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Existence of solutions of infinite systems of integral equations in the Fréchet spaces

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Abstract

In this paper we apply the technique of measures of noncompactness to the theory of infinite system of integral equations in the Fréchet spaces. Our aim is to provide a few generalization of Tychonoff fixed point theorem and prove the existence of solutions for infinite systems of nonlinear integral equations with help of the technique of measures of noncompactness and a generalization of Tychonoff fixed point theorem. Also, we present an example of nonlinear integral equations to show the efficiency of our results. Our results extend several comparable results obtained in the previous literature.

Keywords: Measure of noncompactness; Fréchet space; Tychonoff fixed point theorem; Infinite systems of equations.
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1. Introduction

The theory of infinite systems of integral equations is considered as an important branch of nonlinear analysis. In fact, infinite systems of integral equations are the natural generalization of infinite systems of differential equations which can arise in the theory of branching processes, the theory of neural nets, the theory of dissociation of polymers and real world problems (cf. [29, 30, 31, 32, 33]). Also, infinite systems of integral equations are particular cases of integral equations in Banach spaces which have been considered in many research papers [14, 15, 28, 32].
On the other hand, Measures of noncompactness are very useful tools in the theory of operator equations in Banach spaces. They are frequently used in the theory of functional equations, including

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ordinary differential equations, equations with partial derivatives, integral and integro-differential equations, optimal control theory, etc. In particular, the fixed point theorems derived from them have many applications. There exists an enormous amount of considerable literature devoted to this subject (see for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28]).

There have recently been many papers regarding the relationship between the above concepts, for example, Arab et al. [11], Olszowy [27], Mursaleen and Mohiuddineb [23], Mursaleen and Alotaibi [24], Banaś and Lecko [15], Rzepka and Sadarangani [28] which discussed the solvability of infinite systems of differential and integral equations with the help of measures of noncompactness.

The aim of this paper is to give fixed point theorems for condensing operators in the Fréchet space. Moreover, we study the problem of the existence of solutions for infinite systems of integral equations of the form

$$x_n(t) = f_n(t, x_1(t), \dots, x_n(t)) + \int_0^1 k_n(t,s) Q_n((x_i(s))_{i=1}^{i=\infty}) ds. \tag{1.1}$$

We are going to show that Eq. (1.1) has solution that belongs to space $(L^p[0,1])^\omega$ (denote the countable cartesian product of $L^p[0,1]$ with itself). The obtained results extend several papers (see [3, 4, 5, 7, 8, 11, 12, 14], for example). Finally, an example is presented to show the efficiency of our results.

2. Preliminaries

Here, we recall some basic facts concerning measures of noncompactness. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. The symbol $\overline{X}$, $Conv X$ will denote the closure and closed convex hull of a subset X of E, respectively. Moreover, let $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets of E.

A topological vector space (TVS) is a vector space X over the field $\mathbb{R}$ which is endowed with a topology such that the maps $(x, y) \to x + y$ and $(\alpha, x) \to \alpha x$ are continuous from $X \times X$ and $\mathbb{R} \times X$ to X. A topological vector space is called locally convex if there is a basis for the topology consisting of convex sets (that is, sets A such that if $x, y \in A$ then $tx + (1-t)y \in A$ for $0 < t < 1$).

Definition 2.1. [19] A Fréchet space is a locally convex space which is complete with respect to a translation-invariant metric.

Example 2.2. Let E_i be a Banach space for all $i \in \mathbb{N}$, then $\prod_{i\in\mathbb{N}} E_i$ is a Fréchet space by

$$d(x, y) = \sup\{\frac{1}{2^i} \min\{1, d_i(x_i, y_i)\} : i \in \mathbb{N}\},$$

where $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots) \in \prod_{i\in\mathbb{N}} E_i$.

Definition 2.3. [11] Let $\mathcal{M}$ be a class of subsets of a Fréchet space E, we say $\mathcal{M}$ is an admissible set if $\mathfrak{N}_E \cap \mathcal{M} \neq \emptyset$ and if $X \in \mathcal{M}$, then $Conv(X), \overline{X} \in \mathcal{M}$.

