus \{\theta\}$,
(φ_2) $w - \varphi(w) \in int(P)$ *for every* $w \in int(P)$,
(φ_3) $\lim\limits_{n\to\infty} \varphi^n(w) = \theta$ *for every* $w \in P \setminus \{\theta\}$.

Definition 2.7. [17] Let P be a cone and let (w_n) be a sequence in P. One says that $w_n \to \theta$ if for every $\epsilon \in P$ with $\theta \ll \epsilon$ there exists $n_0 \in \mathbb{N}$ such that $w_n \ll \epsilon$ for all $n \geq n_0$.
A nondecreasing mapping $\psi : P \to P$ is called a ψ-map if

(ψ_1) $\psi(w) = \theta$ *if and only if* $w = \theta$,
(ψ_2) *for every* $w_n \in P$, $w_n \to \theta$ *if and only if* $\psi(w_n) \to \theta$,
(ψ_3) *for every* $w_1, w_2 \in P$, $\psi(w_1 + w_2) \preceq \psi(w_1) + \psi(w_2)$.

Definition 2.8. [5] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.9. [5] Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n). We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

Definition 2.10. [5] Let (X, d) be a cone metric space, (x_n) be a sequence in X. If for any $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X.

Definition 2.11. [5] Let (X, d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 2.12. [21] Every cone metric space (X, d) is a topological space. For $c \gg \theta$, $c \in E$, $x \in X$ let $B(x, c) = \{y \in X : d(y, x) \ll c\}$ and $\beta = \{B(x, c) : x \in X, c \gg \theta\}$. Then $\tau_c = \{U \subseteq X : \forall x \in U, \exists B \in \beta, x \in B \subseteq U\}$ is a topology on X.

Definition 2.13. [21] Let (X, d) be a cone metric space. A map $T : (X, d) \to (X, d)$ is called sequentially continuous if $x_n \in X, x_n \to x$ implies $Tx_n \to Tx$.

Lemma 2.14. [21] Let (X, d) be a cone metric space, and $T : (X, d) \to (X, d)$ be any map. Then, T is continuous if and only if T is sequentially continuous.

Lemma 2.15. [14] Let E be a real Banach space with a cone P. Then

(i) If $a \ll b$ and $b \ll c$, then $a \ll c$.

(ii) If $a \preceq b$ and $b \ll c$, then $a \ll c$.

Lemma 2.16. [5] Let E be a real Banach space with a cone P. Then one has the following.

(i) If $\theta \ll c$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.

(ii) If $a_n,\ b_n$ are sequences in E such that $a_n \to a,\ b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.

Proposition 2.17. [6] If E is a real Banach space with a cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$ then $a = \theta$.

Definition 2.18. [20] Let (X, d) be a metric space and $F : X \times X \to X$ be a given mapping. Let M be a nonempty subset of X^4. We say that M is an F-invariant subset of X^4 if and only if for all $x, y, z, w \in X$ we have

(i) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$ and

(ii) $(x, y, z, w) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

3. Main Results

In this section we always suppose that E is a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P where $int(P) \neq \emptyset$. Also, we mean by φ the φ-map and by ψ the ψ-map, unless otherwise stated.

Definition 3.1. Let (X, d) be a cone metric space and $F : X \times X \to X$, $g : X \to X$ be given mappings. A nonempty subset M of X^4 is called strongly (F, g)-invariant if and only if for all $x, y, z, w \in X$ we have

(i) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$ and

(ii) $(gx, gy, gz, gw) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$ and $(F(x, y), F(y, x), gz, gw) \in M$.

We observe that the set $M = X^4$ is trivially strongly (F, g)-invariant.
The following examples illustrate that a strongly (F, g)-invariant set need not be an F-invariant set.

