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Non-linear Bayesian prediction of generalized order statistics for liftime models

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Abstract

In this paper, we obtain Bayesian prediction intervals as well as Bayes predictive estimators under square error loss for generalized order statistics when the distribution of the underlying population belongs to a family which includes several important distributions.

Keywords: Bayes predictive estimators; Bayesian prediction intervals; order statistics; record values; k-record values; generalized order statistics.
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1. Introduction preliminaries

Kamps [20] introduced the concept of generalized order statistics (GOS) as a unification of several models of ascendingly ordered random variables. The use of this concept, which includes well-known concepts that have been treated separately in statistical literature as special cases, has been growing steadily over the years.

In the fields of reliability analysis and lifetime studies we often consider several models of ascendingly ordered random variables. Many of these models, such as ordinary order statistics, sequential order statistics, record values, Pfeifer's record model and progressive type II censored order statistics are contained in the GOS model. For instance, the r^{th} extreme order statistic represents the lifetime of an r out of n system, whereas the sequential order statistics model is an extension of the ordinary order statistics model and serves as a model describing certain dependencies or interactions among the system components. The progressive type II censored order statistics model is used to analyse data in lifetime tests.

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Using the concept of GOS, known results in submodels can be subsumed, generalized and integrated within a general framework. Well-known distributional and inferential properties of ordinary order statistics and record values turn out to remain valid for GOS. Thus, GOS provide a large class of models with many interesting and useful properties for both the description and the analysis of practical problems.

Ahsanullah ([4], [5]) and Habibullah and Ahsanullah [17] studied the distributional properties of GOS and obtained minimum variance linear unbiased estimators of the parameters of a two-parameter uniform, exponential and Pareto type II distributions in terms of GOS. For the uniform model, Ahsanullah [4] also obtained the best linear invariant estimators of the parameters and in [5] he characterized the exponential distribution in terms of GOS. Kamps and Gather [21] developed a characteristic property of GOS for exponential distributions and Keseling [22] used conditional distributions of GOS to characterize certain continuous distributions.

Prediction is one of the most important topics in statistical inference. Several authors (Geisser ([15], [16]), Al-Hussaini and Jaheen ([8], [9]), Dunsmore and Amin [14] and Al-Hussaini [6]) have predicted future order statistics and records from homogeneous as well as heterogeneous populations.

In this paper, we obtain Bayesian prediction intervals as well as the Bayes predictive estimators for GOS when the distribution of the underlying population belongs to a family which includes several important distributions such as the Weibull, compound Weibull (or three-parameter Burr type XII), Pareto, Beta, Gompertz and compound Gompertz distributions.

The organization of this paper is as follows. In Section 2, we present some preliminaries. In Section 3, we obtain Bayesian prediction intervals and the Bayesian predictive estimators under square error loss for GOS when the underlying population is assumed to belong to a certain family of distributions. As an example we obtain Bayesian prediction intervals and the Bayesian predictive estimators for GOS when the underlying population is Pareto.

2. Preliminaries

Kamps [20] defined GOS as follows: Let F be an absolutely continuous function with density f. Let $n \in N, k > 0$, $\tilde{m} = (m_1, \ldots, m_{n-1}) \in R^{n-1}$, and $M_r = \sum_{j=r}^{n-1} m_j$ for $\gamma_r = k+n-r+M_r > 0$ for all $r = 1, \ldots, n-1$. The components of the random vector $\mathbf{X}(n, \tilde{m}, k) = \Big(X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)\Big)$ are said to be GOS if their joint density function is of the form

$$f_{\mathbf{X}(n,\tilde{m},k)}(\mathbf{x}) = k(\prod_{j=1}^{n-1} \gamma_j)\Big(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i} f(x_i)\Big)(1-F(x_n))^{k-1} f(x_n),$$

for $F^{-1}(0) < x_1 \leq \ldots \leq x_n < F^{-1}(1)$.

Generalized order statistics contain many models of order statistics as special cases.

(i) The order statistics $X_{1:n}, \ldots X_{n:n}$ of a sample $(X_1, \ldots, X_n)$ of size n from cdf F are GOS with parameters $m_1 = \ldots = m_{n-1} = 0$ and $k = 1$.

