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Approximation of a generalized Euler-Lagrange type additive mapping on Lie C*-algebras

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Abstract

Using fixed point method, we prove some new stability results for Lie (α,β,γ)-derivations and Lie C*-algebra homomorphisms on Lie C*-algebras associated with the Euler-Lagrange type additive functional equation

$$\sum_{j=1}^{n} f\Big(-r_jx_j+\sum_{1\le i\le n, i\ne j} r_ix_i\Big)+2\sum_{i=1}^{n} r_if(x_i)=nf\Big(\sum_{i=1}^{n} r_ix_i\Big)$$

where $r_1,\ldots,r_n\in\mathbb{R}$ are given and $r_i,r_j\ne 0$ for some $1\le i<j\le n$.

Keywords: Fixed point theorem; Lie (α,β,γ)-derivation; Lie C*-algebra homomorphisms; generalized Hyers-Ulam stability.
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1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: When is it true that a function, which approximately satisfies a functional equation $\mathfrak{E}$, must be close to an exact solution of $\mathfrak{E}$?

If the problem admits a solution, we say that the equation $\mathfrak{E}$ is stable. Such a problem was formulated by Ulam [18] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [5]. It gave rise to the stability theory for functional equations. Hyers' theorem was

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generalized by Aoki [1] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. Găvruţă [3] generalized the Rassias' result by using a general control function in the spirit of Rassias' approach. Later, the stability of several functional equations has been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6, 7, 8, 16, 17] and references therein).

Now, we deal with the following additive functional equation of Euler-Lagrange type [11]:

$$\sum_{j=1}^{n} f\bigg(-r_j x_j + \sum_{1\le i\le n, i\ne j} r_i x_i\bigg) + 2\sum_{i=1}^{n} r_i f(x_i) = nf\bigg(\sum_{i=1}^{n} r_i x_i\bigg) \tag{1.1}$$

where $r_1, \ldots, r_n \in \mathbb{R}$. Every solution of the functional equation (1.1) is said to be a generalized Euler-Lagrange type additive mapping. Najati and Park [11] investigated the generalized Hyers-Ulam stability of the functional equation (1.1) in Banach modules over a C*-algebra. They also applied their results to investigate C*-algebra homomorphisms in unital C*-algebras. In [9], Kenary et al. proved the generalized Hyers-Ulam stability of the functional equation (1.1) in non-Archimedean Banach spaces by using fixed point method.

In this paper, using some ideas from [4, 10], we apply a fixed point theorem to investigate the stability by using contractively subhomogeneous and expansively superhomogeneous functions for Lie (α, β, γ)-derivations and Lie C*-algebra homomorphisms on Lie C*-algebras associated with the Euler-Lagrange type additive functional equation (1.1).

Next, following [4, 10], we recall some definitions and preliminary results to be used in this paper. A C*-algebra $\mathscr{A}$ endowed with the Lie product $[x, y] = \frac{xy-yx}{2}$ on $\mathscr{A}$, is called a Lie C*-algebra [13, 14]. Let $\mathscr{A}$ and $\mathscr{B}$ be Lie C*-algebras. A $\mathbb{C}$-linear mapping $\mathcal{D} : \mathscr{A} \to \mathscr{A}$ is called a Lie derivation of $\mathscr{A}$ if $\mathcal{D} : \mathscr{A} \to \mathscr{A}$ satisfies

$$\mathcal{D}([x, y]) = [\mathcal{D}(x), y] + [x, \mathcal{D}(y)]$$

for all $x, y \in \mathscr{A}$ [13, 14]. Following [12], a $\mathbb{C}$-linear mapping $\mathscr{D} : \mathscr{A} \to \mathscr{A}$ is called a Lie (α, β, γ)-derivation of $\mathscr{A}$ if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\alpha\mathscr{D}([x, y]) = \beta[\mathscr{D}(x), y] + \gamma[x, \mathscr{D}(y)]$$

for all $x, y \in \mathscr{A}$. A $\mathbb{C}$-linear mapping $H : \mathscr{A} \to \mathscr{B}$ ia called a Lie C*-algebra homomorphism if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in \mathscr{A}$ [2].

The following fixed point theorem will play an important role in proving our main theorems.