Example 3.2. Let $X = \mathbb{R}$, and $F : X \times X \to X$, $g : X \to X$ be defined as $F(x, y) = 3 - x$, $gx = \frac{x}{2}$. Also let $M = \{(a, b, c, d) : b = c = 1; a, b, c, d \in X\}$. Then M is not an F-invariant set as $(0, 1, 1, 0) \in M$ but $(F(0, 1), F(1, 0), F(1, 0), F(0, 1)) = (3, 2, 2, 3) \notin M$. We can easily verify that M is a strongly (F, g)-invariant set.

Example 3.3. Let $X = \mathbb{R}$ and $F : X \times X \to X$ be defined by $F(x, y) = 1 - x^2$. Let $g : X \to X$ be given by $gx = 1 + x$. Then $M = \{(x, y, z, w) \in X^4 : y = z = 0\}$ is not F-invariant as $(1, 0, 0, 1) \in M$ but $(F(1, 0), F(0, 1), F(0, 1), F(1, 0)) = (0, 1, 1, 0) \notin M$. It is easy to see that M is strongly (F, g)-invariant.

Theorem 3.4. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let M be a strongly (F, g)-invariant subset of X^4 such that*

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \preceq \varphi(\psi(d(gx, gu) + d(gy, gv))) \tag{3.1}$$

for all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. If there exist x_0, $y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then F and g have a coupled coincidence point.

Proof . Let x_0, $y_0 \in X$ be such that $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$. We choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$ which is possible since $F(X \times X) \subseteq g(X)$. Continuing this process one can construct sequences (x_n) and (y_n) in X that satisfy $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for all $n \geq 0$. We shall show that

$$(gx_{n+1}, gy_{n+1}, gx_n, gy_n) \in M \tag{3.2}$$

for all $n \geq 0$.

We shall use the mathematical induction. For $n = 0$, (3.2) follows by the choice of x_0 and y_0. Suppose now (3.2) holds for $n = k$, $k \geq 0$. Then $(gx_{k+1}, gy_{k+1}, gx_k, gy_k) \in M$. By using strongly (F, g)-invariance of M, we have

$$(F(x_{k+1}, y_{k+1}), F(y_{k+1}, x_{k+1}), F(x_k, y_k), F(y_k, x_k)) \in M,$$

which implies that, $(gx_{k+2}, gy_{k+2}, gx_{k+1}, gy_{k+1}) \in M$. Thus (3.2) follows for $k + 1$. Hence, by the mathematical induction we conclude that (3.2) holds for $n \geq 0$.
Again, we shall show that

$$(gx_{r+1}, gy_{r+1}, gx_n, gy_n) \in M \tag{3.3}$$

for all $r \geq n$.
Obviously, (3.3) holds for $r = n$. Let us assume that (3.3) holds for some $r = k$, $k \geq n$. Then $(gx_{k+1}, gy_{k+1}, gx_n, gy_n) \in M$ and so by strongly (F, g)-invariance of M, we have

$$(F(x_{k+1}, y_{k+1}), F(y_{k+1}, x_{k+1}), gx_n, gy_n) \in M,$$

which implies that, $(gx_{k+2}, gy_{k+2}, gx_n, gy_n) \in M$. Thus (3.3) follows for $k + 1$. Hence, by the mathematical induction we conclude that (3.3) holds for $r \geq n$.

Now for all $n \in \mathbb{N}$,

$$\begin{aligned}
\psi(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})) &= \psi\begin{pmatrix} d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ +d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \end{pmatrix} \\
&\preceq \varphi(\psi(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n))) \\
&\preceq \varphi^2(\psi(d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1}))) \\
&\;\;\vdots \\
&\preceq \varphi^n(\psi(d(gx_0, gx_1) + d(gy_0, gy_1))).
\end{aligned}$$

Let $\epsilon \in int(P)$, then by (φ_2), $\epsilon_0 = \epsilon - \varphi(\epsilon) \in int(P)$. By (φ_3),

$$\lim_{n\to\infty} \varphi^n(\psi(d(gx_0, gx_1) + d(gy_0, gy_1))) = \theta.$$

So, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n_0$,

$$\psi(d(gx_m, gx_{m+1}) + d(gy_m, gy_{m+1})) \ll \epsilon - \varphi(\epsilon).$$