(ii) The first n record values in a sequence $\{X_\nu, \nu \geq 1\}$ of i.i.d. random variables are GOS with parameters $m_1 = \ldots = m_{n-1} = -1$ and $k = 1$.

(iii) The first n, k-record values $\mathbf{R}_n^{(k)} = (R_1^{(k)}, \ldots R_n^{(k)})$ in an i.i.d. sequence $\{X_\nu, \nu \geq 1\}$ are GOS with parameters $m_1 = \ldots = m_{n-1} = -1$ and $k \geq 1$ a positive integer.

(iv) Pfeifer's record values $X^{(1)}_{\Delta_1},\dots X^{(n)}_{\Delta_n})$ in an array $\{X^{(j)}_i, i\geq 1, j\geq 1\}$ of independent random variables such that $X^{(j)}_i, i\geq 1$, are identically distributed with distribution function $F_j(x)=1-(1-F(x))^{\beta_j}, j\geq 1$, where $\beta_j>0$, are GOS with parameters $m_i=\beta i-\beta_{i+1}-1$ and $k=\beta_n$ (cf. Pfeifer (1982)).

(v) The progressive type II censored order statistics $X^{\tilde{R}}_{1:n:N},\dots X^{\tilde{R}}_{n:n:N})$ where $\tilde{R}=(R_1,\dots,R_n)$ and $R_i\in N_0; 1\leq i\leq n$, are GOS with parameters $m_i=R_i, k=R_n+1$(cf. Balakrishnan et al. [11]).

(vi) The sequential order statistics $X^{(1)}_*,\dots X^{(n)}_*$ of an array of independent random variables $\{Y^{(i)}_j, 1\leq i\leq n, 1\leq j\leq n-i+1\}$ such that $Y^{(i)}_j, 1\leq j\leq n-i+1$ are identically distributed with distribution function $F_i(x)=1-\Big(1-F(x)\Big)^{\alpha_i}$ for $1\leq i\leq n$, are GOS with parameters $m_i=(n-i+1)\alpha_i-(n-i)\alpha_{i+1}-1$ and $k=\alpha_n$.

Cramer and Kamps [13] obtained the joint density function of the first r GOS as well as their marginal univariate and bivariate density functions of GOS. They show that when

$$\gamma_n=k, \gamma_i\neq\gamma_j, i\neq j, 1\leq i,j\leq n,$$

the joint density function of the first r GOS is

$$f_{\mathbf{X}(r,\tilde{m},k)}(\mathbf{x})=c_{r-1}\prod_{i=1}^{r-1}\Big(1-F(x_i)\Big)^{m_i}f(x_i)\Big(1-F(x_r)\Big)^{\gamma_r-1}f(x_r). \tag{2.1}$$

Moreover, the marginal density function of the r^{th} GOS is given by

$$f_{X(r,n,\tilde{m},k)}(x)=c_{r-1}\sum_{i=1}^{r}a_i\Big(1-F(x)\Big)^{\gamma_i-1}f(x),$$

where $c_{r-1}=\prod_{i=1}^{r}\gamma_i$ and $a_i(r)=a_i\prod_{j=1_{j\neq i}}^{r}\dfrac{1}{\gamma_j-\gamma_i}, 1\leq i\leq r\leq n$. Also, for $r<s$, the joint density function of the r^{th} and s^{th} GOS is

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y)=c_{s-1}\frac{f(y)}{1-F(y)}\frac{f(x)}{1-F(x)}\sum_{i=r+1}^{s}a_i^{(r)}(s)\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i}$$
$$\times\sum_{i=1}^{r}a_i\Big(1-F(x)\Big)^{\gamma_i},$$

where $a_i^{(r)}(s)=\displaystyle\prod_{j=r+1_{j\neq i}}^{s}\frac{1}{\gamma_j-\gamma_i}$.