Theorem 1.1. (Banach). Let (X, d) be a complete metric space and consider a mapping $\mathcal{T} : X \to X$ is a strictly contractive mapping, that is

$$d(\mathcal{T}x, \mathcal{T}y) \le Ld(x, y)$$

for all $x, y \in X$ and for some (Lipschitz constant) $0 < L < 1$. Then there exists a unique $a \in X$ such that $\mathcal{T}a = a$. Moreover, for each $x \in X$,

$$\lim_{n\to\infty} \mathcal{T}^n x = a$$

and in fact for each $x \in X$,

$$d(x, a) \le \frac{1}{1-L} d(x, \mathcal{T}x).$$

Let k be a fixed positive integer. We recall that a function $\eta : A \to B$ having a domain A and a codomain $(B, \leq)$ that are both closed under addition is called:
(i) a *homogeneous function of degree* k if $\eta(\lambda x) = \lambda^k \eta(x)$ (for the case $k = 1$, the corresponding function is simply called homogeneous),
(ii) a *contractively subhomogeneous function of degree* k if there exists a constant L with $0 < L < 1$ such that $\eta(\lambda x) \leq L\,\lambda^k \eta(x)$,
(iii) a *expansively superhomogeneous function of degree* k if there exists a constant L with $0 < L < 1$ such that $\eta(\lambda x) \leq \frac{\lambda^k}{L}\eta(x)$ for all $x, y \in A$ and all positive integer $\lambda > 1$.

Remark 1.2. (cf. [4]) If η is contractively subadditive and expansively superadditive separately, then η is contractively subhomogeneous ($\ell = 1$) and expansively superhomogeneous ($\ell = -1$), respectively, and therefore

$$\eta(\lambda^{\ell j} x) \leq (\lambda^{\ell} L)^j \eta(x), \quad j \in \mathbb{N}.$$

Also, if there exists a constant L satisfying $0 < L < 1$ such that a function $\eta : A^n = \overbrace{A \times \cdots \times A}^{n\ times} \to B$ satisfies

$$\eta(x_1, \ldots, \overbrace{\lambda^{\ell} x}^{i^{\rm th}}, \ldots, x_n) \leq \lambda^{\ell} L\, \eta(x_1, \ldots, \overbrace{x}^{i^{\rm th}}, \ldots, x_n)$$

for all $x, x_j \in A$ $(1 \leq j \neq i \leq n)$ and all positive integers λ, then we say that η is n-contractively subhomogeneous if $\ell = 1$, and η is n-expansively superhomogeneous if $\ell = -1$.

Remark 1.3. (cf. [4]) If η is n-contractively subadditive and n-expansively superadditive separately, then η is contractively subhomogeneous of degree n and expansively superhomogeneous of degree n, respectively.

2. Main results

Throughout this section, we will assume that X and Y are linear spaces, $\mathscr{A}$ and $\mathscr{B}$ are Lie C*-algebras and $n_0 \in \mathbb{N}$ is a positive integer. Further, we assume that $\mathbb{T}^1_{1/n_0} := \{e^{i\theta};\ 0 \leq \theta \leq 2\pi/n_0\}$. For convenience, we use the following abbreviations for a given mapping $f : \mathscr{A} \to \mathscr{B}$:

$$\mathscr{D}_{\mu, r_1, \ldots, r_n, f}(x_1, \ldots, x_n) := \sum_{j=1}^{n} f\Big(-\mu r_j x_j + \sum_{1 \leq i \leq n, i \neq j} \mu r_i x_i\Big) + 2\sum_{i=1}^{n} \mu r_i f(x_i) - n f\Big(\sum_{i=1}^{n} \mu r_i x_i\Big)$$

and

$$\mathscr{D}_{\alpha, \beta, \gamma, f}(x, y) := \alpha\, f([x, y]) - \beta\,[f(x), y] - \gamma\,[x, f(y)]$$

for all $x_1, \ldots, x_n, x, y \in X$, all $\mu \in \mathbb{T}^1_{1/n_0}$, $r_1, \ldots, r_n \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{C}$.

Before proceeding to the proof of the main results, we first introduce the following lemmas which will be used in this paper.