We show that

$$\psi(d(gx_m, gx_{n+1}) + d(gy_m, gy_{n+1})) \ll \epsilon, \tag{3.4}$$

for a fixed $m \geq n_0$ and $n \geq m$.
Clearly, this holds for $n = m$. We now suppose that (3.4) holds for some $n \geq m$. Then by using (ψ_3), conditions (3.3) and (3.1), we obtain

$$\begin{aligned}
\psi(d(gx_m, gx_{n+2}) + d(gy_m, gy_{n+2})) &\preceq \psi\begin{pmatrix} d(gx_m, gx_{m+1}) + d(gx_{m+1}, gx_{n+2}) \\ \\ +d(gy_m, gy_{m+1}) + d(gy_{m+1}, gy_{n+2}) \end{pmatrix} \\
&\preceq \psi(d(gx_m, gx_{m+1}) + d(gy_m, gy_{m+1})) \\
&\quad +\psi(d(gx_{m+1}, gx_{n+2}) + d(gy_{m+1}, gy_{n+2})) \\
&\preceq \psi(d(gx_m, gx_{m+1}) + d(gy_m, gy_{m+1})) \\
&\quad +\varphi(\psi(d(gx_m, gx_{n+1}) + d(gy_m, gy_{n+1}))) \\
&\ll \epsilon - \varphi(\epsilon) + \varphi(\epsilon) = \epsilon.
\end{aligned}$$

Therefore, by induction (3.4) holds.
Since ψ is nondecreasing, it follows from (3.4) that

$$\psi(d(gx_m, gx_{n+1})) \preceq \psi(d(gx_m, gx_{n+1}) + d(gy_m, gy_{n+1})) \ll \epsilon$$

for a fixed $m \geq n_0$ and $n \geq m$.
Similarly,

$$\psi(d(gy_m, gy_{n+1})) \ll \epsilon$$

for a fixed $m \geq n_0$ and $n \geq m$.
Therefore, by using (ψ_2) we deduce that (gx_n) and (gy_n) are Cauchy sequences in X. Since X is complete, there exist x^*, $y^* \in X$ such that $gx_n \to x^*$ and $gy_n \to y^*$ as $n \to \infty$. By continuity of g we get $\lim_{n\to\infty} ggx_n = gx^*$ and $\lim_{n\to\infty} ggy_n = gy^*$. Commutativity of F and g now implies that

$$ggx_n = g(F(x_{n-1}, y_{n-1})) = F(gx_{n-1}, gy_{n-1}),\ for\ all\ n \in \mathbb{N}$$

and

$$ggy_n = g(F(y_{n-1}, x_{n-1}) = F(gy_{n-1}, gx_{n-1}),\ for\ all\ n \in \mathbb{N}.$$

Since F is continuous,

$$\begin{aligned}
gx^* = \lim_{n\to\infty} ggx_n &= \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) \\
&= F(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}) \\
&= F(x^*, y^*)
\end{aligned}$$

and

$$\begin{aligned} gy^* = \lim_{n\to\infty} ggy_n &= \lim_{n\to\infty} F(gy_{n-1}, gx_{n-1}) \\ &= F(\lim_{n\to\infty} gy_{n-1}, \lim_{n\to\infty} gx_{n-1}) \\ &= F(y^*, x^*). \end{aligned}$$

Thus, F and g have a coupled coincidence point. □

Taking $\psi = I$, the identity map in Theorem 3.4, we have the following Corollary.

Corollary 3.5. Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let M be a strongly (F, g)-invariant subset of X^4 such that

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \preceq \varphi(d(gx, gu) + d(gy, gv))$$

for all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. If there exist x_0, $y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then F and g have a coupled coincidence point.