Hence for $r<s$, the conditional density function of the s^{th} GOS given the r^{th} GOS is

$$f_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(x|y)=\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\Big(\frac{1-F(y)}{1-F(x)}\Big)^{\gamma_i}\frac{f(y)}{1-F(y)}. \tag{2.2}$$

The predictive density function is

$$H(x_{r+s}|\mathbf{x})=\int_{\Theta}g_r(x_{r+s}|\mathbf{x})q(\boldsymbol{\theta}|\mathbf{x})d\boldsymbol{\theta}, \tag{2.3}$$

where $q(\boldsymbol{\theta}|\mathbf{x})$ denotes the posterior density function of $\boldsymbol{\theta}$ and for $s = 1, \dots . n - r$, $g_r(x_{r+s}|\mathbf{x})$ denotes the conditional density function of the $(r+s)^{th}$ GOS given $\mathbf{x} = (x_1, \dots, x_r)$. Suppose

$$F_{\boldsymbol{\theta}}(x) = 1 - \exp\big(- \lambda_{\boldsymbol{\theta}}(x)\big), x > 0, \tag{2.4}$$

where $\lambda_{\boldsymbol{\theta}}(x)$ is a continuous differentiable function of x such that $\lambda_{\boldsymbol{\theta}}(x) \to 0$ as $x \to 0^+$ and $\lambda_{\boldsymbol{\theta}}(x) \to \infty$ as $x \to \infty$. Then the corresponding density, for $x > 0$ is

$$f_{\boldsymbol{\theta}}(x) = \lambda'_{\boldsymbol{\theta}}(x) \exp(-\lambda_{\boldsymbol{\theta}}(x)). \tag{2.5}$$

We assume that $\boldsymbol{\theta}$ is a random vector with conjugate prior density of the form suggested by Al-Hussaini[6].

$$\pi(\boldsymbol{\theta}; \delta) = C(\boldsymbol{\theta}; \delta) \exp\big(- D(\boldsymbol{\theta}; \delta)\big), \boldsymbol{\theta} \in \Theta, \boldsymbol{\delta} \in \Omega, \tag{2.6}$$

where Ω is the hyperparameter space and $\boldsymbol{\delta}$ is a vector of prior parameters.

Substituting (2.4) and (2.5) into (2.1), the likelihood function becomes

$$L(\boldsymbol{\theta}, \mathbf{x}) = c_{r-1} \exp\Bigg(- \Big(\sum_{j=1}^{r-1} \lambda_{\boldsymbol{\theta}}(x_j)(m_j + 1) + \gamma_r \lambda_{\boldsymbol{\theta}}(x_r)\Big)\Bigg) \prod_{j=1}^{r} \lambda'_{\boldsymbol{\theta}}(x_j). \tag{2.7}$$

From (2.6) and (2.7), the posterior density function takes the form

$$\begin{aligned} q(\boldsymbol{\theta}|\mathbf{x}) = B \exp\Bigg(- \Bigg(\sum_{j=1}^{r-1} \lambda_{\boldsymbol{\theta}}(x_j)(m_j + 1) + \gamma_r \lambda_{\boldsymbol{\theta}}(x_r) + D(\boldsymbol{\theta}; \delta\Bigg)\Bigg) \\ \times C(\boldsymbol{\theta}; \delta) \prod_{j=1}^{r} \lambda'_{\boldsymbol{\theta}}(x_j), \end{aligned} \tag{2.8}$$

where B is a normalizing constant.

$$\begin{aligned} B^{-1} = \int_{\Theta} \exp\Bigg(- \Bigg(\sum_{j=1}^{r-1} \lambda_{\boldsymbol{\theta}}(x_j)(m_j + 1) + \gamma_r \lambda_{\boldsymbol{\theta}}(x_r) + D(\boldsymbol{\theta}; \delta)\Bigg)\Bigg) \\ \times C(\boldsymbol{\theta}; \delta) \prod_{j=1}^{r} \lambda'_{\boldsymbol{\theta}}(x_j) d\boldsymbol{\theta}. \end{aligned} \tag{2.9}$$

For the general lifetime model (2.4) with a vector of parameters $\boldsymbol{\theta}$, the density function of the $(r+s)^{th}$ GOS given the first r GOS, is obtained from (2.2).