Lemma 2.1. (cf. [11]). Let X and Y be linear spaces and let $r_1, \dots, r_n$ be real numbers with $\sum_{k=1}^{n} r_k \neq 0$ and $r_i, r_j \neq 0$ for some $1 \leq i < j \leq n$. Assume that a mapping $f : X \to Y$ satisfies the functional equation (1.1) for all $x_1, \dots, x_n \in X$. Then f is an additive mapping satisfying $f(r_k x) = r_k f(x)$ for all $x \in X$ and $r_k \in \mathbb{R}$ $(1 \leq k \leq n)$.

Using the same method of the proof of Lemma 2.1, we have an alternative result of Lemma 2.1 when $\sum_{k=1}^{n} r_k = 0$.

Lemma 2.2. (cf. [11]). Let X and Y be linear spaces and let $r_1, \dots, r_n$ be real numbers with $r_i, r_j \neq 0$ for some $1 \leq i < j \leq n$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation (1.1) for all $x_1, \dots, x_n \in X$. Then f is an additive mapping satisfying $f(r_k x) = r_k f(x)$ for all $x \in X$ and $r_k \in \mathbb{R}$ $(1 \leq k \leq n)$.

Lemma 2.3. (cf. [2, 4]). Let $f : \mathscr{A} \to \mathscr{A}$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x \in \mathscr{A}$. Then the mapping f is $\mathbb{C}$-linear.

Remark 2.4. Throughout this paper, let $r_1, \dots, r_n$ be real numbers such that $r_i, r_j \neq 0$ for fixed $1 \leq i < j \leq n$ and $\varphi_{ij}(x, y) := \varphi(0, \dots, 0, \underbrace{x}_{i^{\text{th}}}, 0, \dots, 0, \underbrace{y}_{j^{\text{th}}}, 0, \dots, 0)$ for all $x, y \in \mathscr{A}$ and all $1 \leq i < j \leq n$.

In the following theorem, we prove the generalized Hyers-Ulam stability of the functional equation (1.1) on Lie C*-algebras by using contractively subhomogeneous and expansively superhomogeneous functions.

Theorem 2.5. *Assume that there exist a contractively subhomogeneous mapping $\varphi : \mathscr{A}^n \to [0, \infty)$ and a 2-contractively subhomogeneous mapping $\psi : \mathscr{A}^2 \to [0, \infty)$ with a constant $0 < L < 1$ such that a mapping $f : \mathscr{A} \to \mathscr{A}$ with $f(0) = 0$ satisfies*

$$\|\mathscr{D}_{\mu, r_1, \dots, r_n, f}(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n) \tag{2.1}$$

and

$$\|\mathscr{D}_{\alpha, \beta, \gamma, f}(x, y)\| \leq \psi(x, y) \tag{2.2}$$

for all $x_1, \dots, x_n, x, y \in \mathscr{A}$, all $\mu \in \mathbb{T}^1_{1/n_0}$ and some $\alpha, \beta, \gamma \in \mathbb{C}$. Then there exists a unique Lie (α, β, γ)-derivation $\mathfrak{L} : \mathscr{A} \to \mathscr{A}$ which satisfies the functional equation (1.1) and the inequality

$$\begin{aligned} \|f(x) - \mathfrak{L}(x)\| \leq \frac{1}{4(1-L)} \Big\{ & [\varphi_{ij}(\frac{x}{r_i}, \frac{x}{r_j}) + 2\varphi_{ij}(\frac{x}{2r_i}, -\frac{x}{2r_j})] \\ & + [\varphi_{ij}(\frac{x}{r_i}, 0) + 2\varphi_{ij}(\frac{x}{2r_i}, 0)] + [\varphi_{ij}(0, \frac{x}{r_j}) + 2\varphi_{ij}(0, -\frac{x}{2r_j})] \Big\} \end{aligned} \tag{2.3}$$

for all $x \in \mathscr{A}$.

Proof . Consider the set

$$\mathfrak{W} := \left\{ g : \mathscr{A} \to \mathscr{A}, \sup_{x \in \mathscr{A}} \frac{\|g(x) - f(x)\|}{\Phi(x)} < \infty \right\}$$

where

$$\Phi(x) := \frac{1}{2}\Big\{[\varphi_{ij}(\frac{x}{r_i}, \frac{x}{r_j}) + 2\varphi_{ij}(\frac{x}{2r_i}, -\frac{x}{2r_j})] + [\varphi_{ij}(\frac{x}{r_i}, 0) + 2\varphi_{ij}(\frac{x}{2r_i}, 0)] + [\varphi_{ij}(0, \frac{x}{r_j}) + 2\varphi_{ij}(0, -\frac{x}{2r_j})]\Big\}$$