Definition 2.4. [11] Let $\mathcal{M}$ be an admissible subset of a Fréchet space E, we say that $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ is a measure of noncompactness on Fréchet space E if it satisfies the following conditions:

(1°) The family $ker\mu = \{X \in \mathcal{M} : \mu(X) = 0\}$ is nonempty and $ker\mu \subseteq \mathfrak{N}_E$;

(2°) $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$;

(3°) $\mu(\overline{X}) = \mu(X)$;

(4°) $\mu(ConvX) = \mu(X)$;

(5°) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;

(6°) If $\{X_n\}$ is a sequence of closed sets from $\mathcal{M}$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \cdots$, and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

Theorem 2.5. (Darbo [14]) Let C be a nonempty, closed, bounded, and convex subset of the Banach space E and $F : C \to C$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that

$$\mu(FX) \leq k\mu(X),$$

for any nonempty subset of C. Then F has a fixed point in C.

Theorem 2.6. (Tychonoff fixed point theorem [1]) Let E be a Hausdorff locally convex linear topological space, C be a convex subset of E and $F : C \longrightarrow E$ be a continuous mapping such that

$$F(C) \subseteq A \subseteq C,$$

with A compact. Then F has at least one fixed point.

Theorem 2.7. ([11]) Suppose μ_i be a measure of noncompactness on Banach spaces E_i for all $i \in \mathbb{N}$. If we define

$$\mathcal{M} = \{C \subseteq \prod_{i=1}^{\infty} E_i : \sup_i\{\mu_i(\pi_i(C))\} < \infty\},$$

where $\pi_i(C)$ denotes the natural projection of $\prod\limits_{i=1}^{\infty} E_i$ into E_i and $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ by

$$\mu(C) = \sup\{\mu_i(\pi_i(C)) : i \in \mathbb{N}\}, \tag{2.1}$$

then $\mathcal{M}$ is an admissible set and μ is a measure of noncompactness on $X = \prod\limits_{i=1}^{\infty} E_i$.

3. Main result

In this section, we state some main results in Fréchet spaces which generalize and improve Darbo's fixed point theorem, Tychonoff fixed point theorem, the mentioned corresponding results of Arab et al. [11], Aghajani et al. [3] and several authors (see [4, 5, 7, 8, 12, 14])

Theorem 3.1. *Let Ω be a nonempty, closed and convex subset of a Fréchet space E, $\mathcal{M}$ be an admissible set such that $\Omega \in \mathcal{M}$ and $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ be a measure of noncompactness on E. Also, suppose that $F : \Omega \longrightarrow \Omega$ is a continuous mapping such that*

$$\psi(\mu(FX)) \leq \varphi(\mu(X)), \tag{3.1}$$

and $F(X) \in \mathcal{M}$ for any nonempty subset $X \in \mathcal{M}$ where $\psi, \varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions such that ψ is lower semicontinuous and φ is upper semicontinuous on $\mathbb{R}$. Moreover, $\psi(0) = \varphi(0) = 0$ and $\psi(t) > \varphi(t) > 0$ for $t > 0$. Then F has at least one fixed point and the set of all fixed points of F in Ω is compact.

Proof . By induction, we obtain a sequence $\{\Omega_n\}$ such that $\Omega_0 = \Omega$ and $\Omega_n = Conv(F\Omega_{n-1})$, $n \geq 1$. It is obvious that $\Omega_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. If there exists an integer $N \geq 0$ such that $\mu(\Omega_N) = 0$, then Ω_N is compact. Thus, Theorem 2.6 implies that F has a fixed point. Now assume that $\mu(\Omega_n) \neq 0$ for $n \geq 0$. Since $\{\mu(\Omega_n)\}$ is a positive decreasing sequence of real numbers, then there exists $r \geq 0$ such that $\mu(\Omega_n) \to r$ as $n \to \infty$ and by (3.1), we have

$$\psi(r) \leq \lim_{n\to\infty} \psi(\mu(\Omega_{n+1})) = \lim_{n\to\infty} \psi(\mu(Conv(F(\Omega_n)))) = \lim_{n\to\infty} \psi(\mu(F(\Omega_n))) \leq \lim_{n\to\infty} \varphi(\mu(\Omega_n)) \leq \varphi(r).$$

This result, $\psi(0) = \varphi(0) = 0$ and $\psi(t) > \varphi(t) > 0$ for $t > 0$ imply that $r = 0$. Hence we deduce that $\mu(\Omega_n) \longrightarrow 0$ as $n \longrightarrow 0$. Since the sequence (Ω_n) is nested, in view of axiom (6°) of Definition 2.4 we deduce that the set $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, closed and convex subset of the set Ω. Moreover, the set Ω_∞ is invariant under the operator F and belongs to $Ker\mu$. Thus, applying Tychonoff fixed point theorem, F has a fixed point. To complete the proof it remains to verify that $\mu(F_F) = 0$ where $F_F = \{x \in \Omega : Fx = x\}$. Since $F(F_F) = F_F$ and by (3.1), we have

$$\psi(\mu(F_F)) = \psi(\mu(F(F_F))) \leq \varphi(\mu(F_F)).$$

Moreover, $\psi(t) > \varphi(t) > 0$ for $t > 0$, so $\mu(F_F) = 0$ and F_F is relatively compact and since F is a continuous function so the set of fixed points of F in Ω is compact. □