$$\begin{aligned} g_r(x_{r+s}|\boldsymbol{\theta}; \mathbf{x}) = \frac{c_{r+s-1}}{c_{r-1}} \lambda'_{\boldsymbol{\theta}}(x_{r+s}) \sum_{i=r+1}^{r+s} a_i^{(r)}(r+s) \\ \times \exp\Bigg(- \gamma_i \Bigg(\lambda_{\boldsymbol{\theta}}(x_{r+s}) - \lambda_{\boldsymbol{\theta}}(x_r)\Bigg)\Bigg), \end{aligned} \tag{2.10}$$

where

$$a_i^{(r)}(r+s) = \prod_{j=r+1_{j \neq i}}^{r+s} \frac{1}{\gamma_j - \gamma_i}. \tag{2.11}$$

Hence, for $s = 1, \ldots, (n-r)$, the Bayes predictive density function for the $(r+s)^{th}$ GOS is

$$
\begin{aligned}
H(x_{r+s}|\mathbf{x}) = \frac{Bc_{r+s-1}}{c_{r-1}} \sum_{i=r+1}^{r+s} a_i^{(r)}(r+s) \int_{\boldsymbol{\theta}} C(\boldsymbol{\theta};\delta)\lambda'_{\boldsymbol{\theta}}(x_{r+s}) \\
\times \exp\Bigg(- \Big(\gamma_r \lambda_{\boldsymbol{\theta}}(x_r) + \gamma_i[\lambda_{\boldsymbol{\theta}}(x_{r+s}) - \lambda_{\boldsymbol{\theta}}(x_r)] + D(\boldsymbol{\theta};\delta) \\
+ \sum_{j=1}^{r-1} \lambda_{\boldsymbol{\theta}}(x_i)(m_j+1)\Big)\Bigg) \prod_{j=1}^{r} \lambda'_{\theta}(x_j) d\boldsymbol{\theta}, \qquad (2.12)
\end{aligned}
$$

where B and $a_i^{(r)}(r+s)$ are given by (2.9) and (2.11) respectively.

3. Baysian prediction of future GOS'S

In this section, we suppose that the first r GOS $X(1, n, \tilde{m}, k) < \ldots < X(r, n, \tilde{m}, k)$ are observed and that $X(r+1, n, \tilde{m}, k) < \ldots < X(n, n, \tilde{m}, k)$ are to be predicted. Al-Hussaini et al. [10] predicted the future GOS when $\gamma_i = \gamma_j, i \neq j$. We shall obtain Bayesian prediction intervals and the Bayes predictive estimators for GOS $X(r+1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ when $\gamma_i \neq \gamma_j, i \neq j$.

For $s = 1, \ldots, (n-r)$, the survival function for the future $(r+s)^{th}$ GOS is obtained from (2.12) we have

$$
\begin{aligned}
P(X(r+s, n, \tilde{m}, k) > \nu|\mathbf{x}) &= \int_{\nu}^{\infty} H(x_{r+s}|\mathbf{x}) dx_{r+s} \\
&= \frac{Bc_{r+s-1}}{c_{r-1}} \sum_{i=r+1}^{r+s} \frac{a_i^{(r)}(r+s)}{\gamma_i} \int_{\Theta} C(\boldsymbol{\theta};\delta) \prod_{j=1}^{r} \lambda'_{\boldsymbol{\theta}}(x_j) \\
&\quad \times \exp\Bigg(- \Big(\gamma_i[\lambda_{\boldsymbol{\theta}}(\nu) - \lambda_{\boldsymbol{\theta}}(x_r)] + \gamma_r\lambda_{\boldsymbol{\theta}}(x_r) + D(\boldsymbol{\theta};\delta) \\
&\quad + \sum_{j=1}^{r-1} \lambda_{\boldsymbol{\theta}}(x_j)(m_j+1)\Big)\Bigg) d\boldsymbol{\theta}. \qquad (3.1)
\end{aligned}
$$

For $s = 1, \ldots, (n-r)$, the $\tau \times 100\%$ Bayesian prediction bounds for the future $(r+s)^{th}$ GOS are obtained by solving the equations

$$
P(X_{r+s} > L_s(\mathbf{x})|\mathbf{x}) = \frac{1+\tau}{2} = \tau_1 \quad \text{and} \quad P(X_{r+s} > U_s(\mathbf{x})|\mathbf{x}) = \frac{1-\tau}{2} = \tau_2.
$$

Where $L_s(\mathbf{x})$ and $U_s(\mathbf{x})$ are respectively the lower and upper Bayesian predictive bounds.