Corollary 3.6. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let M be a strongly (F, g)-invariant subset of X^4 such that*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \preceq k(d(gx, gu) + d(gy, gv))$$

for some $k \in [0, 1)$ and all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. If there exist x_0, $y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then F and g have a coupled coincidence point.

Proof . The proof can be obtained from Theorem 3.4 by taking $\psi = I$, the identity map and $\varphi(x) = kx$, where $k \in [0, 1)$ is a constant. □

Corollary 3.7. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let M be a strongly (F, g)-invariant subset of X^4 such that*

$$d(F(x, y), F(u, v)) \preceq ad(gx, gu) + bd(gy, gv) \tag{3.5}$$

for some a, $b \in [0, 1)$ with $a+b < 1$ and all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$.
If there exist x_0, $y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then F and g have a coupled coincidence point.

Proof . Let $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. Using (3.5), we have

$$d(F(x, y), F(u, v)) \preceq ad(gx, gu) + bd(gy, gv)$$

and

$$d(F(y, x), F(v, u)) \preceq ad(gy, gv) + bd(gx, gu).$$

Therefore,

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \preceq (a + b)(d(gx, gu) + d(gy, gv)),$$

where $a + b < 1$. The result follows from Corollary 3.6. □

Theorem 3.8. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is a complete subspace of X. Let M be a strongly (F, g)-invariant subset of X^4 such that*

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \preceq \varphi(\psi(d(gx, gu) + d(gy, gv)))$$

for all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. Suppose $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x$, $y_n \to y$ implies $(x, y, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. If there exist x_0, $y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then F and g have a coupled coincidence point.

Proof . Consider Cauchy sequences (gx_n) and (gy_n) as in the proof of Theorem 3.4. Since $(g(X), d)$ is complete, there exist $x^*, y^* \in X$ such that $gx_n \to gx^*$ and $gy_n \to gy^*$. It is to be noted that $(gx_{n+1}, gy_{n+1}, gx_n, gy_n) \in M$ for all $n \geq 0$ and so by the given condition $(gx^*, gy^*, gx_n, gy_n) \in M$ for all $n \geq 0$.

By (ψ_2), for $\theta \ll c$, one can choose a natural number n_0 such that $\psi(d(gx_n, gx^*)) \ll \frac{c}{4}$ and $\psi(d(gy_n, gy^*)) \ll \frac{c}{4}$ for all $n \geq n_0$.

Then,

$$\begin{aligned}
\psi\begin{pmatrix} d(F(x^*, y^*), gx^*) \\ +d(F(y^*, x^*), gy^*) \end{pmatrix} &\preceq \psi\begin{pmatrix} d(F(x^*, y^*), gx_{n+1}) + d(gx_{n+1}, gx^*) \\ +d(F(y^*, x^*), gy_{n+1}) + d(gy_{n+1}, gy^*) \end{pmatrix} \\
&\preceq \psi(d(gx_{n+1}, gx^*) + d(gy_{n+1}, gy^*)) \\
&\quad +\psi\begin{pmatrix} d(F(x^*, y^*), F(x_n, y_n)) \\ +d(F(y^*, x^*), F(y_n, x_n)) \end{pmatrix} \\
&\preceq \psi(d(gx_{n+1}, gx^*)) + \psi(d(gy_{n+1}, gy^*)) \\
&\quad +\varphi(\psi(d(gx_n, gx^*) + d(gy_n, gy^*))) \\
&\prec \psi(d(gx_{n+1}, gx^*)) + \psi(d(gy_{n+1}, gy^*)) \\
&\quad +\psi(d(gx_n, gx^*) + d(gy_n, gy^*)) \\
&\preceq \psi(d(gx_{n+1}, gx^*)) + \psi(d(gy_{n+1}, gy^*)) \\
&\quad +\psi(d(gx_n, gx^*)) + \psi(d(gy_n, gy^*)) \\
&\ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c.
\end{aligned}$$