Corollary 3.2. ([11]) *Let Ω be a nonempty, closed and convex subset of a Fréchet space E, $\mathcal{M}$ an admissible set such that $\Omega \in \mathcal{M}$ and $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+$ is a measure of noncompactness on E. Let $F : \Omega \longrightarrow \Omega$ be a continuous mapping such that*

$$\mu(FX) \leq \varphi(\mu(X)), \tag{3.2}$$

and $F(X) \in \mathcal{M}$ for any nonempty subset $X \in \mathcal{M}$ where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is upper semicontinuous and nondecreasing function such that $\varphi(t) < t$ for $t > 0$ and $\varphi(0) = 0$. Then F has at least one fixed point in the set Ω.

Proof . Take $\psi(t) = t$ in Theorem 3.1. □

We introduce the following useful corollary which will be used in Section 4 and extend recent result of Arab et al. [11]

Corollary 3.3. *Let Ω_i $(i \in \mathbb{N})$ be a nonempty, convex and closed subset of a Banach space E_i, μ_i an arbitrary measure of noncompactness on E_i and $\sup_i\{\mu_i(\Omega_i)\} < \infty$. Let $F_i : \prod_{i=1}^{\infty} \Omega_i \longrightarrow \Omega_i$ $(i \in \mathbb{N})$ be a continuous operator such that*

$$\psi(\mu_i(F_i(\prod_{i=1}^{\infty} X_i))) \leq \varphi(\sup_i \mu_i(X_i)), \tag{3.3}$$

for any subset X_i of Ω_i $(i \in \mathbb{N})$ where $\psi, \varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies the hypotheses of Theorem 3.1 and ψ is nondecreasing. Then there exist $(x_j^*)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \Omega_j$ such that for all $i \in \mathbb{N}$

$$F_i((x_j^*)_{j=1}^{\infty}) = x_i^*. \tag{3.4}$$

Proof . Assume that $\widetilde{F} : \prod_{i=1}^{\infty} \Omega_i \longrightarrow \prod_{i=1}^{\infty} \Omega_i$ as follows

$$\widetilde{F}((x_j)_{j=1}^{\infty}) = (F_1((x_j)_{j=1}^{\infty}), F_2((x_j)_{j=1}^{\infty}), \dots, F_i((x_j)_{j=1}^{\infty}), \dots),$$

for all $(x_j)_{j=1}^{\infty} \in \prod_{i=1}^{\infty} \Omega_i$. It is obvious that F is continuous. It suffices to show that the hypothesis (3.1) of Theorem 3.1 holds where μ is defined by Theorem 2.7. Take an arbitrary nonempty subset X of $\prod_{i=1}^{\infty} \Omega_i$. Now, by (2°) and (3.3) we obtain

$$\begin{aligned}
\psi(\mu(\widetilde{F}(X))) &\le \psi(\mu(\prod_{i=1}^{\infty} F_i(\prod_{j=1}^{\infty} \pi_j(X)))) \\
&= \sup_i \psi(\mu_i(F_i((\prod_{j=1}^{\infty} \pi_j(X))))) \\
&\le \sup_i \varphi(\sup_j \mu_j(\pi_j(X))) \\
&\le \sup_i \varphi(\sup_j(\mu_j(\pi_j(X)))) \\
&\le \sup_i \varphi(\mu(X)) \\
&\le \varphi(\mu(X)).
\end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are satisfied, hence $\widetilde{F}$ has a fixed point and there exist $(x_j^*)_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \Omega_j$ such that

$$(x_j^*)_{j=1}^{\infty} = \widetilde{F}((x_j^*)_{j=1}^{\infty}) = (F_1((x_j^*)_{j=1}^{\infty}), F_2((x_j^*)_{j=1}^{\infty}), \dots, F_j((x_j^*)_{j=1}^{\infty}), \dots)$$

and that (3.4) holds. □

4. Existence of solutions of infinite systems of integral equations

In this section we are going to show how the result contained in section 3 can be applied to infinite systems of nonlinear integral equations.