for all $x \in \mathscr{A}$, and introduce the following metric on $\mathfrak{W}$:

$$d(g, h) = \sup_{x\in\mathscr{A}} \frac{\|g(x) - h(x)\|}{\Phi(x)}.$$

Then it is easy to see that $(\mathfrak{W}, d)$ is a complete metric space. Now we consider the mapping $\mathcal{J} : \mathfrak{W} \to \mathfrak{W}$ given by

$$\mathcal{J}g(x) := \frac{1}{2}g(2x), \quad \text{for all } g \in \mathfrak{W} \text{ and } x \in \mathscr{A}. \tag{2.4}$$

Let $g, h \in \mathfrak{W}$ and let $\rho \in \mathbb{R}_+$ be an arbitrary constant with $d(g, h) \leq \rho$. From the definition of d, we have

$$\frac{\|g(x) - h(x)\|}{\Phi(x)} \leq \rho$$

for all $x \in \mathscr{A}$. By the assumption and the last inequality, we get

$$\frac{\|\mathcal{J}g(x) - \mathcal{J}h(x)\|}{\Phi(x)} = \frac{\|g(2x) - h(2x)\|}{2\Phi(x)} \leq \frac{L\|g(2x) - h(2x)\|}{\Phi(2x)} \leq L\rho$$

for some $L < 1$ and for all $x \in \mathscr{A}$. Hence, it holds that $d(\mathcal{J}g, \mathcal{J}h) \leq L\rho$, that is, $d(\mathcal{J}g, \mathcal{J}h) \leq Ld(g, h)$ for all $g, h \in \mathfrak{W}$. This means that $\mathcal{J}$ is a strictly contractive self-mapping of $\mathfrak{W}$, with the Lipschitz constant L.

For each $1 \leq k \leq n$ with $k \neq i, j$, substituting $x_k = 0$ and $\mu = 1$ in the functional inequality (2.1), we obtain

$$\|f(-r_ix_i + r_jx_j) + f(r_ix_i - r_jx_j) - 2f(r_ix_i + r_jx_j) + 2r_if(x_i) + 2r_jf(x_j)\| \leq \varphi_{ij}(x_i, x_j) \tag{2.5}$$

for all $x_i, x_j \in \mathscr{A}$. Letting $x_i = 0$ in (2.5), we get

$$\|f(-r_jx_j) - f(r_jx_j) + 2r_jf(x_j)\| \leq \varphi_{ij}(0, x_j) \tag{2.6}$$

for all $x_j \in \mathscr{A}$. Similarly, letting $x_j = 0$ in (2.5), we get

$$\|f(-r_ix_i) - f(r_ix_i) + 2r_if(x_i)\| \leq \varphi_{ij}(x_i, 0) \tag{2.7}$$

for all $x_i \in \mathscr{A}$. It follows from (2.5), (2.6) and (2.7) that

$$\begin{aligned}&\|f(-r_ix_i + r_jx_j) + f(r_ix_i - r_jx_j) - 2f(r_ix_i + r_jx_j) + 2r_if(x_i) + 2r_jf(x_j)\\ &\quad -(f(-r_ix_i) - f(r_ix_i) + 2r_if(x_i)) - (f(-r_jx_j) - f(r_jx_j) + 2r_jf(x_j))\|\\ &\quad \leq \varphi_{ij}(x_i, x_j) + \varphi_{ij}(x_i, 0) + \varphi_{ij}(0, x_j)\end{aligned} \tag{2.8}$$

for all $x_i, x_j \in \mathscr{A}$. Replacing x_i and x_j by $\frac{x}{r_i}$ and $\frac{y}{r_j}$ in (2.8), we obtain

$$\begin{aligned}\|f(-x+y)+&f(x-y)-2f(x+y)+f(x)+f(y)-f(-x)-f(-y)\|\\&\le \varphi_{ij}(\frac{x}{r_i},\frac{y}{r_j})+\varphi_{ij}(\frac{x}{r_i},0)+\varphi_{ij}(0,\frac{y}{r_j})\end{aligned} \tag{2.9}$$