So, $\frac{c}{i} - \psi(d(F(x^*, y^*), gx^*) + d(F(y^*, x^*), gy^*)) \in P$, for all $i \geq 1$. Since $\frac{c}{i} \to \theta$ as $i \to \infty$ and P is closed, $-\psi(d(F(x^*, y^*), gx^*) + d(F(y^*, x^*), gy^*)) \in P$. But $P \cap (-P) = \theta$ gives that

$$\psi(d(F(x^*, y^*), gx^*) + d(F(y^*, x^*), gy^*)) = \theta.$$

By (ψ_1), we get

$$d(F(x^*, y^*), gx^*) + d(F(y^*, x^*), gy^*) = \theta.$$

This shows that $d(F(x^*, y^*), gx^*) = d(F(y^*, x^*), gy^*) = \theta$ and so $F(x^*, y^*) = gx^*$, $F(y^*, x^*) = gy^*$. Thus, F and g have a coupled coincidence point. □

If we let ψ be the identity map in Theorem 3.8, then we have the following Corollary.

Corollary 3.9. Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is a complete subspace of X. Let M be a strongly (F, g)-invariant subset of X^4 such that

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \preceq \varphi(d(gx,gu) + d(gy,gv))$$

for all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. Suppose $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x,\ y_n \to y$ implies $(x, y, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. If there exist $x_0,\ y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then F and g have a coupled coincidence point.

Corollary 3.10. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is a complete subspace of X. Let M be a strongly (F, g)-invariant subset of X^4 such that*

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \preceq k(d(gx,gu) + d(gy,gv))$$

for some $k \in [0,1)$ and all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. Suppose $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x,\ y_n \to y$ implies $(x, y, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. If there exist $x_0,\ y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then F and g have a coupled coincidence point.

Proof . The proof can be obtained from Theorem 3.8 by taking $\psi = I$, the identity map and $\varphi(x) = kx$, where $k \in [0,1)$ is a constant. □

Corollary 3.11. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is a complete subspace of X. Let M be a strongly (F, g)-invariant subset of X^4 such that*

$$d(F(x,y), F(u,v)) \preceq ad(gx,gu) + bd(gy,gv)$$

for some $a,\ b \in [0,1)$ with $a+b < 1$ and all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. Suppose $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x,\ y_n \to y$ implies $(x, y, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. If there exist $x_0,\ y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then F and g have a coupled coincidence point.

Proof . The proof follows from Theorem 3.8 by an argument similar to that used in Corollary 3.7. □

Theorem 3.12. *In addition to hypothesis of either Theorem 3.4 or Theorem 3.8, suppose that any two elements x and y of X satisfy $(gx, gy, gy, gx) \in M$ or $(gy, gx, gx, gy) \in M$ and g is one-one. Then F and g have a coupled coincidence point of the form (x^*, x^*) for some $x^* \in X$.*

Proof . We first note that the set of coupled coincidence points of F and g is nonempty. We will show that if (x^*, y^*) is a coupled coincidence point of F and g, then $x^* = y^*$. Suppose that $d(gx^*, gy^*) \neq \theta$. Then, by using (φ_1) we have

$$\begin{aligned}
\psi(d(gx^*, gy^*) + d(gy^*, gx^*)) &= \psi(d(F(x^*, y^*), F(y^*, x^*)) + d(F(y^*, x^*), F(x^*, y^*))) \\
&\preceq \varphi(\psi(d(gx^*, gy^*) + d(gy^*, gx^*))) \\
&\prec \psi(d(gx^*, gy^*) + d(gy^*, gx^*)),
\end{aligned}$$

a contradiction. Therefore, $d(gx^*, gy^*) = \theta$ which gives that $gx^* = gy^*$. Since g is one-one, it follows that $x^* = y^*$. □

Now, we present some examples to support our results.