We will use a measure of noncompactness in the space $L^p[0,1]$. In order to define this measure, take an arbitrary set X of $\mathfrak{M}_{L^p[0,1]}$. For $x \in X$ and $\varepsilon > 0$ let us put

$$\begin{aligned}
\omega(x,\varepsilon) &= \sup\{\|\tau_h x - x\|_p : |h| < \varepsilon\}, \\
\omega(X,\varepsilon) &= \sup\{\omega(x,\varepsilon) : x \in X\}
\end{aligned}$$

where

$$\tau_h x(t) = \begin{cases} x(t+h) & 0 \le t+h \le 1 \\ 0 & otherwise \end{cases}$$

for all $t, h \in [0,1]$. Moreover,

$$\omega_0(X) = \lim_{\varepsilon \longrightarrow 0} \omega(X, \varepsilon).$$

It can be shown [14] that the mapping $\omega_0 = \omega_0(X)$ is the measure of noncompactness in the space $L^P[0,1]$.

Definition 4.1. A function $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to have the Carathéodory property if

(i) For all $x \in \mathbb{R}$ the function $t \to f(t,x)$ is measurable on $\mathbb{R}_+$.

(ii) For almost all $t \in \mathbb{R}_+$ the function $x \to f(t,x)$ is continuous on $\mathbb{R}$.

Theorem 4.2. (*Minkowki's Inequality for Integrals*) [6] Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and let f be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$. If $f \geq 0$ and $1 \leq p < \infty$, then

$$\Big[\int \Big(\int f(x,y) d\nu(y) \Big)^p d\mu(x) \Big]^{\frac{1}{p}} \leq \int \Big(\int f(x,y)^p d\mu(x) \Big)^{\frac{1}{p}} d\nu(y).$$

Let us consider the Equation (1.1) under the following assumptions:

(a$_1$) $f_n : [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ satisfies the Carathéodory conditions and $f_n(.,0,\dots,0) \in L^p[0,1]$.

(a$_2$) There exist a non-decreasing, continuous and concave function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(t) < t$ for all $t > 0$, $\phi(0) = 0$ and $a \in L^p[0,1]$ such that

$$|f_n(t, x_1, \dots, x_n) - f_n(s, y_1, \dots, y_n)| \leq |a(t) - a(s)| + \sqrt[p]{\phi(\max_{1 \leq i \leq n} |x_i - y_i|^p)}, \ a.e. \tag{4.1}$$

for all $n \in \mathbb{N}$

(a$_3$) $k_n : [0,1] \times [0,1] \longrightarrow \mathbb{R}_+$ $(n \in \mathbb{N})$ is measurable if there exists $g \in L^p[0,1]$ such that $|k_n(t,s)| \leq g(t)$ for all $t, s \in [0,1]$ and $n \in \mathbb{N}$.

(a$_4$) The operator Q_n $(n \in \mathbb{N})$ acts continuously from the space $(L^p[0,1])^\omega$ into $L^p[0,1]$ and there exists a nondecreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\|Q_n x\|_p \leq \psi(\sup \|x_i\|_p) \tag{4.2}$$

for any $x = (x_i)_1^\infty \in L^p[0,1])^\omega$ with $\sup_{1 \leq i \leq \infty} \|x_i\|_p < \infty$ and $n \in \mathbb{N}$.

(a$_5$) There exists a positive solution r_0 of the inequality

$$\sqrt[p]{\phi(r^p)} + \|f_n(.,0)\|_p + \psi(r)\|K_n\| \leq r, \tag{4.3}$$

for all $n \in \mathbb{N}$ where

$$(K_n x)(t) = \int_0^1 k_n(t,s) x(s) ds.$$

Theorem 4.3. *Under assumptions* (a$_1$)-(a$_5$), *the Equation* (1.1) *has at least a solution in the space* $(L^p[0,1])^\omega$.

Remark 4.4. Under the assumptions (a_3) and (a_5) the linear Fredholm integral operator $K_n : L^p[0,1] \longrightarrow L^p[0,1]$ ($n \in \mathbb{N}$) is a continuous operator.