for all $x, y \in \mathscr{A}$. Putting $y = x$ in (2.9), we have

$$\|2f(x)-2f(-x)-2f(2x)\| \le \varphi_{ij}(\frac{x}{r_i},\frac{x}{r_j})+\varphi_{ij}(\frac{x}{r_i},0)+\varphi_{ij}(0,\frac{x}{r_j}) \tag{2.10}$$

for all $x \in \mathscr{A}$. Replacing x and y by $\frac{x}{2}$ and $-\frac{x}{2}$ in (2.9), respectively, we get

$$\|f(x)+f(-x)\| \le \varphi_{ij}(\frac{x}{2r_i},-\frac{x}{2r_j})+\varphi_{ij}(\frac{x}{2r_i},0)+\varphi_{ij}(0,-\frac{x}{2r_j}) \tag{2.11}$$

for all $x \in \mathscr{A}$. It follows from (2.10) and (2.11) that

$$\|f(2x)-2f(x)\| \le \Phi(x) \tag{2.12}$$

for all $x \in \mathscr{A}$. Thus

$$\frac{\|\frac{1}{2}f(2x)-f(x)\|}{\Phi(x)} \le \frac{1}{2} \tag{2.13}$$

for all $x \in \mathscr{A}$. Hence $d(\mathcal{J}f, f) \le \frac{1}{2}$. Due to Theorem 1.1, there exists a unique mapping $\mathfrak{L} \in \mathfrak{W}$ such that $\mathfrak{L}(2x) = 2\mathfrak{L}(x)$ for all $x \in \mathscr{A}$, i.e., $\mathfrak{L}$ is a unique fixed point of $\mathcal{J}$. Moreover,

$$\mathfrak{L}(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m} \tag{2.14}$$

for all $x \in \mathscr{A}$. Also

$$d(f,\mathfrak{L}) \le \frac{1}{1-L}d(f,\mathcal{J}f) \le \frac{1}{2(1-L)},$$

i.e., inequality (2.3) holds for all $x \in \mathscr{A}$.

It follows from (2.1), (2.14) and the contractively subhomogeneity of φ that

$$\begin{aligned}\|\mathscr{D}_{\mu,r_1,\ldots,r_n,\mathfrak{L}}(x_1,\ldots,x_n)\| &= \lim_{m\to\infty}\frac{1}{2^m}\|\mathscr{D}_{\mu,r_1,\ldots,r_n,\,f}(2^m x_1,\ldots,2^m x_n)\|\\&\le \lim_{m\to\infty}\frac{1}{2^m}\varphi(2^m x_1,\ldots,2^m x_n)\\&\le \lim_{m\to\infty} L^m\varphi(x_1,\ldots,x_n)=0\end{aligned}$$

holds for all $x_1, \ldots, x_n \in \mathscr{A}$ and $\mu \in \mathbb{T}^1_{1/n_0}$. So $\mathscr{D}_{\mu,r_1,\ldots,r_n,\mathfrak{L}}(x_1,\ldots,x_n) = 0$ for all $x_1, \ldots, x_n \in \mathscr{A}$ and $\mu \in \mathbb{T}^1_{1/n_0}$. If we put $\mu = 1$ in the last equality, then $\mathfrak{L}$ is additive by Lemma 2.2 and $\mathfrak{L}(r_k x) = r_k\,\mathfrak{L}(x)$ for all $x \in \mathscr{A}$ and for all $1 \le k \le n$. So letting $x_i = x$ and $x_k = 0$ for all $1 \le k \le n$, $k \ne i$ in the last equality, we obtain $\mathfrak{L}(\mu x) = \mu\mathfrak{L}(x)$. Now by using Lemma 2.3, we infer that the mapping $\mathfrak{L} \in \mathfrak{W}$ is $\mathbb{C}$-linear.