Example 3.13. Let $E = \mathbb{R}^2$, the Euclidean plane and $P = \{(x,x) \in \mathbb{R}^2 : x \geq 0\}$ a cone in E. Let $X = \mathbb{R}$ with usual order and define $d : X \times X \to E$ by

$$d(x,y) = (\mid x - y \mid, \mid x - y \mid)$$

for all $x, y \in X$. Then (X, d) is a partially ordered complete cone metric space. Consider $F(x,y) = -\frac{x}{9}$ for all $x, y \in X$ and $gx = \frac{x}{3}$ for all $x \in X$. Then $F(X \times X) \subseteq g(X) = X$. Further F and g are continuous and commuting.
Let $\psi, \varphi : P \to P$ be defined by $\psi(x,x) = (\frac{x}{4}, \frac{x}{4})$ and $\varphi(x,x) = (\frac{3x}{5}, \frac{3x}{5})$.

We show that for all $(x,y,u,v) \in X^4 = M$,

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \preceq \varphi(\psi(d(gx,gu) + d(gy,gv))).$$

Now, we have
$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)))$

$$\begin{aligned} &= \psi\left(d\left(\frac{-x}{9}, \frac{-u}{9}\right) + d\left(\frac{-y}{9}, \frac{-v}{9}\right)\right) \\ &= \psi\left(\left(\frac{\mid x-u \mid}{9}, \frac{\mid x-u \mid}{9}\right) + \left(\frac{\mid y-v \mid}{9}, \frac{\mid y-v \mid}{9}\right)\right) \\ &= \left(\frac{\mid x-u \mid}{36} + \frac{\mid y-v \mid}{36}, \frac{\mid x-u \mid}{36} + \frac{\mid y-v \mid}{36}\right). \end{aligned} \tag{3.6}$$

Again,

$$\begin{aligned} \varphi(\psi(d(gx,gu) + d(gy,gv))) &= \varphi\left(\psi\left(d\left(\tfrac{x}{3}, \tfrac{u}{3}\right) + d\left(\tfrac{y}{3}, \tfrac{v}{3}\right)\right)\right) \\ &= \varphi\left(\psi\left(\left(\frac{\mid x-u \mid}{3}, \frac{\mid x-u \mid}{3}\right) + \left(\frac{\mid y-v \mid}{3}, \frac{\mid y-v \mid}{3}\right)\right)\right) \\ &= \left(\frac{\mid x-u \mid}{20} + \frac{\mid y-v \mid}{20}, \frac{\mid x-u \mid}{20} + \frac{\mid y-v \mid}{20}\right). \end{aligned} \tag{3.7}$$

It follows from conditions (3.6) and (3.7) that

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \preceq \varphi(\psi(d(gx,gu) + d(gy,gv)))$$

for all $(x,y,u,v) \in X^4 = M$. Thus, we have all the conditions of Theorem 3.4. Moreover, F and g have a coupled coincidence point at $(0,0)$.

Example 3.14. Let $E = \mathbb{R}^2$, the Euclidean plane a and $P = \{(x,x) \in \mathbb{R}^2 : x \geq 0\}$ a cone in E. Let $X = \mathbb{R}$ with usual order and define $d : X \times X \to E$ by

$$d(x,y) = (\mid x - y \mid, \mid x - y \mid)$$

for all $x, y \in X$. Then (X, d) is a partially ordered complete cone metric space. Define $F : X \times X \to X$ as follows:

$$F(x,y) = \begin{cases} \frac{x-y}{6}, & if \ \ x \geq y \\ \\ 0, & if \ \ x < y, \end{cases}$$

for all $x, y \in X$ and $g : X \to X$ with $gx = 1 - \frac{x}{2}$ for all $x \in X$. Then $F(X \times X) \subseteq g(X) = X$. Let $\psi, \varphi : P \to P$ be defined by $\psi(x, x) = (\frac{x}{2}, \frac{x}{2})$ and $\varphi(x, x) = (\frac{3x}{4}, \frac{3x}{4})$.