Proof . Let us fix arbitrarily $n \in \mathbb{N}$. $F_n : (L^p[0,1])^\omega \longrightarrow L^p[0,1]$ ($n \in \mathbb{N}$) is defined by

$$F_n((x_j)_{j=1}^{\infty})(t) = f_n(t, x_1(t), \cdots, x_n(t)) + \int_0^1 k_n(t,s) Q_n((x_i(s))_{i=1}^{i=\infty}) ds. \tag{4.4}$$

By using conditions $(a_1) - (a_5)$ and since f_n is concave, for arbitrary fixed $t \in [0,1]$, we have

$$\begin{aligned} &|F_n((x_j)_{j=1}^{\infty})(t)| \\ &\leq |f_n(t, x_1(t), \cdots, x_n(t)) - f_n(t, 0, \ldots, 0)| + |f_n(t, 0, \ldots, 0)| + |\int_0^1 k_n(t,s) Q_n((x_i(s))_{i=1}^{i=\infty}) ds| \\ &\leq \sqrt[p]{\phi(\max_{1 \leq i \leq n} |x_i(t)|^p)} + |f_n(t, 0, \ldots, 0)| + |\int_0^1 k_n(t,s) Q_n((x_i(s))_{i=1}^{i=\infty}) ds| \quad a.e. \end{aligned}$$

Thus

$$\|F_n((x_j)_{j=1}^{\infty})\|_p \leq \sqrt[p]{\phi(\max_{1 \leq i \leq n} \|x_i\|_p^p)} + \|f_n(.,0)\|_p + \|K_n\| \psi(\|(x_i)_{i=1}^{i=\infty}\|_p). \tag{4.5}$$

Hence $F_n((x_j)_{j=1}^{\infty}) \in L^p[0,1]$ for any $(x_j)_{j=1}^{\infty} \in (L^p[0,1])^\omega$ and F_n is well defined and from (4.5), we have $F_n((\overline{B}_{r_0})^\omega) \subseteq \overline{B}_{r_0}$, where r_0 is a constant appearing in assumption (4.3). Also, F_n is continuous in $(L^p[0,1])^\omega$ because $f_n(t,.)$, K_n and Q_n are continuous for almost all $t \in [0,1]$.
If we define $k_{n,s} : [0,1] \longrightarrow \mathbb{R}_+$ by $k_{n,s}(t) := k_n(t,s)$ for all $s \in [0,1]$, then we show that $\omega_0(\{k_{n,s} : s \in [0,1]\}) = 0$.
To do this, fix arbitrary $\varepsilon > 0$. We define the function $\vartheta : [0,1] \longrightarrow \mathbb{R}$ as follows

$$\vartheta(s) = \int_0^1 |k_n(t,s)|^p dt. \tag{4.6}$$

Since there exists $g \in L^p[0,1]$ such that $|k_n(t,s)| \leq g(t)$ for all $t, s \in [0,1]$, so ϑ is continuous and there exists $\delta_1 > 0$ such that $|\vartheta(s) - \vartheta(t)| < \varepsilon$ for all $s, t \in [0,1]$ with $|s - t| < \delta$. Moreover, there exist $s_1, \ldots, s_m$ and $\delta_2 > 0$ such that $[0,1] \subseteq \cup_{i=1}^m B_{\delta_1}(s_i)$ and

$$\|\tau_h k_{n,s_i} - k_{n,s_i}\|_p \leq \varepsilon$$

where $|h| \leq \delta_2$. Since $\{k_{n,s_1}, \ldots, k_{n,s_m}\}$ is a compact subset of $L^p[0,1]$ and $\omega_0(\{k_{n,s_1}, \ldots, k_{n,s_m}\}) = 0$ for every $s \in [0,1]$ and $|h| \leq \delta_2$, there exists s_{i_0} such that $|s - s_{i_0}| \leq \varepsilon$ and

$$\begin{aligned} \|\tau_h k_{n,s} - k_{n,s}\|_p &= \Big(\int_0^1 |k_n(t,s) - k_n(t+h,s)|^p dt \Big)^{\frac{1}{p}} \\ &\leq \Big(\int_0^1 |k_n(t,s) - k_n(t,s_i)|^p dt \Big)^{\frac{1}{p}} + \Big(\int_0^1 |k_n(t,s_i) - k_n(t+h,s_i)|^p dt \Big)^{\frac{1}{p}} \\ &\quad + \Big(\int_0^1 |k_n(t+h,s) - k_n(t+h,s_i)|^p dt \Big)^{\frac{1}{p}} \\ &\leq 2|\vartheta(s) - \vartheta(s_{i_0})|^p + \|\tau_h k_{n,s_i} - k_{n,s_i}\|_p \\ &\leq 2\varepsilon^p + \varepsilon. \end{aligned}$$