It follows from the 2-contractively subhomogeneity of ψ, (2.2) and (2.14) that

$$\begin{aligned}\|\mathscr{D}_{\alpha,\beta,\gamma,\mathfrak{L}}(x,y)\| &= \lim_{m\to\infty}\frac{1}{4^m}\|\mathscr{D}_{\alpha,\beta,\gamma,\,f}(2^m x,2^m y)\| \\ &\le \lim_{m\to\infty}\frac{1}{4^m}\psi(2^m x,2^m y) \le \lim_{m\to\infty}\frac{1}{4^m}(2L)^{2m}\psi(x,y)=0\end{aligned}$$

for all $x,y\in\mathscr{A}$ and some $\alpha,\beta,\gamma\in\mathbb{C}$. Then we have

$$\alpha\,\mathfrak{L}([x,y]) = \beta\,[\mathfrak{L}(x),y]+\gamma\,[x,\mathfrak{L}(y)] \tag{2.15}$$

for all $x,y\in\mathscr{A}$ and some $\alpha,\beta,\gamma\in\mathbb{C}$. Thus, the mapping $\mathfrak{L}\in\mathfrak{W}$ is a unique Lie (α,β,γ)-derivation on Lie C*-algebra $\mathscr{A}$ satisfying (2.3). This completes the proof of the theorem. □

Theorem 2.6. *Assume that there exist an expansively superhomogeneous mapping $\varphi:\mathscr{A}^n\to[0,\infty)$ and a 2-expansively superhomogeneous mapping $\psi:\mathscr{A}^2\to[0,\infty)$ with a constant $0<L<1$ such that a mapping $f:\mathscr{A}\to\mathscr{A}$ with $f(0)=0$ satisfies (2.1) and (2.2) for all $x_1,\dots,x_n,x,y\in\mathscr{A}$, all $\mu\in\mathbb{T}^1_{1/n_0}$ and some $\alpha,\beta,\gamma\in\mathbb{C}$. Then there exists a unique Lie (α,β,γ)-derivation $\mathfrak{L}:\mathscr{A}\to\mathscr{A}$ which satisfies the functional equation (1.1) and the inequality*

$$\begin{aligned}\|f(x)-\mathfrak{L}(x)\| \le \frac{L}{4(1-L)}\Big\{&[\varphi_{ij}(\frac{x}{r_i},\frac{x}{r_j})+2\varphi_{ij}(\frac{x}{2r_i},-\frac{x}{2r_j})] \\ &+[\varphi_{ij}(\frac{x}{r_i},0)+2\varphi_{ij}(\frac{x}{2r_i},0)]+[\varphi_{ij}(0,\frac{x}{r_j})+2\varphi_{ij}(0,-\frac{x}{2r_j})]\Big\}\end{aligned} \tag{2.16}$$

for all $x\in\mathscr{A}$.

Proof . Let $(\mathfrak{W},d)$ be a complete metric space defined in the proof of Theorems 2.5. Now, we consider the mapping $\mathcal{J}:\mathfrak{W}\to\mathfrak{W}$ defined by

$$\mathcal{J}g(x):=2g(\frac{x}{2}),\quad \text{for all } g\in\mathfrak{W} \text{ and } x\in\mathscr{A}. \tag{2.17}$$

One can show that $d(\mathcal{J}g,\mathcal{J}h)\le Ld(g,h)$ for all $g,h\in\mathfrak{W}$. By (2.12), we have

$$\|2f(\frac{x}{2})-f(x)\|\le\Phi(\frac{x}{2}) \tag{2.18}$$

for all $x\in\mathscr{A}$. Thus

$$\frac{\|2f(\frac{x}{2})-f(x)\|}{\Phi(x)}\le\frac{L}{2} \tag{2.19}$$

for all $x\in\mathscr{A}$. Hence $d(\mathcal{J}f,f)\le\frac{L}{2}$. Due to Theorem 1.1, there exists a unique mapping $\mathfrak{L}\in\mathfrak{W}$ such that $\mathfrak{L}(2x)=2\mathfrak{L}(x)$ for all $x\in\mathscr{A}$, i.e., $\mathfrak{L}$ is a unique fixed point of $\mathcal{J}$. Moreover,

$$\mathfrak{L}(x)=\lim_{m\to\infty}2^m f(\frac{x}{2^m}) \tag{2.20}$$

for all $x\in\mathscr{A}$. Also

$$d(f,\mathfrak{L})\le\frac{1}{1-L}d(f,\mathcal{J}f)\le\frac{L}{2(1-L)},$$

which implies that (2.16) holds for all $x\in\mathscr{A}$.