Now, the Bayes predictive estimator for the $(r+s)^{th}$ GOS under square error loss (SEL) can be obtained by (2.12) as

$$
\begin{aligned}
\tilde{x}_{r+s} &= E(X(r+s, n, \tilde{m}, k)|\mathbf{x}) \\
&= \int_{x_r}^{\infty} x_{r+s} H(x_{r+s}|\mathbf{x}) dx_{r+s}, \qquad (3.2)
\end{aligned}
$$

3.1. Examples

The family of Pareto distributions is used regularly in reliability and survival analysis. The pdf, cdf of a Pareto distribution are respectively

$$\begin{aligned} f(x|\alpha,\beta) &= \alpha\beta^{\alpha}(x+\beta)^{-(\alpha+1)}, \quad x,\alpha,\beta>0,\\ F(x|\alpha,\beta) &= 1-(\frac{x+\beta}{\beta})^{-\alpha}, \quad x,\alpha,\beta>0. \end{aligned}$$

Now

$$\boldsymbol{\theta}=(\alpha,\beta), \quad \lambda_{\boldsymbol{\theta}}(x)=\alpha\ln\frac{x+\beta}{\beta}, \quad \lambda'_{\boldsymbol{\theta}}(x)=\frac{\alpha}{x+\beta}, \tag{3.3}$$

Substituting (3.3) into (2.7), the likelihood function of the first r GOS is

$$\begin{aligned} L(\alpha,\beta,\mathbf{x}) &= c_{r-1}\alpha^{r}\exp\Big(-\alpha\Big(\gamma_r\ln\frac{x_r+\beta}{\beta}+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}\Big)\Big)\\ &\quad\times\exp\Big(-\sum_{j=1}^{r}\ln(x_j+\beta)\Big), \end{aligned} \tag{3.4}$$

where $\mathbf{x}=(x_1,\ldots,x_r)$.

For $s=1,\ldots,(n-r)$, the conditional density function in (2.10) for the $(r+s)^{th}$ GOS given the first r GOS is obtained as

$$g_r(x_{r+s}|\mathbf{x})=\frac{\alpha c_{r+s-1}}{c_{r-1}(x_{r+s}+\beta)}\sum_{i=r+1}^{r+s}a_i^{(r)}(r+s)\exp\big(-\alpha\gamma_i\ln\frac{x_{r+s}+\beta}{x_r+\beta}\big). \tag{3.5}$$

3.1.1. Pareto (α,β) model when α is the unknown parameter

We assume that the parameter α is a random variable with a gamma conjugate prior density.

$$\pi(\alpha)=\frac{b^{a}}{\Gamma(a)}\alpha^{a-1}\exp(-b\alpha), \alpha>0, \tag{3.6}$$

It follows from (3.4) and (3.6) that the posterior density function of the parameter α can be expressed as

$$q(\alpha|\mathbf{x})=B\alpha^{r+a-1}\exp\Big(-\alpha\Big(b+\gamma_r\ln\frac{x_r+\beta}{\beta}+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}\Big)\Big), \tag{3.7}$$

where

$$b+\gamma_r\ln\frac{x_r+\beta}{\beta}+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}>0$$

and B is a normalizing constant given by

$$B^{-1}=\frac{\Gamma(r+a)}{\Big(b+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}+\gamma_r\ln\frac{x_r+\beta}{\beta}\Big)^{r+a}}.$$

Hence, the Bayes predictive density function for the $(r+s)^{th}$ GOS from (3.5) and (3.7) is obtained as

$$H(x_{r+s}|\mathbf{x}) = \frac{B_1}{x_{r+s}+\beta}\sum_{i=r+1}^{r+s} a_i^{(r)}(r+s) \times\Big(1+\frac{\gamma_i \ln\frac{x_{r+s}+\beta}{x_r+\beta}}{b+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}+\gamma_r\ln\frac{x_r+\beta}{\beta}}\Big)^{-(r+a+1)}, \tag{3.8}$$

where

$$B_1 = \frac{c_{r+s-1}(r+a)}{c_{r-1}\Big(b+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}+\gamma_r\ln\frac{x_r+\beta}{\beta}\Big)}.$$