So, we have

$$\begin{aligned} \omega(k_{n,s}, \delta_2) &\leq 2\varepsilon^p + \varepsilon, \\ \omega(\{k_{n,s} : s \in [0,1]\}, \delta_2) &\leq 2\varepsilon^p + \varepsilon, \end{aligned}$$

and

$$\omega_0(\{k_{n,s} : s \in [0,1]\}) = 0.$$

Now, For any nonempty subset X_j of $\overline{B}_{r_0}$ for all $j \in \mathbb{N}$, we claim that $[\omega_0(F_nX)]^p \leq \phi([\omega_0(X)]^p)$. To verify this, let $\varepsilon > 0$, $(x_j)_{j=1}^{\infty} \in X$ and $t, h \in [0,1]$ with $|h| \leq \varepsilon$, thus we have

$$\begin{aligned}
&|(F_n(x_j)_{j=1}^{\infty})(t+h) - (F_n(x_j)_{j=1}^{\infty})(t)| \\
\leq &|f_n(t+h, x_1(t+h), \cdots, x_n(t+h)) + \int_0^1 k_n(t+h,s)Q_n((x_i(s))_{i=1}^{i=\infty})ds \\
&- f_n(t, x_1(t), \cdots, x_n(t)) - \int_0^1 k_n(t,s)Q_n((x_i(s))_{i=1}^{i=\infty})ds| \\
\leq &|f_n(t+h, x_1(t+h), \cdots, x_n(t+h)) - f_n(t, x_1(t), \cdots, x_n(t))| \\
&+ |\int_0^1 k_n(t+h,s)Q_n((x_i(s))_{i=1}^{i=\infty})ds - \int_0^1 k_n(t,s)Q_n((x_i(s))_{i=1}^{i=\infty})ds| \\
\leq &|a(t+h) - a(t)| + \sqrt[p]{\phi[\max_{1\leq i\leq n} |x_i(t+h) - x_i(t)|^p]} \\
&+ \int_0^1 |k_n(t+h,s) - k_n(t,s)||Q_n((x_i(s))_{i=1}^{i=\infty})|ds.
\end{aligned}$$

So,

$$\begin{aligned}
&\|\tau_h F_n(x_j)_{j=1}^{\infty} - F_n(x_j)_{j=1}^{\infty}\|_p \\
&= \Big(\int_0^1 |(F_n(x_j)_{j=1}^{\infty})(t+h) - (F_n(x_j)_{j=1}^{\infty})(t)|^p dt\Big)^{\frac{1}{p}} \\
&\leq \Big(\int_0^1 |a(t+h) - a(t)|^p dt\Big)^{\frac{1}{p}} + \Big(\int_0^1 (\sqrt[p]{\phi[\max_{1\leq i\leq n} |x_i(t+h) - x_i(t)|^p]})^p dt\Big)^{\frac{1}{p}} \\
&\quad + \Big(\int_0^1 |\int_0^1 |k_n(t+h,s) - k_n(t,s)||Q_n((x_i(s))_{i=1}^{i=\infty})|ds|^p dt\Big)^{\frac{1}{p}} \\
&\leq \omega(a,\varepsilon) + \phi(\max_{1\leq i\leq n} \|\tau_h x_i - x_i\|^p)^{\frac{1}{p}} + \omega(\{k_{n,s} : s \in [0,1]\}, \varepsilon)\psi(\|(x_i)_{i=1}^{i=\infty}\|_p).
\end{aligned} \tag{4.7}$$

Indeed,

$$\begin{aligned}
&\Big(\int_0^1 (\sqrt[p]{\phi[\max_{1\leq i\leq n} |x_i(t+h) - x_i(t)|^p]})^p dt\Big)^{\frac{1}{p}} \\
&= \Big(\int_0^1 \phi[\max_{1\leq i\leq n} |x_i(t+h) - x_i(t)|^p] dt\Big)^{\frac{1}{p}} \\
&= \Big(\phi[\int_0^1 \max_{1\leq i\leq n} |x_i(t+h) - x_i(t)|^p dt]\Big)^{\frac{1}{p}} \\
&= [\phi(\max_{1\leq i\leq n} \|\tau_h x_i - x_i\|_p^p)]^{\frac{1}{p}},
\end{aligned}$$