The remaining assertion goes through in a similar way to the corresponding part of Theorem 2.5. □

Theorem 2.7. *Assume that there exists a contractively subhomogeneous mapping $\phi : \mathscr{A}^{n+2} \to [0,\infty)$ with a constant $0 < L < 1$ such that a mapping $f : \mathscr{A} \to \mathscr{A}$ with $f(0) = 0$ satisfies*

$$\|\mathscr{D}_{\mu,r_1,\ldots,r_n,f}(x_1,\ldots,x_n) + \mathscr{D}_{\alpha,\beta,\gamma,f}(x,y)\| \le \phi(x_1,\ldots,x_n,x,y) \tag{2.21}$$

for all $x_1,\ldots,x_n,x,y \in \mathscr{A}$, all $\mu \in \mathbb{T}^1_{1/n_0}$ and some $\alpha,\beta,\gamma \in \mathbb{C}$. Then there exists a unique Lie (α,β,γ)-derivation $\mathfrak{L} : \mathscr{A} \to \mathscr{A}$ which satisfies the functional equation (1.1) and the inequality

$$\begin{aligned}\|f(x) - \mathfrak{L}(x)\| \le \frac{1}{4(1-L)}\Big\{&[\phi_{ij}(\frac{x}{r_i},\frac{x}{r_j}) + 2\phi_{ij}(\frac{x}{2r_i},-\frac{x}{2r_j})]\\ &+ [\phi_{ij}(\frac{x}{r_i},0) + 2\phi_{ij}(\frac{x}{2r_i},0)] + [\phi_{ij}(0,\frac{x}{r_j}) + 2\phi_{ij}(0,-\frac{x}{2r_j})]\Big\}\end{aligned} \tag{2.22}$$

for all $x \in \mathscr{A}$.

Proof . For each $1 \le k \le n$ with $k \ne i,j$, substituting $x_k = x = y = 0$ and $\mu = 1$ in the functional inequality (2.21), we obtain

$$\begin{aligned}&\|f(-r_ix_i + r_jx_j) + f(r_ix_i - r_jx_j) - 2f(r_ix_i + r_jx_j) + 2r_if(x_i) + 2r_jf(x_j)\|\\ &\le \phi(\overbrace{0,\ldots,0,\underbrace{x}_{i^{\text{th}}},0,\ldots,0,\underbrace{y}_{j^{\text{th}}},0,\ldots,0}^{n\ times},0,0)\end{aligned} \tag{2.23}$$

for all $x_i, x_j \in \mathscr{A}$. For convenience, let

$$\phi_{ij}(x,y) := \phi(\overbrace{0,\ldots,0,\underbrace{x}_{i^{\text{th}}},0,\ldots,0,\underbrace{y}_{j^{\text{th}}},0,\ldots,0}^{n\ times},0,0)$$

for all $x, y \in \mathscr{A}$ and all $1 \le i < j \le n$. By the same way as in the proof of Theorem 2.5, we obtain

$$\|f(2x) - 2f(x)\| \le \Psi(x) \tag{2.24}$$

for all $x \in \mathscr{A}$, where

$$\begin{aligned}\Psi(x) :=\frac{1}{2}\Big\{&[\phi_{ij}(\frac{x}{r_i},\frac{x}{r_j}) + 2\phi_{ij}(\frac{x}{2r_i},-\frac{x}{2r_j})]\\ &+ [\phi_{ij}(\frac{x}{r_i},0) + 2\phi_{ij}(\frac{x}{2r_i},0)] + [\phi_{ij}(0,\frac{x}{r_j}) + 2\phi_{ij}(0,-\frac{x}{2r_j})]\Big\}.\end{aligned}$$

We introduce the same definition for $\mathfrak{W}$ as in the proof of Theorem 2.5 (by replacing Φ by Ψ) such that $(\mathfrak{W}, d)$ becomes a complete metric space. Let $\mathcal{T} : \mathfrak{W} \to \mathfrak{W}$ be the mapping defined by

$$\mathcal{T}g(x) := \frac{1}{2}g(2x), \quad \text{for all } g \in \mathfrak{W} \text{ and } x \in \mathscr{A}.$$

Then, we have $d(\mathcal{T}g, \mathcal{T}h) \le Ld(g,h)$ for all $g, h \in \mathfrak{W}$. It follows from (2.24) that $d(\mathcal{T}f, f) \le \frac{1}{2}$. The rest of this proof is similar to the proof of Theorems 2.5 and 2.6. This completes the proof. □