It follows that the survival function for the $(r+s)^{th}$ GOS is

$$\begin{aligned} P(X(r+s,n,\tilde{m},k)\geq \nu|\mathbf{x}) &= \int_{\nu}^{\infty} H(x_{r+s}|\mathbf{x})dx_{r+s} \\ &= \frac{c_{r+s-1}}{c_{r-1}}\sum_{i=r+1}^{r+s}\frac{a_i^{(r)}(r+s)}{\gamma_i} \\ &\quad\times\Big(1+\frac{\gamma_i\ln\frac{\nu+\beta}{x_r+\beta}}{b+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}+\gamma_r\ln\frac{x_r+\beta}{\beta}}\Big)^{-(r+a)}. \end{aligned} \tag{3.9}$$

From (3.2), putting $u = x_{r+s}+\beta$, the Bayes predictive estimators for the future $(r+s)^{th}$ GOS under SEL are obtained as

$$\tilde{x}_{r+s} = -\frac{\beta c_{r+s-1}}{c_{r-1}}\sum_{i=r+1}^{r+s}\frac{a_i^{(r)}(r+s)}{\gamma_i}+B_1\sum_{i=r+1}^{r+s}a_i^{(r)}(r+s) \times\int_{x_r+\beta}^{\infty}\Big(1+\frac{\gamma_i\ln\frac{u}{x_r+\beta}}{b+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}+\gamma_r\ln\frac{x_r+\beta}{\beta}}\Big)^{-(r+a+1)}du.$$

From (2.3) in [12] we have

$$\tilde{x}_{r+s} = -\frac{\beta c_{r+s-1}}{c_{r-1}}\prod_{i=r+1}^{r+s}\frac{1}{\gamma_i}+B_1\sum_{i=r+1}^{r+s}a_i^{(r)}(r+s) \times\int_{x_r+\beta}^{\infty}\Big(1+\frac{\gamma_i\ln\frac{u}{x_r+\beta}}{b+\sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta}+\gamma_r\ln\frac{x_r+\beta}{\beta}}\Big)^{-(r+a+1)}du. \tag{3.10}$$

3.1.2. Pareto (α, β) model when α, β are both unknown
Here, we assume that the joint prior density for the parameters is of the form $\pi(\alpha, \beta) = \pi_2(\beta \mid \alpha)\pi_1(\alpha)$ where $\pi_1(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}\exp(-b\alpha)$, $\alpha > 0$ and $\pi_2(\beta \mid \alpha) = l\alpha(c)^{l\alpha}(\beta + c)^{-(l\alpha+1)}, \beta, \alpha > 0$. Thus

$$\pi(\alpha, \beta) = \frac{b^a l}{\Gamma(a)}\alpha^a \exp\big(-\ln(\beta + c)\big)\exp\Big(-\alpha(b + l\ln(\beta + c) - l\ln c)\Big) \tag{3.11}$$

In other words, $\alpha \sim \Gamma(a, b)$ and $\beta|\alpha \sim Pareto(l\alpha, c)$. From (3.4) and (3.11), the joint posterior density of the parameters α and β is obtained as

$$\begin{aligned} q(\alpha, \beta|\mathbf{x}) = B\alpha^{r+a}\exp\Big(-\alpha\big(b + \gamma_r \ln\frac{x_r + \beta}{\beta} + \sum_{j=1}^{r-1}(m_j + 1)\ln\frac{x_j + \beta}{\beta} \\ + l\ln(\beta + c) - l\ln c\big)\Big)\exp\Big(-\Big(\sum_{j=1}^{r}\ln(x_j + \beta) + \ln(\beta + c)\Big)\Big), \end{aligned} \tag{3.12}$$

where

$$b + \gamma_r \ln\frac{x_r + \beta}{\beta} + \sum_{j=1}^{r-1}(m_j + 1)\ln\frac{x_j + \beta}{\beta} + l\ln(\beta + c) - l\ln c > 0$$

and B is a normalizing constant. We have

$$\begin{aligned} B^{-1} = \int_0^\infty \Gamma(r + a + 1)\Big(b + \gamma_r \ln\frac{x_r + \beta}{\beta} + \sum_{j=1}^{r-1}(m_j + 1)\ln\frac{x_j + \beta}{\beta} \\ + l\ln(\beta + c) - l\ln c\Big)^{-(r+a+1)} \\ \times \exp\Big(-\Big(\sum_{j=1}^{r}\ln(x_j + \beta) + \ln(\beta + c)\Big)\Big)d\beta. \end{aligned} \tag{3.13}$$