and by Minkowski's Inequality for Integrals, we have

$$\begin{aligned}
&\Big(\int_0^1 |\int_0^1 |k_n(t+h,s)-k_n(t,s)||Q_n((x_i(s))_{i=1}^{i=\infty})|ds|^p dt\Big)^{\frac{1}{p}}\\
&= \int_0^1 \Big(\int_0^1 |k_n(t+h,s)-k_n(t,s)|^p|Q_n((x_i(s))_{i=1}^{i=\infty})|^p dt\Big)^{\frac{1}{p}} ds\\
&\le \int_0^1 \omega(k_{n,s},\varepsilon)|Q_n((x_i(s))_{i=1}^{i=\infty})|ds\\
&\le \omega(\{k_{n,s}: s\in[0,1]\},\varepsilon)\|Q_n(x_i)_{i=1}^{i=\infty}\|_p\\
&\le \omega(\{k_{n,s}: s\in[0,1]\},\varepsilon)\psi(\|(x_i)_{i=1}^{i=\infty}\|_p).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\tau_h F_n(x_j)_{j=1}^{\infty} - F_n(x_j)_{j=1}^{\infty}\|_p\\
&\le \omega(a,\varepsilon) + [\phi(\max_{1\le i\le n}\|\tau_h x_i - x_i\|_p^p)]^{\frac{1}{p}} + \omega(\{k_{n,s}: s\in[0,1]\},\varepsilon)\psi(\|(x_i)_{i=1}^{i=\infty}\|_p)\\
&\le \omega(a,\varepsilon) + \Big(\phi(\max_{1\le i\le n}[\omega(x_i,\epsilon)]^p)\Big)^{\frac{1}{p}} + \omega(\{k_{n,s}: s\in[0,1]\},\varepsilon)\psi(\|(x_i)_{i=1}^{i=\infty}\|_p).
\end{aligned}$$

By using the above estimate we have

$$\omega(F_n(\prod_{i=1}^{\infty} X_i),\varepsilon) \le \omega(a,\varepsilon) + \Big(\phi(\max_{1\le j\le n}[\omega(X_j,\epsilon)]^p)\Big)^{\frac{1}{p}} + \omega(\{k_{n,s}: s\in[0,1]\},\varepsilon)\psi(r_0).$$

Since $\{a\}$ is a compact set and $\omega_0(\{k_{n,s}: s\in[0,1]\})=0$, so we have $\omega(a,\varepsilon)\to 0$ and $\omega(\{k_{n,s}: s\in[0,1]\},\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Then we obtain

$$[\omega_0(F_n(\prod_{i=1}^{\infty} X_i))]^p \le \phi([\max_{1\le i\le n}\omega_0(X_i)]^p).$$

Now, by considering the functions $\psi,\varphi:[0,\infty)\to[0,\infty)$ defined by

$$\psi(t)=t^p,\ \textit{and}\ \varphi(t)=\phi(t^p),$$

we get

$$\psi(\omega_0(F_n(\prod_{i=1}^{\infty} X_i))) \le \varphi(\max_{1\le i\le n}\omega_0(X_i)).$$

Thus from Corollary 3.3 the functional integral equation (1.1) has at least a solution in $(L^p[0,1])^{\omega}$. □

Example 4.5. Consider the following integral equation of the form

$$x_n(t) = \frac{1}{\sqrt[3]{t}} + \frac{1}{n}\sum_{i=1}^{i=n}\ln(|x_i(t)|+1) + \sum_{i=1}^{\infty}\frac{1}{2^i}\int_0^t (t-s)^2\frac{x_i(s)}{e^{x_i(s)}+n}ds \qquad 0\le t\le 1 \tag{4.8}$$

In this example, we have

$$\begin{aligned}
&k_n(t,s) = (t-s)^2\chi_E(t,s) \quad (E=\{(t,s): 0\le s\le t\le 1\})\\
&f_n(t,x_1,x_2,\dots,x_n) = \frac{1}{\sqrt[3]{t}} + \frac{1}{n}\sum_{i=1}^{i=n}\ln(|x_i|+1),\\
&Q_n(x_i)_{i=1}^{i=\infty} = \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{x_i}{e^{x_i}+n}.
\end{aligned}$$