Theorem 2.8. *Assume that there exists an expansively superhomogeneous mapping* $\phi : \mathscr{A}^{n+2} \to [0,\infty)$ *with a constant* $0 < L < 1$ *such that a mapping* $f : \mathscr{A} \to \mathscr{A}$ *with* $f(0) = 0$ *satisfies* (2.21) *for all* $x_1, \ldots, x_n, x, y \in \mathscr{A}$, *all* $\mu \in \mathbb{T}^1_{1/n_0}$ *and some* $\alpha, \beta, \gamma \in \mathbb{C}$. *Then there exists a unique Lie* (α, β, γ)*-derivation* $\mathfrak{L} : \mathscr{A} \to \mathscr{A}$ *which satisfies the functional equation* (1.1) *and the inequality*

$$\|f(x) - \mathfrak{L}(x)\| \le \frac{L}{4(1-L)}\Big\{[\phi_{ij}(\frac{x}{r_i}, \frac{x}{r_j}) + 2\phi_{ij}(\frac{x}{2r_i}, -\frac{x}{2r_j})] + [\phi_{ij}(\frac{x}{r_i}, 0) + 2\phi_{ij}(\frac{x}{2r_i}, 0)] + [\phi_{ij}(0, \frac{x}{r_j}) + 2\phi_{ij}(0, -\frac{x}{2r_j})]\Big\}$$

for all $x \in \mathscr{A}$.

Proof . The proof is similar to the proof of Theorem 2.7. □

Next, we investigate the Lie C*-algebra homomorphisms on Lie C*-algebras associated with the functional equation (1.1).

Theorem 2.9. *Assume that there exist a contractively subhomogeneous mapping* $\varphi : \mathscr{A}^n \to [0,\infty)$ *and a* 2*-contractively subhomogeneous mapping* $\psi : \mathscr{A}^2 \to [0,\infty)$ *with a constant* $0 < L < 1$ *such that a mapping* $f : \mathscr{A} \to \mathscr{B}$ *with* $f(0) = 0$ *satisfies* (2.1) *for all* $x_1, \ldots, x_n \in \mathscr{A}$ *and all* $\mu \in \mathbb{T}^1_{1/n_0}$, *and*

$$\|f([x,y]) - [f(x), f(y)]\| \le \psi(x,y) \tag{2.25}$$

for all $x, y \in \mathscr{A}$. *Then there exists a unique Lie C**-algebra homomorphism* $\mathfrak{L} : \mathscr{A} \to \mathscr{B}$ *satisfying* (2.3) *for all* $x \in \mathscr{A}$.

Proof . By the same method as in Theorem 2.5, we obtain a $\mathbb{C}$-linear mapping $\mathfrak{L} : \mathscr{A} \to \mathscr{B}$ satisfying (2.3). The mapping is given by $\mathfrak{L}(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}$ for all $x \in \mathscr{A}$. It follows from (2.25) that

$$\begin{aligned}\|\mathfrak{L}([x,y]) - &[\mathfrak{L}(x), \mathfrak{L}(y)]\| \\ &= \lim_{m\to\infty} \frac{1}{4^m}\|f(4^m[x,y]) - [f(2^m x), f(2^m y)]\| \\ &= \lim_{m\to\infty} L^{2m}\psi(x,y) = 0\end{aligned}$$

for all $x, y \in \mathscr{A}$. Thus, $\mathfrak{L}$ is a Lie C*-algebra homomorphism. This completes the proof. □

Theorem 2.10. *Assume that there exist an expansively superhomogeneous mapping* $\varphi : \mathscr{A}^n \to [0,\infty)$ *and a* 2*-expansively superhomogeneous mapping* $\psi : \mathscr{A}^2 \to [0,\infty)$ *with a constant* $0 < L < 1$ *such that a mapping* $f : \mathscr{A} \to \mathscr{B}$ *with* $f(0) = 0$ *satisfies* (2.1) *and* (2.25) *for all* $x_1, \ldots, x_n, x, y \in \mathscr{A}$ *and all* $\mu \in \mathbb{T}^1_{1/n_0}$. *Then there exists a unique Lie C**-algebra homomorphism* $\mathfrak{L} : \mathscr{A} \to \mathscr{B}$ *satisfying* (2.16) *for all* $x \in \mathscr{A}$.

Proof . The proof is similar to the proof of Theorems 2.6 and 2.9. □

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