From (3.5) and (3.12), the Bayes predictive density function of the $(r + s)^{th}$ GOS is

$$\begin{aligned} H(x_{r+s}|\mathbf{x}) = \frac{c_{r+s-1}(r + a + 1)}{c_{r-1}I_0}\sum_{i=r+1}^{r+s} a_i^{(r)}(r + s) \\ \times \int_0^\infty (x_{r+s} + \beta)^{-1}\Big(b + \gamma_r \ln\frac{x_r + \beta}{\beta} + \gamma_i \ln\frac{x_{r+s} + \beta}{x_r + \beta} \\ + \sum_{j=1}^{r-1}(m_j + 1)\ln\frac{x_j + \beta}{\beta} + l\ln(\beta + c) - l\ln c\Big)^{-(r+a+2)} \\ \times \exp\Big(-\Big(\sum_{j=1}^{r}\ln(x_j + \beta) + \ln(\beta + c)\Big)\Big)d\beta, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} I_0 = \int_0^\infty \Big(b + \gamma_r \ln\frac{x_r + \beta}{\beta} + \sum_{j=1}^{r-1}(m_j + 1)\ln\frac{x_j + \beta}{\beta} + l\ln(\beta + c) \\ - l\ln c\Big)^{-(r+a+1)}\exp\Big(-\Big(\sum_{j=1}^{r}\ln(x_j + \beta) + \ln(\beta + c)\Big)\Big)d\beta. \end{aligned} \tag{3.15}$$

By (3.14),

$$\begin{aligned}
P(X(r+s,n,\tilde{m},k) \geq \nu|\mathbf{x}) &= \int_{\nu}^{\infty} H(x_{r+s}|\mathbf{x})dx_{r+s} \\
&= \frac{c_{r+s-1}}{c_{r-1}I_0} \sum_{i=r+1}^{r+s} \frac{a_i^{(r)}(r+s)}{\gamma_i} \int_0^{\infty} \Big(b+\gamma_r \ln\frac{x_r+\beta}{\beta} + l\ln(\beta+c) \\
&\quad + \sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta} - l\ln c + \gamma_i \ln\frac{\nu+\beta}{x_r+\beta}\Big)^{-(r+a+1)} \\
&\quad \times \exp\Big(-\Big(\sum_{j=1}^{r}\ln(x_j+\beta)+\ln(\beta+c)\Big)\Big)d\beta, \qquad (3.16)
\end{aligned}$$

where I_0 is given by (3.15).

As mentioned above, for $s = 1, ..., n-r$ the lower and upper $\tau \times 100\%$ Bayesian prediction bounds for the future $(r+s)^{th}$, GOS in the both cases, can be obtained numerically by equating $\Pr(X(r+s,n,\tilde{m},k) \geq \nu \mid \mathbf{x})$ in (3.10) and (3.16), to $(\frac{1+\tau}{2})$ and $(\frac{1-\tau}{2})$ respectively.

By (3.2) and (3.14), the Bayes predictive estimator for the future $(r+s)^{th}$, GOS under SEL is

$$\begin{aligned}
\tilde{x}_{r+s} &= \frac{c_{r+s-1}(r+a+1)}{c_{r-1}I_0} \sum_{i=r+1}^{r+s} a_i^{(r)}(r+s) \\
&\quad \times \int_{x_r}^{\infty}\int_0^{\infty} \frac{x_{r+s}}{x_{r+s}+\beta}\Big(b+\gamma_r\ln\frac{x_r+\beta}{\beta}+\gamma_i\ln\frac{x_{r+s}+\beta}{x_r+\beta} \\
&\quad + \sum_{j=1}^{r-1}(m_j+1)\ln\frac{x_j+\beta}{\beta} + l\ln(\beta+c) - l\ln c\Big)^{-(r+a+2)} \\
&\quad \times \exp\Big(-\Big(\sum_{j=1}^{r}\ln(x_j+\beta)+\ln(\beta+c)\Big)\Big)d\beta dx_{r+s}.
\end{aligned}$$

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