It can readily be seen that f_n satisfies assumption a_1 and hypothesis (a_2) with $p < 3$, $a(t) = \dfrac{1}{\sqrt[3]{t}}$ and $\phi(t) = (\ln \sqrt[p]{t} + 1)^p$, indeed,

$$\begin{aligned}
|f_n(t, x_1, \ldots, x_n) - f_n(s, y_1, \ldots, y_n)| =& |\frac{1}{\sqrt[3]{t}} - \frac{1}{\sqrt[3]{s}}| + |\frac{1}{n}\sum_{i=1}^{i=n} |\ln(|x_i| + 1) - \ln(|y_i| + 1)| \\
\leq& |\frac{1}{\sqrt[3]{t}} - \frac{1}{\sqrt[3]{s}}| + \frac{1}{n}\sum_{i=1}^{i=n} |\ln(\frac{|x_i| + 1}{|y_i| + 1})| \\
\leq& |\frac{1}{\sqrt[3]{t}} - \frac{1}{\sqrt[3]{s}}| + \frac{1}{n}\sum_{i=1}^{i=n} |\ln(1 + \frac{|x_i - y_i|}{|y_i| + 1})| \\
\leq& |\frac{1}{\sqrt[3]{t}} - \frac{1}{\sqrt[3]{s}}| + \ln(1 + (\max_{1\leq i\leq n} |x_i - y_i|)| \\
=& |a(t) - a(s)| + \sqrt[p]{\phi(\max_{1\leq i\leq n} |x_i - y_i|^p)}.
\end{aligned}$$

Also, $k_n(t, s)$ $(n \in \mathbb{N})$ satisfies hypothesis (a_3) with $g(t) = t^2$, $k_n(., s) \in L^p[0, 1]$ for each $s \in [0, 1]$ and $g \in L^p[0, 1]$ and $\|K_n\| \leq 1$. Now, we show that Q_n is continuous operator of $(L^p[0, 1])^\omega$ into $L^p[0, 1]$. To establish this claim, let us fix $x \in (L^p[0, 1])^\omega$, $n \in \mathbb{N}$ and $\varepsilon = \frac{1}{2^n}$ and take arbitrary $((y_j)_{j=1}^\infty) \in (L^p[0, 1])^\omega$, such that $\sup\{\frac{1}{2^i}\min\{1, \|x_i - y_i\|_p\} : i \in \mathbb{N}\} < \varepsilon$. Then we have

$$\begin{aligned}
|Q_n(x_i)_{i=1}^{i=\infty} - Q_n(y_i)_{i=1}^{i=\infty}| \leq& \sum_{i=1}^{n} \frac{1}{2^i}|\frac{x_i}{e^{x_i} + n} - \frac{y_i}{e^{y_i} + n}| + \sum_{i=n+1}^{\infty} \frac{1}{2^i}|\frac{x_i}{e^{x_i} + n} - \frac{y_i}{e^{y_i} + n}| \\
\leq& 4\varepsilon + \sum_{i=1}^{n} \frac{1}{2^i}(\frac{|x_i - y_i|}{e^{x_i} + n} + |\frac{y_i|e^{x_i} - e^{y_i}|}{(e^{x_i} + n)(e^{y_i} + n)}|) \\
\leq& 4\varepsilon + \sum_{i=1}^{n} \frac{1}{2^i}(|x_i - y_i| + |e^{x_i} - e^{y_i}|).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|Q_n(x_i)_{i=1}^{i=\infty} - Q_n(y_i)_{i=1}^{i=\infty}\|_p \leq& 4\varepsilon + \sum_{i=1}^{n} \frac{1}{2^i}(\|x_i - y_i\|_p + \|e^{x_i} - e^{y_i}\|_p) \\
\leq& 6\varepsilon + 2\vartheta(\varepsilon),
\end{aligned}$$

where

$$\vartheta(\varepsilon) = sup\{\|e^{x_i} - e^{y_i}\|_p : 1 \leq i \leq n, \|y_i\| \leq b + \varepsilon\},$$

with $b = \sup_i\{\|x_i\|_p + \varepsilon\}$. Hence, we obtain $\vartheta(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and Q_n is a continuous operator. Moreover, for all $x \in (L^p[0, 1])^\omega$ we deduce

$$|Q_n(x_i)_{i=1}^{i=\infty}| \leq \sum_{i=1}^{\infty} \frac{1}{2^i}|\frac{x_i}{e^{x_i} + n}| \leq 1,$$

and this operator satisfies hypothesis (a_4) with $\psi(t) = 1$. On the other hand, with choosing $p \in [1, 3)$ we can compute r_0 which satisfies the following inequality:

$$\sqrt[p]{\phi(r^p)} + \|f_n(., 0)\|_p + \psi(r)\|K_n\| \leq \ln(r + 1) + \sqrt[p]{|\frac{3}{p - 3}|} + 1 \leq r.$$

Consequently, all the conditions of Theorem 4.3 are satisfied. Hence, the functional integral equation (4.8) has at least a solution in $(L^p[0,1])^\omega$ for $1 \leq p < 3$.

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