We show that for all $(x, y, u, v) \in X^4 = M$,

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \preceq \varphi(\psi(d(gx, gu) + d(gy, gv))).$$

Now, we have

Case-I $(y > x$ and $v > u)$. Then

$$\begin{aligned}
&\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \\
&= \psi\left(d(0,0) + d\left(\frac{y-x}{6}, \frac{v-u}{6}\right)\right) \\
&= \psi\left(\frac{| y-x-v+u |}{6}, \frac{| y-x-v+u |}{6}\right) \\
&= \left(\frac{| y-x-v+u |}{12}, \frac{| y-x-v+u |}{12}\right) \\
&\preceq \left(\frac{| x-u |}{12} + \frac{| y-v |}{12}, \frac{| x-u |}{12} + \frac{| y-v |}{12}\right). \qquad (3.8)
\end{aligned}$$

Again,

$$\begin{aligned}
\varphi(\psi(d(gx, gu) + d(gy, gv))) &= \varphi\left(\psi\left(d\left(1 - \tfrac{x}{2}, 1 - \tfrac{u}{2}\right) + d\left(1 - \tfrac{y}{2}, 1 - \tfrac{v}{2}\right)\right)\right) \\
&= \varphi\left(\psi\left(\left(\frac{| x-u |}{2}, \frac{| x-u |}{2}\right) + \left(\frac{| y-v |}{2}, \frac{| y-v |}{2}\right)\right)\right) \\
&= \left(3\frac{| x-u |}{16} + 3\frac{| y-v |}{16}, 3\frac{| x-u |}{16} + 3\frac{| y-v |}{16}\right). \qquad (3.9)
\end{aligned}$$

It follows from conditions (3.8) and (3.9) that

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \preceq \varphi(\psi(d(gx, gu) + d(gy, gv))).$$

Case-II $(y > x$ and $u \geq v)$. Then

$$\begin{aligned}
&\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \\
&= \psi\left(d\left(0, \frac{u-v}{6}\right) + d\left(\frac{y-x}{6}, 0\right)\right) \\
&= \psi\left(\left(\frac{u-v}{6}, \frac{u-v}{6}\right) + \left(\frac{y-x}{6}, \frac{y-x}{6}\right)\right) \\
&= \left(\frac{u-v+y-x}{12}, \frac{u-v+y-x}{12}\right) \\
&\preceq \left(\frac{| x-u |}{12} + \frac{| y-v |}{12}, \frac{| x-u |}{12} + \frac{| y-v |}{12}\right) \\
&\prec \left(3\frac{| x-u |}{16} + 3\frac{| y-v |}{16}, 3\frac{| x-u |}{16} + 3\frac{| y-v |}{16}\right) \\
&= \varphi(\psi(d(gx, gu) + d(gy, gv))).
\end{aligned}$$

Case-III $(x \geq y$ and $u \geq v)$. Then

$$\begin{aligned}
&\psi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))) \\
&= \psi\left(d\left(\frac{x-y}{6},\frac{u-v}{6}\right)+d(0,0)\right) \\
&= \left(\frac{\mid x-y-u+v \mid}{12},\frac{\mid x-y-u+v \mid}{12}\right) \\
&\preceq \left(\frac{\mid x-u \mid}{12}+\frac{\mid y-v \mid}{12},\frac{\mid x-u \mid}{12}+\frac{\mid y-v \mid}{12}\right) \\
&\preceq \left(3\frac{\mid x-u \mid}{16}+3\frac{\mid y-v \mid}{16},3\frac{\mid x-u \mid}{16}+3\frac{\mid y-v \mid}{16}\right) \\
&= \varphi(\psi(d(gx,gu)+d(gy,gv))).
\end{aligned}$$

The case $x \geq y$ and $v > u$ is similar to Case-II. It is easy to see that all other conditions of Theorem 3.8 are satisfied for $M = X^4$. Thus, we have all the conditions of Theorem 3.8 and $(2,2)$ is a coupled coincidence point of F and g.

Remark 3.15. It is worth mentioning that in above two examples F does not satisfy mixed g-monotone property.

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